Home assignment 3 — Symmetry TIF310/FYM310, period II, 2022

Deadline Sunday Jan. 8. 2023

Hand in solutions produced by TEX by mail to martin.cederwall@chalmers.se or printed in a box outside room Origo 6102. Good luck!

3.1. In 2-dimensional Minkowski space, let $u = \frac{1}{\sqrt{2}}(x^0+x^1)$, $v = \frac{1}{\sqrt{2}}(x^0-x^1)$. What is the metric in this coordinate system? Show that any transformation to new coordinates $u' = f(u)$, $v' = g(v)$, where f and g are strictly monotonous real functions, is a conformal transformation, *i.e.*, a transformation under which $g_{mn} \mapsto e^{2\phi} g_{mn}$ for some function ϕ.
Consider the infinitesimal form of such transformations. In $d > 2$ Minkowski space, the conformal algebra is $\mathfrak{so}(2,d)$, but in $d = 2$ it is infinite-dimensional. Which transformations form an $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ subalgebra?

3.2. Let a Dynkin diagram be defined as a circle of $n+1$ nodes, where each node is connected to its two neighbours with single lines. Show that it can not be the Dynkin diagram of a finite-dimensional Lie algebra, *i.e.*, that a set of roots with the corresponding scalar products do not form a basis of Euclidean $(n+1)$-dimensional space. (The algebra A_n^+ defined by this Dynkin diagram is indeed infinite-dimensional, a so called affine Lie algebra.)

3.3. Inspect the root space of G_2, and verify that all roots of G_2 are weights of $\mathfrak{sl}(3)$, and that it consists of the weights for the adjoint (the roots) of $\mathfrak{sl}(3)$ together with the weight for the two 3-dimensional representations, which we can call $\mathbf{3}$ and $\bar{\mathbf{3}}$. There should be a formulation of G_2 with manifest $\mathfrak{sl}(3)$ and generators $J_m{}^n$, K_m and L^m. Construct the brackets between these by making some Ansatz (on tensor form, only using invariant tensors of $\mathfrak{sl}(3)$) and checking the Jacobi identities.
Hints: Addition of weights can give clues to what can appear on the right hand sides. Only some of the Jacobi identities are "non-trivial", in the sense that they do not follow from the $\mathfrak{sl}(3)$ *covariance.*

3.4. The energy-momentum (or stress-energy) tensor can be derived as the variation of an action with respect to the metric as

$$T^{mn} = \frac{2}{\sqrt{|g|}} \frac{\partial \mathscr{L}}{\partial g_{mn}} ,$$

where $\mathscr{L}$ is the Lagrangian density. This applies also for a theory defined in flat space, but then the metric has to be reinstated so that coordinate invariance of the action is manifest. Use this definition to derive the energy-momentum tensor for Maxwell theory, and identify the usual forms of the energy density as T^{00} and the Poynting vector as T^{0i}. Show that the Maxwell energy-momentum tensor is traceless precisely in $d = 4$, and relate this property to the invariance of the action under a rescaling of the metric. This is a sign of conformal invariance.
Hint: be careful with precisely how the metric enters $\mathscr{L}$*.*

3.5. Consider a (Newtonian) particle moving in an isotropic harmonic potential in n space dimensions, so that $V(x) = \frac{1}{2}kx^i x^i$. Show that the symmetry is $U(n)$ rather than the (expected) $O(n)$.
Hint: It is useful to write down the Hamiltonian in terms of "creation and annihilation operators".
Such "hidden symmetries" can be used to relate solutions with equal energy to each other.