Home assignment 3 — Symmetry TIF310/FYM310, period II, 2021

Deadline Friday Jan. 7

Hand in solutions produced by TEX by mail to martin.cederwall@chalmers.se or printed in a box outside room Origo 6102. Good luck!

3.1. Let a Dynkin diagram be defined as a circle of $n+1$ nodes, where each node is connected to its two neighbours with single lines. Show that it can not be the Dynkin diagram of a finite-dimensional Lie algebra, *i.e.*, that a set of roots with the corresponding scalar products do not form a basis of Euclidean $(n+1)$-dimensional space. (The algebra A_n^+ defined by this Dynkin diagram is indeed infinite-dimensional, a so called affine Lie algebra.)

3.2. The Chern–Simons action in three dimensions is defined as

$$S = \frac{k}{4\pi}\int \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

where A is a connection 1-form taking values in the Lie algebra of some gauge group G. Show that the action is invariant under gauge transformations, up to a total derivative, and that the equations of motion are $F = 0$. (This is a *topological* field theory. Although there are no local degrees of freedom, this type of action has many applications in physics and mathematics.)

3.3. Show that the Poisson bracket is a Lie bracket. If Q and Q' are conserved charges, show that also $\{Q, Q'\}$ is a conserved charge, and that conserved charges therefore generate a Lie algebra.

3.4. A construction of the 14-dimensional Lie algebra G_2. Inspect the root space of G_2, and verify that all roots of G_2 are weights of $\mathfrak{sl}(3)$, and that it consists of the weights for the adjoint (the roots) of $\mathfrak{sl}(3)$ together with the weight for the two 3-dimensional representations, which we can call **3** and $\bar{\mathbf{3}}$. There should be a formulation of G_2 with manifest $\mathfrak{sl}(3)$ and generators $J_m{}^n$, K_m and L^m. Construct the brackets by making some Ansatz and checking the Jacobi identities. (Hint: Only some of the Jacobi identities are "non-trivial", in the sense that they do not follow from the $\mathfrak{sl}(3)$ covariance.)

3.5. Symmetries of the Kepler problem. Consider the motion of a Newtonian particle with mass m in the central potential $V(\vec{r}) = -\frac{k}{r}$. Show that the components of the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ fulfil $\{L_i, H\} = 0$, and are conserved charges. Which is the Lie algebra generated by these charges? Consider the Runge–Lenz vector

$$\vec{A} = \vec{p} \times \vec{L} - km\hat{r}\ .$$

The dimensionless vector $\frac{\vec{A}}{km}$ is the so called eccentricity vector. Show that $\vec{A}$ is conserved. It is convenient to rescale the Runge–Lenz vector to

$$\vec{B} = \frac{\vec{A}}{\sqrt{2m|E|}} ,$$

where E is the energy, for $E \neq 0$. Investigate the algebra of conserved charges under the Poisson bracket. It may be different in the cases $E < 0$, $E = 0$ and $E > 0$. Such "hidden symmetries" may be used to relate solutions to the equations of motion with the same energy to each other. Discuss (without explicit calculation) whether *e.g.* a circular orbit can be transformed into an elliptic (but non-circular) one using the corresponding hidden symmetry group.