Home assignment 2 — Symmetry TIF310/FYM310 2022

Deadline Friday Dec. 9

Hand in solutions produced by TEX by mail to martin.cederwall@chalmers.se or printed in a box outside room Origo 6102. Good luck!

Maximum 2 points per problem.

2.1. Find the static, spherically symmetric, solution to the Klein–Gordon equation with $m \neq 0$ in $d = 4$ Minkowski space corresponding to a point source at the origin.

2.2. Show that the action for electromagnetic fields

$$S = -\frac{1}{4}\int d^4x\, \sqrt{|g|} F^{mn}F_{mn}$$

is unchanged when the metric g_{mn} is replaced by the same metric multiplied by some (non-zero) function, *i.e.*, $g_{mn}(x) \mapsto e^{2\phi(x)}g_{mn}(x)$.

2.3. The tensor product of two spinor representations contains antisymmetric tensors. This is due to the existence of invariant tensors $\gamma^{m_1\ldots m_p}$.
a) As an example, take spinors in 9 dimensions. The dimension of the spinor representation S is 16. Count the number of matrices $\gamma^{m_1\ldots m_p}$ for different values of p, and determine, by demanding that the numbers sum up to the dimensions of the symmetric and antisymmetric parts of $S \otimes S$, which of these (with one index lowered to $(\gamma^{m_1\ldots m_p})_{\alpha\beta}$) are symmetric and which are antisymmetric in $\alpha\beta$.
b) Tensor products of more than two spinors are more difficult. There may be identities, so called Fierz identities, for products of more than one matrix $\gamma^{m_1\ldots m_p}$. In 3-dimensional Minkowski space, *i.e.*, for γ-matrices of $\mathfrak{so}(1,2)$, show that $\gamma_{m(\alpha\beta}\gamma^m{}_{\gamma\delta)} = 0$. One way of checking this is to contract the expression with all matrices $M^{(\gamma\delta)}$ and using the properties of the γ matrices. There may be simpler ways. Is there some invariant tensor with the same index structure?

2.4. Consider the rank 2 simple Lie algebra G_2. This algebra has a 7-dimensional representation. Construct it by acting with lowering operators on some highest weight state. (A good starting point can be to draw a picture of the root lattice, and then construct the coroots and fundamental weights.)

2.5. A non-linear realisation. Consider the quotient space $SL(2,\mathbb{R})/U(1)$, defined as equivalence classes of elements in $SL(2,\mathbb{R})$ modulo the right action of a $U(1)$. Two elements g and g' in $SL(2,\mathbb{R})$ (2×2 real matrices with unit determinant) are considered equivalent if they are related by a $U(1)$ transformation as $g' = gh$, where

$$h = e^{\theta j}\ , \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ .$$

Show that elements in $SL(2,\mathbb{R})$ are in the same equivalence class as an element of the form

$$g = \frac{1}{\sqrt{y}}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\ . \qquad (2.1)$$

Use this parametrisation to derive the transformation of the complex number $z = x + iy$ for such a representative of the equivalence class under the left action $g \mapsto Mg$ with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}) \; , \quad ad - bc = 1 \; .$$

(Hint: In order to get Mg back to an element of the form (2.1)*, you will need to use the equivalence.)*
Show that the metric

$$ds^2 = \frac{dz d\bar{z}}{(\mathrm{Im}\, z)^2}$$

is invariant under $SL(2, \mathbb{R})$. (This is the so called Poincaré upper half plane, describing a 2-dimensional hyperbolic space with constant curvature.) Discuss what the $SL(2, \mathbb{R})$ isometry means. Is this a maximally symmetric space?