Home assignment 1, 2022 — Symmetry TIF310/FYM310

Deadline Friday Nov. 18

Hand in solutions, produced by TEX, by mail to martin.cederwall@chalmers.se (or printed in a box outside room Origo 6102, which requires transversing locked doors). Good luck!

1.1. Show that the group $SU(2)$ is the 3-dimensional sphere S^3.
(Hint: show that any element $M \in SU(2)$ can be represented as a 2×2 complex matrix

$$M = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} , \quad a\bar{a} + b\bar{b} = 1 \text{ .)}$$

1.2. Consider the Lie algebra $\mathfrak{so}(4, \mathbb{R})$ of rotations in 4 euclidean dimensions.
a) Show that $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.
b) Describe a 4-dimensional vector as an element of a representation module of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

1.3. Consider the discrete group acting on vectors in $\mathbb{R}^2 = \mathbb{C}$ which is generated by reflections in the real line and in the line through the origin and the point $e^{2\pi i/3}$. Call these two group elements a and b. The statement that they "generate" a group means that one considers all elements obtained by applying them repeatedly, in some order. Each of the generators (being a reflection) squares to the identity element, $a^2 = I = b^2$. They generate cycles of length 2. What is the length of the cycle generated by ab? How many elements does the group contain? Make a complete table of the group composition, and show that it in fact is the "symmetric group" S_3 of permutations of three elements.

1.4. *a)* The Lie algebra $\mathfrak{so}(10)$ of rotations in 10 dimensions has an irreducible 45-dimensional representation. Describe the representation module in terms of tensors.
b) Same question, but for an irreducible 54-dimensional representation (this may require the use of some invariant tensor).
c) What is the tensor product of the 10-dimensional vector representation with itself?

1.5. The Poincaré algebra in d dimensions consists of rotations (elements in $\mathfrak{so}(d)$) together with translations. Find a maximal proper ideal of the Poincaré algebra. If it is "divided out" (*i.e.*, if one forms the Lie algebra on the equivalence classes modulo the ideal), which algebra is obtained?