Lecture notes, Symmetry TIF310/FYM310

Contents

1. Preliminaries—Vector spaces and tensors

1.1. Vector spaces

A vector space V is a set of objects which can be added and multiplied by numbers. Many of the objects in this course will belong to vector spaces. This applies to the Lie algebras as well as their representation modules.

A vector space V over a field $\mathbb{K}$ (in this course, $\mathbb{K}$ will always be $\mathbb{R}$ or $\mathbb{C}$) is a set of objects that can be added and multiplied by elements in $\mathbb{K}$. The addition is commutative and associative: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$, $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$, for all $\vec{a},\vec{b},\vec{c}\in V$. It has an identity element, the null vector $\vec{0}$, such that $\vec{a}+\vec{0}=\vec{a}$ and an inverse $-\vec{a}$ such that $\vec{a}+(-\vec{a})=\vec{0}$. The multiplication satisfies $x(y\vec{a})=(xy)\vec{a}$, $x(\vec{a}+\vec{b})=x\vec{a}+x\vec{b}$ and $1\vec{a}=\vec{a}$. In what follows, the arrows over vectors will be dropped.

All these properties are natural, and satisfied by vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$. Examples of vector spaces are, in addition to $\mathbb{R}^n$ or $\mathbb{C}^n$, spaces of functions, *e.g.* polynomials of some given degree, or solutions to some homogeneous equations. For our purposes, we can think of any (finite-dimensional) vector space as $\mathbb{K}^n$.

A basis can be chosen as $\{e_i\}_{i=1}^{\dim V}$, where there is no linear dependence between the basis elements, and every vector in V can be written as $v=\sum_{i=1}^{\dim V}v^ie_i$, which we write as $v=v^ie_i$, with an invisible summation sign every time an index is repeated.

A vector space has a priori no notion of a scalar product. But one can always define the dual vector space V^* to a vector space V. We use the notation $\langle u,v\rangle$ for the natural scalar product, given by $\langle e^{*i},e_j\rangle=\delta^i_j$, so that $\langle u,v\rangle=u_iv^i$. The dual vector space can be defined as the set of linear functions $V\to\mathbb{K}$, which is itself a vector space.

The choice of basis is of course completely arbitrary. A change of basis from $\{e_i\}$ to $\{e'_i\}$ amounts to a linear transformation $e'_i={M_i}^je_j$, where M is a matrix with $\det M\neq 0$. This is a transformation in the group $GL(\dim V,\mathbb{K})$ (or simply $GL(V)$, the set of linear maps, endomorphisms, from V to itself), see Section 2.1. The same vector $v=v^ie_i=v'^ie'_i$ then has components

$$v'^i={(M^{-1})_j}^iv^j \tag{1.1}$$

in the new basis. A dual vector then has components $u'_i={M_i}^ju_j$. The scalar product $\langle u,v\rangle$ remains unchanged.

It is thus important that summation always is performed with one index up and one down, *i.e.*, between a vector index and a dual vector (covector) index. A confusion which may arise from the acquaintance with vectors in $\mathbb{R}^3$ is that they have come equipped with a metric, a length function $V\times V\to\mathbb{R}$: $v\times w\mapsto(v,w)=\sum_{i=1}^3v^iw^i$. This is not built into the concept of a vector space, but is an additional structure. A scalar product relies on introducing a matrix η_{ij} with $\det\eta\neq 0$, and defining $(v,w)=\eta_{ij}v^iw^j$. Equivalently, a metric is a (non-degenerate) map $V\to V^*$. The metric η will depend on the choice of basis, and is only invariant under an orthogonal subgroup of $GL(V)$, see Section 2.1. In the example with $\mathbb{R}^3$ and the Euclidean metric, it is normally taken to be the unit matrix, $\eta_{ij}=\delta_{ij}$, which makes it "invisible", which may add to the confusion.

Vector space (over the same field $\mathbb{K}$) can be "added" and multiplied.

The direct sum $V\oplus W$ of the vector spaces V and W have elements $v\oplus w$ with $v\in V$, $w\in W$. This can be thought of as a pair (v,w). If one think of elements in V and W as column vectors of length m and n (the respective dimensions), an element in $V\oplus W$ is a column vector of length $m+n$. Example: $\mathbb{R}^m\oplus\mathbb{R}^n=\mathbb{R}^{m+n}$.

The tensor product $V\otimes W$ is defined as the vector space with basis $\{e_i\otimes e'_{i'}\}$, where $\{e_i\}$ and $\{e'_{i'}\}$ are bases for V and W, respectively. The number of basis elements is thus mn, where m and n are the respective dimensions of V and W. If one thinks of elements in V and W as vectors of length m and n, an element in $V\otimes W$ can be thought of as an $m\times n$ matrix. Example: $\mathbb{R}^m\otimes\mathbb{R}^n=\mathbb{R}^{mn}$. (Another way of saying this is that any element in $V\otimes W$ can be written as a sum of terms $v\otimes w$, with the equivalence relations $kv\otimes w\approx v\otimes kw=k(v\otimes w))$, where $k\in\mathbb{K}$, and $(v+v')\otimes w=v\otimes w+v'\otimes w$, $v\otimes(w+w')=v\otimes w+v\otimes w'$.)

1.2. Tensors

Tensor formalism is a notational device, as well as a way of keeping track of transformation properties. Say that we have a vector v in some vector space V. As above, it can be written in terms of the elements in some basis as $v=v^ie_i$. In tensor notation, v is written as "v^i", with an explicit free index i. It can be considered as a list of the components in the basis $\{e_i\}$. The vector v^i is assumed to behave as in eq. (1.1) under a change of basis, which is an element of $GL(V)$. Analogously, a covector is written w_i. Note, again, that there is *a priori* no way to translate between upper and lower indices. A general tensor belongs to the (iterated) tensor product of a number of copies of V and V^*, and is written in terms of its components as $t^{i_1\ldots i_p}_{j_1\ldots j_q}$. Its transformation rule is

$$t'^{i_1\ldots i_p}_{j_1\ldots j_q}=(M^{-1})_{k_1}{}^{i_1}\ldots(M^{-1})_{k_p}{}^{i_p}M_{j_1}{}^{l_1}\ldots M_{j_q}{}^{l_q}t^{k_1\ldots k_p}_{l_1\ldots l_q}\ . \qquad (1.2)$$

The allowed operations with (general) tensors are symmetrisations and antisymmetrisations in groups of indices and contractions between upper and lower indices. Any result of such operations is guaranteed to give a result which again is a tensor.

Often, a vector space comes equipped with some extra structure, for example a metric, as mentioned above. This structure is typically such that $GL(V)$ is restricted to some subgroup (see Section 2.1), and tensors are considered as transforming under representations (see Section 2.4) of the subgroup in question. Say, for example, that there is a metric g_{ij}. It will be invariant under an orthogonal subgroup of $GL(V)$. The rules for manipulating tensors, stated above, do not change, but one is also allowed to include g_{ij} in the expressions. The tensors are then considered as tensors with respect to orthogonal transformations. If some other structure is introduced, it will always involve additional invariant (under some subgroup of $GL(V)$) tensors, that can be used in the same fashion.

Later in Section 4, we will also consider tensors under the infinite-dimensional group of general coordinate transformations.

2. Groups and algebras

2.1. Groups

A *group* is a set G with a binary operation (a "product") xy defined on any pair of elements $x, y \in G$, that fulfils the requirements

- The product is associative: $(xy)z = x(yz)$, $\forall x, y, z \in G$;
- G contains a unit element I: $Ix = x = xI$, $\forall x \in G$
- Any element $x \in G$ has a unique inverse x^{-1}, such that $x^{-1}x = I = xx^{-1}$.

Examples:

(if needed, check the product, its properties, and the existence of unit element and inverse!)

- The group $\mathbb{Z}$ of integers under addition. The unit element is 0.
- The cyclic groups $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, where the elements are $\{0, 1, \ldots, n-1\}$ and the product is defined by addition modulo n.
- The non-zero real numbers $\mathbb{R}^*$ under multiplication.

All the examples above are *abelian* groups, which means that $xy = yx$ for all pairs of elements. Example of non-abelian groups are

- The symmetric group S_n of permutations of n elements.
- The group of rotations in 3 euclidean dimensions.

Some other definitions:

A *subgroup* $H \subset G$ is a subset of elements such that H itself is a group. A proper subgroup of G is a subgroup which is neither G itself nor the trivial subgroup $\{I\}$.

A *normal subgroup* $H \subset G$ is a subgroup which is invariant under *conjugation*, *i.e.*, $h \in H$, $g \in G \Rightarrow ghg^{-1} \in H$.

The *direct product* of two groups G and H is the set of ordered pairs $x = (g, h)$ with product $xx' = (gg', hh')$. (Check identity and inverse!)

The *left (right) coset* G/H ($H\backslash G$) is the set of elements $x \in G$ modulo the equivalence relation $x \approx xh$ ($x \approx hx$) for all $h \in H$.

2.2. Lie Groups

A *Lie group* is a group which is also a manifold.

Examples:

- The group $GL(n, \mathbb{R})$, the group of (general) linear transformations on $\mathbb{R}^n$. Elements can be represented as real $(n \times n)$-matrices. Note that matrix multiplication is associative. The same holds for $GL(n, \mathbb{C})$.
- The group of special linear transformations, $SL(n, \mathbb{R})$, is defined as the subgroup of $GL(n, \mathbb{R})$ consisting of elements x with $\det x = 1$.
- The orthogonal group $O(n, \mathbb{R})$ (or $O(n, \mathbb{C})$) of matrices M with $M^{-1} = M^t$.
- The special orthogonal group $SO(n, \mathbb{R})$ (or $SO(n, \mathbb{C})$) of orthogonal matrices with unit determinant.
- The unitary group $U(n)$ of unitary $(n \times n)$-matrices. Unitary means that $M^{-1} = M^\dagger$.
- The special unitary group $SU(n)$ of unitary matrices with unit determinant.
- The symplectic group $Sp(2n, \mathbb{R})$ (or $Sp(2n, \mathbb{R})$) of $(2n \times 2n)$-matrices M preserving a non-degenerate skew-symmetric form ω. Let
$$\epsilon = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{2.1}$$
and $\omega(x, y) = x^t \epsilon y$ for $x, y \in \mathbb{R}^{2n}$. Then one demands that $\omega(Mx, My) = \omega(x, y)$. This amounts to $M^t \epsilon M = \epsilon$.

This list contains what is often called the "classical" matrix groups.

Exercises

2.1. Is $\mathbb{Z}$ a group under multiplication?

2.2. Is $\mathbb{N}$, the natural numbers (including 0), a group under addition?

2.3. Do the rational numbers $\mathscr{Q}$ form a group under multiplication?

2.4. How many elements are there in the *symmetric group* S_n of permutations of n elements? Do the even permutations form a subgroup? A normal subgroup?

2.5. What is the dimension of the group of orthogonal rotations in d dimensions?

2.6. If $m, n \in \mathbb{Z}$, is $\mathbb{Z}_{mn} = \mathbb{Z}_m \times \mathbb{Z}_n$?

2.7. Show that the left and right cosets coincide if H is a normal subgroup of G. Show that G/H then is a group.

2.8. What is the dimension of $Sp(2n, \mathbb{R})$?

2.9. Show that $Sp(2, \mathbb{R}) \simeq SL(2, \mathbb{R})$.

2.3. From Lie groups to Lie algebras

Even the classical groups are complicated objects. Recall *e.g.* the parametrisation of $SO(3, \mathbb{R})$ in terms of Euler angles. It is much easier to consider the *tangent space* at the identity element. By dealing with elements that are (infinitesimally) close to the identity, most of the structure of a Lie group is encoded in the *Lie algebra*, which is a vector space. This is precisely what one does when one describes rotation in terms of a rotation vector $\vec{\omega}$, instead of finite rotations in finite amount of time.

The idea is to write a group element g as $g = e^a$, where a is an element in the tangent space to G, which will become the Lie algebra $\mathfrak{g}$. As an analogy, think of the group of translations in $\mathbb{R}$ (this group is really $\mathbb{R}$ under addition). Consider a function $f(x)$. Assuming differentiability etc., it can be given as a Maclaurin series

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{d^k f}{dx^k}(0) = e^{x \frac{d}{dy}} f(y)|_{y=0} \ . \tag{2.2}$$

The finite translation is written as the exponentiation of an infinitesimal one, and we can say that translations are *generated* by the derivative. This hold equally in $\mathbb{R}^n$.

For a matrix x, the exponential function is defined as

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \ , \tag{2.3}$$

and the series converges. The inverse function, the logarithm, is given by the series

$$\log(1 + x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} \ , \tag{2.4}$$

and it is defined at least in some neighbourhood of the unit matrix.

An arbitrary element infinitesimally close to the unit element can be written as $x = e^{\epsilon a} \approx 1 + \epsilon a$, where ϵ is infinitesimally small. Given an element a in the tangent space, the group elements $y(t) = e^{ta}$ form an abelian 1-dimensional subgroup $G_a \subset G$. In the neighbourhood of the unit element, G_a looks like $\mathbb{R}$ (if $t \in \mathbb{R}$; if $t \in \mathbb{C}$ the group is $\mathbb{C}$).

Before we turn to the important question of what happens to the group multiplication, let us take an example. Take an orthogonal matrix M and let $M = e^A$. Then, $M^t = (e^A)^t = e^{A^t}$, and the condition $M^tM = 1$ turns into $1 = e^{A^t}e^A$. If $A^t = -A$, this is fulfilled. The tangent space (at the identity) of the space of orthogonal matrices is the vector space of antisymmetric matrices.

Exercise:

2.10. What are the corresponding statements for special linear matrices, for unitary and special unitary matrices, for symplectic matrices? Show that the properties are preserved by the commutator.

Answer:

$$\begin{array}{lcl} J \in \mathfrak{sl}(n) & : & \mathrm{tr}J = 0 \\ J \in \mathfrak{u}(n) & : & J + J^\dagger = 0 \\ J \in \mathfrak{su}(n) & : & J + J^\dagger = 0,\ \mathrm{tr}J = 0 \\ J \in \mathfrak{sp}(n) & : & J^t\epsilon + \epsilon J = 0 \end{array} \tag{2.5}$$

Now, we need some kind of product on the tangent space, that encodes the same amount of information as the group product, at least in the vicinity of the unit element. If two element $x = e^{\epsilon a}$, $x' = e^{\epsilon' a'} \in G$ close to the unit element commute, $xy = yx$, the corresponding 2-parameter subgroup $G_{a,a'}$ is abelian and $G_{a,a'}$ is (locally) the group of translations in 2 dimensions. Then also $aa' = a'a$ Let us look for an expression that encodes the deviation from commutativity. Take the expression $H(x, y) = xyx^{-1}y^{-1}$, and let $x = e^a$, $y = e^b$. Expanding to bilinear order in a, b, we get

$$\begin{aligned} H(x, y) &= (1 + a + \tfrac{1}{2}a^2)(1 + b + \tfrac{1}{2}b^2)(1 - a + \tfrac{1}{2}a^2)(1 - b + \tfrac{1}{2}b^2) + \ldots \\ &= 1 + [a, b] + \ldots \approx e^{[a,b]} \ . \end{aligned} \tag{2.6}$$

The group product "induces" an antisymmetric product, the commutator, or *Lie bracket*. The important and defining property of the Lie bracket, which follows from associativity of the group elements (see exercise 2.12), is the Jacobi identity

$$J(a, b, c) \equiv [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \ . \tag{2.7}$$

The Jacobi identity is automatically satisfied for matrix commutators.

One may wonder if the structure of the Lie algebra encoded by the Lie bracket captures the full structure of the group, since it deals only with group multiplication in the vicinity of the identity. The answer is that it "almost" does, and what it misses is only "global" information. What this means will become clear in examples. The "local" structure of the group is fully encoded in the Lie algebra. Given a Lie group G, the corresponding Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ is uniquely defined. However, different Lie groups can correspond to the same Lie algebra. More about that later.

Definition: A *Lie algebra* $\mathfrak{g}$ is a vector space equipped with a skew-symmetric bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (the Lie bracket) that fulfils the Jacobi identity.

A *Lie subalgebra* $\mathfrak{h} \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ which is itself a Lie algebra. If H is a Lie subgroup of G, $\mathrm{Lie}(H)$ is a Lie subalgebra of $\mathrm{Lie}(G)$. A proper subalgebra of $\mathfrak{g}$ is one that is neither $\mathfrak{g}$ itself or the trivial subalgebra.

An *ideal* $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a Lie subalgebra which is preserved by the Lie bracket, *i.e.*, $g \in \mathfrak{g}$, $h \in \mathfrak{h} \Rightarrow [g,h] \in \mathfrak{h}$. If $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, $\mathfrak{g}$ mod $\mathfrak{h}$ is a Lie algebra (show this!). Note the correspondence between Lie subgroups and Lie subalgebras, and between normal subgroups and ideals.

The *direct sum* of two Lie algebras $\mathfrak{g} \oplus \mathfrak{g}'$ is the direct sum of the vector spaces $\mathfrak{g}$ and $\mathfrak{g}'$ with the Lie bracket $[(x,x'),(y,y')] = ([x,y],[x',y'])$. It is a Lie algebra (show this!).

A Lie algebra is called *simple* if it has no proper ideals, and has dimension ≥ 2. (The 1-dimensional Lie algebra is not defined to be simple.) If a Lie algebra is the direct sum of simple Lie algebras, it is called *semi-simple*. Likewise for Lie groups, a Lie group is called simple if it has no normal subgroups (besides $\{I\}$ and the group itself.

Let T_α, $\alpha = 1,\ldots,\dim\mathfrak{g}$ be a basis for $\mathfrak{g}$. Then one has $[T_\alpha,T_\beta] = f_{\alpha\beta}{}^\gamma T_\gamma$. The numbers $f_{\alpha\beta}{}^\gamma$ are called the *structure constants* of $\mathfrak{g}$.

<u>Example:</u> In $\mathfrak{gl}(n)$, take a basis $T^i{}_j$ of matrices which have the matrix element 1 at row i and column j and 0 otherwise, *i.e.*, $(T^i{}_j)_k{}^l = \delta^i_k\delta^l_j$. Then,

$$\begin{aligned}[][T^i{}_j,T^k{}_l]_m{}^n &= \delta^i_m\delta^p_j\delta^k_p\delta^n_l - \delta^k_m\delta^p_l\delta^i_p\delta^n_j \\ &= \delta^k_j\delta^i_m\delta^n_l - \delta^i_l\delta^k_m\delta^n_j \\ &= \delta^k_j(T^i{}_l)_m{}^n - \delta^i_l(T^k{}_j)_m{}^n \ ,\end{aligned} \tag{2.8}$$

so that

$$\begin{aligned}[][T^i{}_j,T^k{}_l] &= \delta^k_jT^i{}_l - \delta^i_lT^k{}_j \\ &= (\delta^k_j\delta^i_m\delta^n_l - \delta^i_l\delta^n_j\delta^k_m)T^m{}_n \ ,\end{aligned} \tag{2.9}$$

and the structure constants are $f^i{}_j,{}^k{}_{l,m}{}^n = \delta^k_j\delta^i_m\delta^n_l - \delta^i_l\delta^n_j\delta^k_m$.

<u>Exercises:</u>

2.11. Show that $SL(2,\mathbb{C})$ is simple.

2.12. Let $x,y,z \in \mathscr{A}$, where $\mathscr{A}$ is an associative algebra, *i.e.*, $(x\star y)\star z = x\star(y\star z)$. Form a skew-symmetric product by $[x,y] = x\star y - y\star x$. Show that it satisfies the Jacobi identity.

2.13. Form a basis and determine the structure constants for $\mathfrak{so}(n)$.

2.14. Form bases for $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{su}(n)$ and $\mathfrak{sp}(2n,\mathbb{R})$.

2.15. Show that $\mathfrak{gl}(n)$ and $\mathfrak{u}(n)$ are not semi-simple, *e.g.* by finding proper ideals.

2.16. Show that if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, $\mathfrak{g}$ mod $\mathfrak{h}$ is a Lie algebra.

2.4. Representations

A representation is a way to realise a Lie algebra as matrices acting on a vector space. Suppose we have an "abstract" Lie algebra $\mathfrak{g}$ with a basis $\{T_\alpha\}$ in which the structure constants are $f_{\alpha\beta}{}^\gamma$. Let

V be a vector space with basis $\{u_i\}$, $i = 1, \ldots, \dim V$, and ϱ a linear map from $\mathfrak{g}$ to $GL(V)$. If the matrices $(\varrho(T_\alpha))_i{}^j$ fulfil the same algebra (under commutation) as the T_α's, *i.e.*,

$$[\varrho(T_\alpha), \varrho(T_\beta)] = f_{\alpha\beta}{}^\gamma \varrho(T_\gamma) \,, \tag{2.10}$$

we call ϱ a representation of $\mathfrak{g}$. (Note that $[\cdot, \cdot]$ on the left hand side means commutator, on the right hand side Lie bracket.) The condition can be written without reference to a basis as

$$[\varrho(g), \varrho(h)] = \varrho([g, h]) \,, \forall g, h \in \mathfrak{g} \,. \tag{2.11}$$

In mathematical terms, this means that ϱ is a "Lie algebra homomorphism".

The vector space V is called a (representation) module, and $\dim V$ the dimension of the representation. (In the physics community, the term representation is often used for the module.) We will only deal with finite-dimensional representations.

The direct sum of two representations is a representation: Take two modules V, V' and let $v \in V$, $v' \in V'$. Then $w = (v, v') \in V \oplus V' = W$. The representation on W is given by

$$\varrho_W(g) \cdot (v, v') = (\varrho_V(g) \cdot v, \varrho_{V'}(g) \cdot v') \,. \tag{2.12}$$

The tensor product of two representations is also a representation: Take two modules V, V' with bases $\{e_i\}$ and $\{e'_{i'}\}$. The tensor product of the vector space (over $\mathbb{R}$ or $\mathbb{C}$) is the vector space $W = V \otimes V'$ with basis $\{e''_{ii'}\}$, where $e''_{ii'} = e_i \otimes e'_{i'}$. The representation on W is then given as

$$\varrho_W(g) = \varrho_V(g) \otimes 1 + 1 \otimes \varrho_{V'}(g) \,. \tag{2.13}$$

A representation is called *irreducible* if it is not the direct sum of lower-dimensional representations.

We have already encountered some representations. From the construction of the classical matrix groups and algebras, it follows for example that $\mathfrak{sl}(n)$, $\mathfrak{su}(n)$ and $\mathfrak{so}(n)$ have n-dimensional representations. Every Lie algebra has a trivial, 1-dimensional representation, for which $\varrho(g) = 0 \in GL(1)$. Every Lie algebra $\mathfrak{g}$ also has a $\dim \mathfrak{g}$-dimensional representation, the adjoint representation, defined by the algebra itself. In the adjoint representation, $\varrho(g) = \operatorname{ad} g$, where $\operatorname{ad} g \cdot h = [g, h]$, *i.e.*, $(\varrho(T_\alpha))_\beta{}^\gamma = -f_{\alpha\beta}{}^\gamma$,

Exercises

2.17. Verify by explicit calculation the the adjoint representation is a representation.

2.18. Verify that the expressions for the representation matrices in the direct sum and tensor product are correct. What are the dimensions of the representations obtained?

2.19. Show that $\mathfrak{sl}(n)$ has a representation of dimension $\frac{1}{2}n(n+1)$. Is it irreducible?

2.20. Show that $\mathfrak{so}(n)$ has an irreducible representation of dimension $\frac{1}{2}n(n+1)-1$.

2.5. The Cartan–Killing form

Take a (semi-simple) Lie algebra $\mathfrak{g}$ and a representation ϱ. Then one can construct a map $K : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ (or $\mathbb{C}$), the *Cartan–Killing form*, as

$$K(g,g') = \mathrm{tr}(\varrho(g)\varrho(g')) \ . \tag{2.14}$$

It can be shown that K is non-degenerate for (and only for) semi-simple Lie algebras, and that its dependence on ϱ only amount to a scaling. We can, if we like, choose to evaluate it in the adjoint representation, where it becomes $K(g,g') = \mathrm{tr}(\mathrm{ad}\,g\,\mathrm{ad}\,g')$. It is invariant in the sense that $K([h,g],g') + K(g,[h,g']) = 0$ for all $g,g',h \in \mathfrak{g}$.

The Cartan–Killing form provides a natural metric on the Lie algebra. Namely, consider $\gamma_{\alpha\beta} = K(T_\alpha,T_\beta)$. This metric and its inverse $\gamma^{\alpha\beta}$ can be used to raise, lower and contract adjoint indices.

One may then form the invariant quadratic operator (in any representation)

$$C_2 = \gamma^{\alpha\beta} T_\alpha \circ T_\beta \ . \tag{2.15}$$

It should be understood as the matrix $C_{2,\varrho} = \gamma^{\alpha\beta}\varrho(T_\alpha)\varrho(T_\beta)$ when acting in a representation ϱ. It is called the quadratic Casimir operator, and the invariance implies that it takes the same value when acting on any state in a given irreducible representation module. Its eigenvalues in different representations typically differ.

Exercises

2.21. Show that the Cartan–Killing form is invariant.

2.6. $SL(2)$

Consider the group $SL(2,\mathbb{C})$. It consists of complex matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \quad ad - bc = 1 \ . \tag{2.16}$$

Its Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ consists of traceless matrices

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \tag{2.17}$$

A convenient basis is $\{h, e, f\}$, where

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.18}$$

The non-vanishing commutators are (verify!)

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h. \tag{2.19}$$

This basis is called the Chevalley-Serre basis. The element h spans the (abelian) Cartan subalgebra, e is a raising operator and f a lowering operator. These concepts are useful for analysing (semi-simple) Lie algebras in general.

The real Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is of course described by the same basis and the same commutators.

Now, take a look at $\mathfrak{su}(2)$, containing anti-hermitean matrices of the form

$$B = \begin{pmatrix} ix & iz \\ i\bar{z} & -ix \end{pmatrix}, \quad x \in \mathbb{R}, z \in \mathbb{C}. \tag{2.20}$$

We can write $B = i(xh + ze + \bar{z}f)$. The basis elements of $\mathfrak{su}(2)$ are linear combinations of the basis elements of $\mathfrak{sl}(2)$. However, the coefficients involve complex numbers. The Lie algebras are equivalent as complex Lie algebras, but inequivalent as real Lie algebras. $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$ are different real forms of the same complex Lie algebra. We will see that $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$, the algebra of orthogonal rotations in 3 euclidean dimensions, while $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(1,2)$, the algebra of Lorentz transformations in 3-dimensional Minkowski space. The $\mathfrak{su}(2)$ element in eq. (2.20) can also be written as $B = ix^i\sigma_i$, where $x^1 = \operatorname{Re} z$, $x^2 = \operatorname{Im} z$, $x^3 = x$, and σ_i are the Pauli σ matrices with commutation relations $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$. So, $\tau_i = -\frac{i}{2}\sigma_i$ fulfil $[\tau_i, \tau_j] = \epsilon_{ijk}\tau_k$. This is the same structure constants as the Lie algebra $\mathfrak{so}(3)$ spanned by the antisymmetric (3×3)-matrices $(J_i)_{jk} = -\epsilon_{ijk}$, *i.e.*,

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.21}$$

The Lie algebra isomorphism $\mathfrak{su}(2) \simeq \mathfrak{so}(3)$ does *not* imply that the groups $SU(2)$ and $SO(3)$ are isomorphic (the same). To see this, let us find group elements corresponding to a rotation by an angle θ around the third axis. In $SO(3)$ this is

$$g_{SO(3)}(\theta) = e^{\theta J_3} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{2.22}$$

The corresponding element in $SU(2)$ is

$$g_{SU(2)}(\theta) = e^{\theta \tau_3} = \begin{pmatrix} e^{-\frac{i}{2}\theta} & 0 \\ 0 & e^{\frac{i}{2}\theta} \end{pmatrix} . \tag{2.23}$$

While a rotation by 2π gives $g_{SO(3)}(2\pi) = I$, it gives $g_{SU(2)}(2\pi) = -I$. The subgroup $\{I, -I\}$ is called the *center* of $SU(2)$, consisting of elements that commute with all elements in the group. It is thus a normal subgroup, and can be divided out. We arrive at the precise statement $SU(2)/\mathbb{Z}_2 \simeq SO(3)$. Another way of saying this is that $SU(2)$ is the *double cover* of $SO(3)$. $SO(3)$ does not have a 2-dimensional (spinor) representation, but its double cover does.

Exercises:

2.22. Exponentiate the element A above to obtain an element in the group $SL(2)$. Verify that $e^A = \cos(\sqrt{\det A})I + \frac{\sin(\sqrt{\det A})}{\sqrt{\det A}}A$. What if $\det A = 0$?

2.23. Verify the commutation relations of the $\mathfrak{so}(3)$ generators using the properties of the Levi-Civita tensor.

2.24. Show that $SU(2) = S^3$. Show that $SU(2)/U(1) = S^2$.

2.25. Calculate the Cartan–Killing form for $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(3)$, and interpret the result in terms of $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(1,2)$.

2.7. Representations of $\mathfrak{sl}(2)$. Tensors

The Chevalley–Serre basis is useful for dealing with representations of Lie algebras. The Cartan subalgebra, for $\mathfrak{sl}(2)$ spanned by h, is an abelian subalgebra. This means that all elements in the Cartan subalgebra (and their representation matrices) can be diagonalised simultaneously, and vectors in a representation module may conveniently be classified according to their eigenvalue.

For $\mathfrak{sl}(2)$, we look for eigenvectors of $\varrho(h)$ in a module V. We can decompose V as

$$V = \bigoplus_{\lambda\in\Lambda} V_\lambda , \tag{2.24}$$

where $\varrho(h)V_\lambda = \lambda V_\lambda$ and Λ is the set of eigenvalues of $\varrho(h)$ in the module. Since $[h,e] = 2e$, $[h,f] = -2f$, $\varrho(e)$ raises the eigenvalue by 2 and $\varrho(f)$ lowers it by 2. From now, we skip the notation $\varrho(e)$ etc., and use e etc. instead as a shorthand.

$$\begin{aligned} eV_\lambda &\subset V_{\lambda+2} \;, \\ fV_\lambda &\subset V_{\lambda-2} \;. \end{aligned} \tag{2.25}$$

If the representation is finite-dimensional, there must be some highest value of λ (and also a lowest one). Call this value μ and consider a single such vector v_μ. This is called a *highest weight state*. We have $ev_\mu = 0$. All other vectors in V are obtained by repeatedly acting with the lowering operator f. For some value of p, we must have $f^{p+1}v_\mu = 0$, but $f^p v_\mu \neq 0$. The latter is the lowest weight state. For this to be consistent, one must also have $ef^{p+1}v_\mu = 0$. Then we can use the commutation relations and the fact that e annihilates v_μ to obtain

$$ef^{p+1}v_\mu = \sum_{k=0}^{p} f^{p-k}hf^k v_\mu = \sum_{k=0}^{p}(\mu - 2k)f^p v_\mu = -(p+1)(p-\mu)f^p v_\mu \;. \tag{2.26}$$

This can only vanish if μ is a non-negative integer, and $p = \mu$.

All finite-dimensional representations of $\mathfrak{sl}(2)$ are labelled by the highest weight μ which can take the values $0, 1, 2, \ldots$. The lowest weight is then $-\mu$. The dimension of the representation with highest weight μ is $\dim V = \mu + 1$. The value $\mu = 0$ gives the trivial representation. $\mu = 1$ gives the 2-dimensional representation already encountered, with V_1 spanned by $(1,0)^t$ and V_{-1} by $(0,1)^t$. $\mu = 2$ gives the adjoint representation, which we already know is spanned by traceless matrices.

Let us also examine the value of the quadratic Casimir operator, when acting in different irreducible representations. We can calculate the Cartan–Killing metric using any irreducible representation. Using the (defining) 2-dimensional one, we get the non-vanishing components $(h,h) = 2$, $(e,f) = 1$. The metric $\gamma_{\alpha\beta}$ has two positive eigenvalues and one negative one. This shows that $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(1,2)$, the Lorentz algebra in 3 dimensions. The quadratic Casimir operator (with a suitably chosen normalisation) becomes

$$C_2 = \tfrac{1}{2}\gamma^{\alpha\beta}T_\alpha T_\beta = \tfrac{1}{2}(\tfrac{1}{2}h^2 + ef + fe) \;. \tag{2.27}$$

We know that it takes the same value on different states in an irreducible module, so it suffices to evaluate it on the highest weight state v_μ. We then obtain

$$\begin{aligned} C_2 v_\mu &= \tfrac{1}{2}(\tfrac{1}{2}h^2 + ef)v_\mu = \tfrac{1}{2}(\tfrac{1}{2}h^2 + [e,f])v_\mu \\ &= \tfrac{1}{2}(\tfrac{1}{2}h^2 + h)v_\mu = \frac{\mu(\mu+2)}{4}v_\mu \;. \end{aligned} \tag{2.28}$$

By converting to angular momentum or spin, as conventionally normalised in physics, $\ell = \mu/2$, we get the eigenvalue of the quadratic Casimir operator

$$C_2 = \frac{\mu(\mu+2)}{4} = \ell(\ell+1) \ . \qquad (2.29)$$

We will now express elements in modules of $\mathfrak{sl}(2)$ as tensors. Let $i = 1, 2$ be an index labelling vectors in the 2-dimensional, fundamental, module. Such a vector is written v_i. Now, we can take tensor products of the 2-dimensional representation with itself. For example a tensor with two indices, symmetrised or anti-symmetrised (see exercise 2.28), $v_{(ij)}$ or $v_{[ij]}$. However, since $v_{[ij]}$ is proportional to ϵ_{ij} (see exercise 2.31), it is invariant, and gives the trivial representation. A symmetric tensor has 3 independent components. This is the adjoint representation. We have earlier defined the adjoint as traceless tensors. They have the index structure $v_i{}^j$. If we form $v_i{}^k\epsilon_{kj}$, it becomes

$$v\epsilon = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -\beta & \alpha \\ \alpha & \gamma \end{pmatrix} \ , \qquad (2.30)$$

which is symmetric. This can be continued to higher tensors. A completely symmetric tensor with p indices, $v_{i_1 \ldots i_p}$ transforms under the representation with highest weight p. Its dimension is $p+1$.

Consider the element $-I$ of the group $SL(2)$ (in the defining 2-dimensional representation). It has eigenvalue 1 on all elements in the modules with even highest weight (*i.e.*, on tensors with even number of spinor indices), and $-I$ for odd highest weight (odd number of spinor indices). The subgroup $\{I, -I\} = \mathbb{Z}_2 \subset SL(2)$ is called the *center* of $SL(2)$, consisting of all group elements that commute with all elements in the group. Forming $SL(2)/\{I, -I\}$, we get a group which does not have the representations with odd highest weight. This is $SO(1, 2)$. Its *double cover* is in turn $SL(2)$, which we also can call $Spin(1, 2)$.

<u>Exercises</u>

2.26. Show that ϵ_{ij} is an invariant tensor under $\mathfrak{sl}(2)$ (but not under $\mathfrak{gl}(2)$).

2.27. Show that $\epsilon_{i_1 \ldots i_n}$ is an invariant tensor under $\mathfrak{sl}(n)$ (but not under $\mathfrak{gl}(n)$).

2.28. Show that the symmetric and antisymmetric parts of the tensor product of any representation of a Lie algebra with itself form separate representations (which need not be irreducible).

2.29. Show that the number of independent components in a completely symmetric $\mathfrak{sl}(2)$ tensor with p indices is $p+1$.

2.30. Show that the number of independent components in a completely symmetric $\mathfrak{sl}(n)$ tensor with p indices is $\binom{n+p-1}{p}$.

2.8. $SL(3)$

Start by finding a Cartan subalgebra among the traceless (3×3)-matrices. This is a maximal set of mutually commuting elements. We can take

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} . \tag{2.31}$$

To each h_i we want to associate a raising operator e_i and a lowering operator f_i, such that $\{h_i, e_i, f_i\}$ is a basis for an $\mathfrak{sl}(2)$ subalgebra. They are easily found:

$$\begin{aligned} e_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , \\ f_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \end{aligned} \tag{2.32}$$

The remaining generators are

$$e_3 = [e_1, e_2] = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad f_3 = [f_2, f_1] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} . \tag{2.33}$$

Note that $[e_1, f_2] = 0 = [e_2, f_1]$. The two $\mathfrak{sl}(2)$ subalgebras do not commute with each other, instead one has

$$[h_1, e_2] = -e_2 , \quad [h_2, e_1] = -e_1 , \tag{2.34}$$

and opposite signs for the f's.

Let us evaluate the Cartan–Killing form (in the 3-dimensional representation). The non-vanishing entries are

$$\begin{aligned} &K(h_1, h_1) = 2 , \quad K(h_2, h_2) = 2 , \quad K(h_1, h_2) = -1 , \\ &K(e_1, f_1) = 1 , \quad K(e_2, f_2) = 1 , \quad K(e_3, f_3) = 1 . \end{aligned} \tag{2.35}$$

Of special interest is the restriction to the Cartan subalgebra, which gives the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} . \tag{2.36}$$

A is called the Cartan matrix.

2.9. $Sp(4)$

$\mathfrak{sl}(3)$ is not unique as a simple Lie algebra of rank 2. Another example is $\mathfrak{sp}(4)$. This Lie algebra has already been defined as the matrices leaving an antisymmetric matrix (a symplectic form) invariant. We have $J^t\Omega+\Omega J=0$ for some antisymmetric matrix Ω, which we conveniently choose as

$$\Omega=\begin{pmatrix}0&\epsilon\\ \epsilon&0\end{pmatrix}\;, \tag{2.37}$$

where ϵ is the 2×2 antisymmetric matrix with $\epsilon_{12}=1$. If we let J have the block structure

$$J=\begin{pmatrix}A&B\\ C&D\end{pmatrix}\;, \tag{2.38}$$

we get the conditions $\epsilon A+D^t\epsilon=0$, $\epsilon B+B^t\epsilon=0$, $\epsilon C+C^t\epsilon=0$. We have already seen that this implies that B and C are traceless. If $A=\begin{pmatrix}a&b\\ c&d\end{pmatrix}$, $D=\begin{pmatrix}-d&b\\ c&-a\end{pmatrix}$. A Chevalley–Serre basis can be taken as

$$\begin{aligned}
h_1&=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\;,\quad & h_2&=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&0\end{pmatrix}\;,\\
e_1&=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}\;,\quad & e_2&=\begin{pmatrix}0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\;,\\
f_1&=\begin{pmatrix}0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}\;,\quad & f_2&=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix}\;,
\end{aligned} \tag{2.39}$$

The non-vanishing commutators are

$$\begin{aligned}
&[h_1,e_1]=2e_1\;,\;[h_1,f_1]=-2f_1\;,\;[e_1,f_1]=h_1\;,\\
&[h_2,e_2]=2e_2\;,\;[h_2,f_2]=-2f_2\;,\;[e_2,f_2]=h_2\;,\\
&[h_1,e_2]=-2e_2\;,\;[h_1,f_2]=2f_2\;,\\
&[h_2,e_1]=-e_2\;,\;[h_1,f_2]=f_2\;,
\end{aligned} \tag{2.40}$$

and in addition $[e_1,e_2]$, $[e_1,[e_1,e_2]]$ are non-vanishing (and the same expressions with f's). (Note that $[e_1,f_2]=0=[e_2,f_1]$.) The relations (2.40) can be given a unified form

$$[h_i,e_j]=A_{ij}e_j\;,\;[h_i,f_j]=-A_{ij}f_j\;,\;[e_i,f_j]=\delta_{ij}h_j\;, \tag{2.41}$$

where A is the Cartan matrix

$$A=\begin{pmatrix}2&-2\\ -1&2\end{pmatrix}\;. \tag{2.42}$$

The Cartan–Killing metric, which we now normalise as $K(g,h) = \frac{1}{2}\mathrm{tr}(gh)$ in the 4-dimensional representation, becomes

$$\begin{aligned} &K(h_1,h_1) = 2 \; , \quad K(e_1,f_1) = 1 \; , \\ &K(h_2,h_2) = 1 \; , \quad K(e_2,f_2) = \tfrac{1}{2} \; , \\ &K(h_1,h_2) = -1 \; . \end{aligned} \tag{2.43}$$

Restricted to the h's, we get the matrix

$$\bar{A} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \; . \tag{2.44}$$

Something about $\mathfrak{sp}(4) \simeq \mathfrak{so}(5)$...

Exercises

2.31. Show that the Serre relations $(\mathrm{ad}\, e_i)^{1-A_{ij}} e_j = 0$ are consistent given the relations $[h_i, e_j] = A_{ij} e_j$.

2.10. Cartan subalgebra and root system

We encode the structure of the algebra in the brackets between elements in the $\mathfrak{sl}(2)$ subalgebras corresponding to simple roots as

$$[h_i, e_j] = A_{ij} e_j \; , \; [h_i, f_j] = -A_{ij} f_j \; , \; [e_i, f_j] = \delta_{ij} h_j \; . \tag{2.45}$$

Here, $i, j = 1, \ldots, \mathrm{rank}\, \mathfrak{g}$. The matrix A is called the Cartan matrix.

The meaning of the Cartan matrix is to encode the eigenvalues of the Cartan generators. These eigenvalues need to be consistent with some relations that make the algebra finite-dimensional. Namely, consider starting from the generator e_j and acting repeatedly with e_i $(i \neq j)$ to obtain generators $(\mathrm{ad}\, e_i)^k e_j$, $k = 0, 1, \ldots$. The sequence has to stop at some point, and give a finite-dimensional module of the $\mathfrak{sl}(2)$ algebra spanned by $\{h_i, e_i, f_i\}$, in which e_j is the lowest weight state. From what we have learned about $\mathfrak{sl}(2)$ representations, this implies that all off-diagonal entries in the Cartan matrix must be non-negative integers. It is then consistent with the relations (2.45) to set

$$(\mathrm{ad}\, e_i)^{1-A_{ij}} e_j = 0 \; . \tag{2.46}$$

The corresponding identity for the lowering generators of course also holds, $(\mathrm{ad}\, f_i)^{1-A_{ij}} f_j = 0$. The relations (2.46) are called Serre relations. Together with the defining relations (2.45), they provide all information needed to fully define a semi-simple Lie algebra $\mathfrak{g}$.

In fact, the possible values of the off-diagonal entries in A are much more restricted than that. We will soon relate the Cartan matrix and the Cartan–Killing metric, but first we need some preparation.

We have already defined the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and decomposed the Lie algebra $\mathfrak{g}$ in subspaces with different eigenvalues under the adjoint action of the elements in the Cartan subalgebra. We thus have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha \ . \tag{2.47}$$

The vectors of eigenvalues are called roots. We can define the subspaces as

$$\mathfrak{g}_\alpha = \{g \in \mathfrak{g} \,:\, [h,g] = \langle h, \alpha \rangle g, \forall h \in \mathfrak{h}\} \ . \tag{2.48}$$

Here, $\langle \cdot, \cdot \rangle$ is the natural product between an element in $\mathfrak{h}$ and an element in the dual space $\mathfrak{h}^*$. If this expression seems abstract, one can think of any basis $\{a_\mu\}$ of $\mathfrak{h}$, so that $h = h^\mu a_\mu$, and write the eigenvalue equation for $g_\alpha \in \mathfrak{g}_\alpha$ as $[a_\mu, g_\alpha] = \alpha_\mu g_\alpha$, *i.e.*, $[h, g_\alpha] = h^\mu \alpha_\mu g_\alpha$. We see that the roots lie in $\mathfrak{h}^*$.

The restriction of the Cartan–Killing form to the Cartan subalgebra gives a metric on $\mathfrak{h}$, which also means that it can be used to convert between $\mathfrak{h}$ and $\mathfrak{h}^*$ (what a physicist may call raising and lowering indices). The inverse of the Killing form is a natural metric on $\mathfrak{h}^*$. Let h_α be the element in $\mathfrak{h}$ obtained by using the (inverse of) the Cartan–Killing metric to map a root $\alpha \in \mathfrak{h}^*$ to $\mathfrak{h}$. From now on, we use the same notation $(\cdot, \cdot)$ for the scalar products on $\mathfrak{h}$ and $\mathfrak{h}^*$, so that $(h_\alpha, h_\beta) = (\alpha, \beta)$. (And occasionally for the Cartan–Killing metric itself, before it is restricted to $\mathfrak{h}$.)

The set of roots α_i such that $e_i \in \mathfrak{g}_{\alpha_i}$ are called simple roots. Consider again the $\mathfrak{sl}(2)$ subalgebra spanned by $\{h_i, e_i, f_i\}$ for some simple root α_i. Taking the scalar product of some element h with $[e_i, f_i] = h_i$, we get

$$(h, [e_i, f_i]) = \begin{cases} (h, h_i) \\ ([h, e_i], f_i) = \langle h, \alpha_i \rangle (e_i, f_i) = (h, h_{\alpha_i})(e_i, f_i) \end{cases} \tag{2.49}$$

where the invariance of the metric is used in the second line. This holds for any $h \in \mathfrak{h}$. Since the metric is non-degenerate, it implies that

$$h_i = (e_i, f_i) h_{\alpha_i} \ . \tag{2.50}$$

The invariance of the metric implies that $(e_i, f_i) = \frac{1}{2}(h_i, h_i)$. (Namely, acting with e_i on $(h_i, f_i) = 0$ gives $0 = -2(e_i, f_i) + (h_i, h_i)$. In each separate $\mathfrak{sl}(2)$ subalgebra, the metric can be redefined by some constant preserving this relation. But in $\mathfrak{g}$ one may get different values for (h_i, h_i) for different i, as in the example with $\mathfrak{sp}(4)$.) Thus, $h_i = \frac{1}{2}(h_i, h_i) h_{\alpha_i}$. Squaring this relation gives $(h_i, h_i) = \frac{4}{(\alpha_i, \alpha_i)}$, and thus

$$h_i = \frac{2}{(\alpha_i, \alpha_i)} h_{\alpha_i} \ . \tag{2.51}$$

Going back to the calculation of eq. (2.49), and inserting the Cartan generator corresponding to another simple root for h, one gets

$$(h_i,[e_j,f_j]) = \begin{cases} (h_i,h_j) = (\frac{2}{(\alpha_i,\alpha_i)}h_{\alpha_i},\frac{2}{(\alpha_j,\alpha_j)}h_{\alpha_j}) = \frac{4(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)(\alpha_j,\alpha_j)} \\ ([h_i,e_j],f_j) = A_{ij}(e_j,f_j) = A_{ij}\frac{2}{(\alpha_j,\alpha_j)} \end{cases} \tag{2.52}$$

We now have the important relation between the Cartan matrix and the scalar products of the simple roots:

$$A_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} \ . \tag{2.53}$$

The overall scaling of the roots is a matter of convention, but the relative lengths are important. A standard convention is to set the length squared of the longest roots to 2. We have seen two examples of rank 2.

In $\mathfrak{sl}(3)$, the two simple roots (and also the third positive root) have the same length. In such cases, eq. (2.53) simplifies to $A_{ij} = (\alpha_i,\alpha_j)$, and the Cartan matrix is the metric in the basis of simple roots. With

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{2.54}$$

we see that both simple roots have length $\sqrt{2}$ and that the angle between them is $\frac{2\pi}{3}$.

In $\mathfrak{sp}(4)$, we had

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \ . \tag{2.55}$$

With the convention that the longest root has length $\sqrt{2}$, this gives

$$\begin{aligned} (\alpha_1,\alpha_1) &= 1 \ , \quad (\alpha_1,\alpha_2) = -1 \ , \\ (\alpha_2,\alpha_1) &= -1 \ , \quad (\alpha_2,\alpha_2) = 2 \ , \end{aligned} \tag{2.56}$$

which gives the metric in the basis of the simple roots. The first root is short, and the angle between the roots is $\frac{3\pi}{4}$.

We can now go back to the statement that the off-diagonal entries of the Cartan matrix must be non-positive integers. In view of exercise 2.31, the generator e_j is the lowest state in an $\mathfrak{sl}(2)$-module corresponding to root i with dimension $1 - A_{ij}$. We have

$$A_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} = 2\sqrt{\frac{(\alpha_j,\alpha_j)}{(\alpha_i,\alpha_i)}}\cos\theta_{ij} \ , \tag{2.57}$$

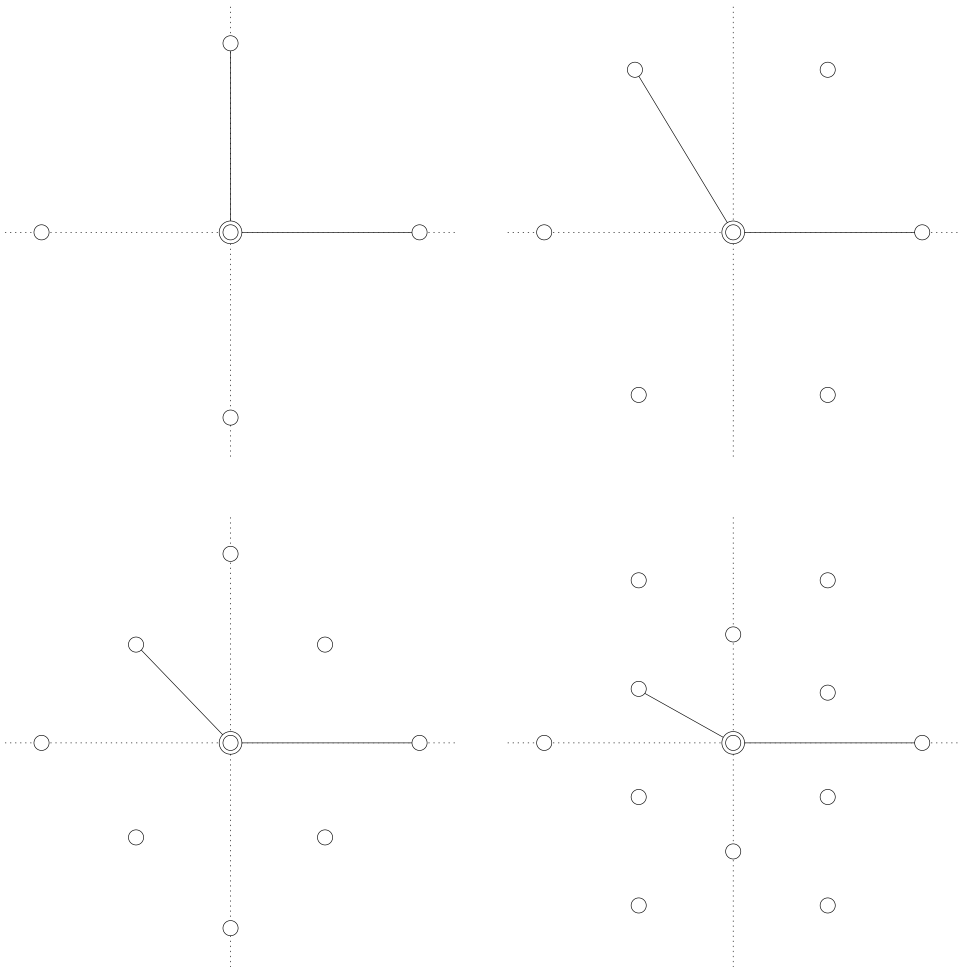
where θ_{ij} is the angle between the roots α_i and α_j. Therefore,

$$A_{ij}A_{ji} = 4\cos^2\theta_{ij} \ . \tag{2.58}$$

This number must be an integer, which only leaves the possibilities $0, 1, 2, 3$ (the value 4 is excluded, since it would correspond to a linear dependence among the simple roots). Assume that root i is long, $(\alpha_i, \alpha_i) = 2$. The different values correspond to the angles and lengths of root j:

$$\begin{array}{rcrlrlrlrl} \cos^2\theta_{ij} = & 0 & : & \theta = \frac{\pi}{2}\,, & (\alpha_j,\alpha_j) = & \text{undetermined}\,, & A_{ij} = & 0\,, & A_{ji} = & 0\,, \\ & \frac{1}{4} & : & \frac{2\pi}{3}\,, & & 2\,, & & -1\,, & & -1\,, \\ & \frac{1}{2} & : & \frac{3\pi}{4}\,, & & 1\,, & & -1\,, & & -2\,, \\ & \frac{3}{4} & : & \frac{5\pi}{6}\,, & & \frac{2}{3}\,, & & -1\,, & & -3\,, \end{array} \qquad (2.59)$$

(The solutions with θ in the second quadrant are chosen, since otherwise eq. (2.57) would yield positive A_{ij}.) These are the only possible relations between lengths and angles between any two simple roots in finite-dimensional semisimple Lie algebras. The corresponding rank 2 subalgebras are $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$, $\mathfrak{sl}(3)$, $\mathfrak{sp}(4)$ and G_2, respectively. Their root spaces are depicted in the figures below.



It is convenient to introduce the *coroot* $\alpha^\vee$ corresponding to the root α by

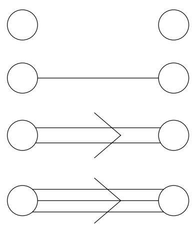
$$\alpha^\vee = \frac{2}{(\alpha,\alpha)}\alpha\ . \tag{2.60}$$

Then, $h_i = h_{\alpha_i^\vee}$, and $A_{ij} = (\alpha_i^\vee, \alpha_j)$. The relation $[h_i, e_j] = A_{ij}e_j$ can be written as $[h_{\alpha_i^\vee}, e_j] = (\alpha_i^\vee, \alpha_j)e_j$, and by linearity,

$$[h_{\alpha^\vee}, g_\beta] = (\alpha^\vee, \beta)g_\beta\ , \quad g_\beta \in \mathfrak{g}_\beta\ . \tag{2.61}$$

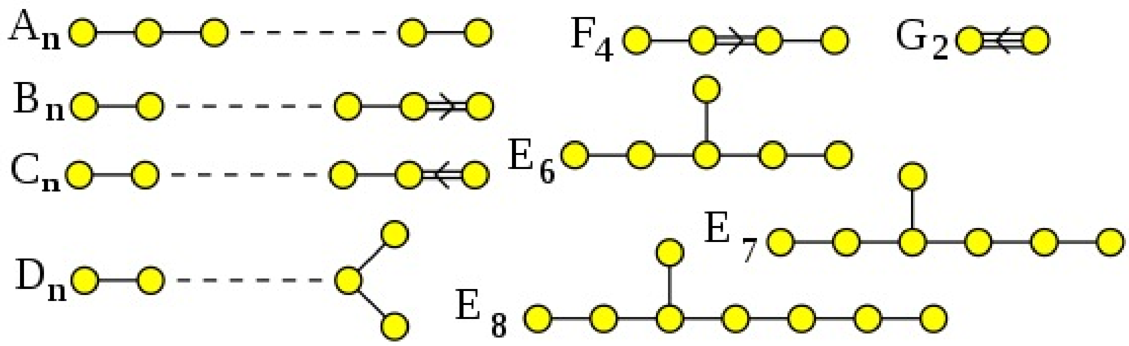
2.11. Classification

We saw that there are four possible relations between two given simple roots in any finite-dimensional semisimple Lie algebra. There is a convenient way of encoding the relations diagrammatically, in a so called Dynkin diagram. Let the root α_i have (length)^2 2. Then the list of possible lengths of α_j and angles between α_i and α_j are the ones listed above. Introduce a node for each simple root, so that the number of nodes is rank $\mathfrak{g}$. Then draw $-A_{ji}$ lines between the nodes (*i.e.*, 0, 1, 2 or 3). To distinguish which of two so connected nodes represents the shorter root, an arrow is drawn towards it. This leads to the possible "couplings" between any two nodes, corresponding to the four cases:



There is some more work before one can determine all allowed Dynkin diagrams describing finite-dimensional simple Lie algebras. The criterion is that a system of vectors, the simple roots, should form a basis for rank $\mathfrak{g}$-dimensional Euclidean space. See for example exercise 2.46 for a configuration (a closed loop) that is not allowed (in the sense that it needs to be embedded in a space with non-positive definite metric, and leads to an infinite-dimensional Lie algebra). The final list, including the classical matrix algebras in the A-, B-, C- and D-series, is given below. In addition to the classical matrix algebras, there are a few exceptional algebras: G_2, F_4 and E_n, $n = 6, 7, 8$.

$A_n \simeq \mathfrak{sl}(n+1)$
$B_n \simeq \mathfrak{so}(2n+1)$
$C_n \simeq \mathfrak{sp}(2n)$
$D_n \simeq \mathfrak{so}(2n)$
E_6, E_7, E_8
F_4
G_2



(Figure courtesy of Wikimedia Commons.)

2.12. Weights and representations

A finite-dimensional representation of $\mathfrak{g}$ must have some highest weight state $|\lambda\rangle$. We label it by its eigenvalues under (the representation matrices of) elements in the Cartan subalgebra:

$$h_i|\lambda\rangle = h_{\alpha_i^\vee}|\lambda\rangle = (\alpha_i^\vee, \lambda)|\lambda\rangle \ . \tag{2.62}$$

In order for the representation to be finite-dimensional, there must exist some smallest non-negative integers p_i, $i = 1, \ldots, \text{rank}\,\mathfrak{g}$, such that $f_i^{p_i+1}|\lambda\rangle = 0$. This is completely analogous to the discussion for $\mathfrak{sl}(2)$ representations. It leads in the same way to $p_i = (\alpha_i^\vee, \lambda)$.

Thus, a highest weight λ is a linear combination with non-negative integer coefficients of the *fundamental weights* Λ_i,

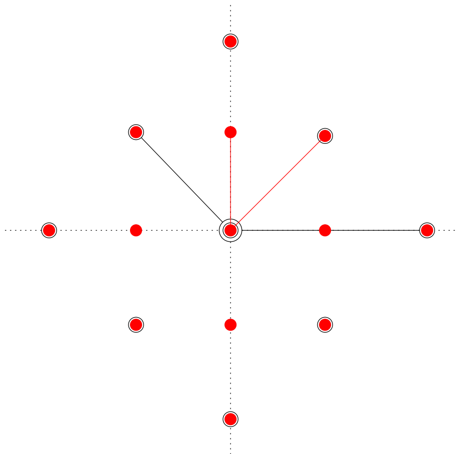
$$\lambda = \sum_{i=1}^{\text{rank}\,\mathfrak{g}} p_i \Lambda_i \ , \tag{2.63}$$

such that $(\alpha_i^\vee, \Lambda_j) = \delta_{ij}$. Such a weight λ is called a dominant integral weight, and $\{p_i\}$ are called the Dynkin labels of the representation. The Dynkin labels must be positive, since otherwise the corresponding $\mathfrak{sl}(2)$ representations would not be finite-dimensional, following from the analysis in Section 2.7. The fundamental weights form a basis for the weight lattice, which is the dual lattice to the coroot lattice. Given a lattice L the dual lattice L^* is defined as consisting of all vectors whose scalar product with all elements in L are integer. Note that the root lattice is a sublattice of the weight lattice.

If $\mathfrak{g}$ has a representation R with highest weight λ, there is also a dual, or conjugate, representation $\bar{R}$ with lowest weight $-\lambda$, where the representation module of $\bar{R}$ is the dual space to the module of R. This may or may not be the same representation as R. Representations with $\bar{R} = R$ are called self-conjugate. Examples of self-conjugate representations are the adjoint for any semi-simple Lie algebra (thanks to the existence of the Killing metric) and the vector representation of $SO(n)$ and its tensor products (thanks to the existence of a metric). Examples of conjugate pairs of representations which are not self-conjugate are the fundamental n-dimensional representations of $\mathfrak{sl}(n)$ and its dual, which in tensor notations are represented with superscripts and subscripts.

The weight lattices of $\mathfrak{sp}(4)$ is depicted in the figure below, with the root lattice superimposed. The fundamental weights are indicated by red lines. The $\mathfrak{sl}(3)$ weight lattice coincides with the G_2 root lattice. The G_2 weight lattice coincides with the root lattice.

Note that in the $\mathfrak{sp}(4)$ weight lattice, there is a weight which is half the long root, but not one which is half the short root. A good exercise is to examine how this happens due to the definition of the weight lattice as the dual to the *coroot* lattice.



In our normalisation, the root lattice of $\mathfrak{sl}(2)$ consists of the numbers $\sqrt{2}\mathbb{Z}$ and the weight lattice of the numbers $\frac{1}{\sqrt{2}}\mathbb{Z}$. Consider the $SU(2)$ group element $e^{i\pi h}$. It has eigenvalue 1 on states whose weights are roots, and -1 on the weights between the roots. This is the previously mentioned element in the center of $SU(2)$, generating the group $\mathbb{Z}_2$. Dividing it out means that one loses the representations labelled by weights between the roots. It is striking that even if the Lie algebra itself does not know about the global structure of the group, its representations seem to do. One may classify the representations in conjugacy classes, labelled by $L/R = \mathbb{Z}_2$, where R is the root lattice and L the weight lattice. (In physics, we would talk about odd or even spin, bosons and fermions.)

For any semi-simple Lie algebra, the weight lattice L is an abelian group under addition of vectors, and the root lattice R is a subgroup, which is automatically a normal subgroup, since addition is commutative. Therefore L/R is a discrete abelian group, whose elements correspond to conjugacy classes of representations. Since eigenvalues (weights) add under the tensor product of representations, the conjugacy classes of representations respect the group product in L/R under tensor product.

<u>Exercises</u>

2.32. Show that the root lattice is a sublattice of the weight lattice.

2.33. Show that the weight lattice of G_2 coincides with the root lattice.

2.34. Identify the fundamental weights for $\mathfrak{sl}(3)$.

2.35. Construct the two 3-dimensional $\mathfrak{sl}(3)$-modules by starting from the highest weights with Dynkin labels (10) and (01) and acting with lowering operators. (Hint: new null states will be encountered in the construction, *e.g.* $f_2^2 f_1|\Lambda_1\rangle$.) If we denote a representation by the Dynkin labels of its highest weight, we can write $\mathbf{3} = (10)$, $\overline{\mathbf{3}} = (01)$. Determine, by some method, the tensor products $(10) \otimes (10)$ and $(10) \otimes (01)$ as direct sums of irreducible representations. Illustrate with sums of weights in a picture.

2.36. Which is the highest weight in the adjoint representation of $\mathfrak{sl}(3)$?

2.37. How many conjugacy classes of representations are there for $SU(3)$? Make a conjecture for the center of $SU(3)$, and try to verify it. What is the center of $SU(n)$?

2.38. Construct the 4- and 5- dimensional modules of $\mathfrak{sp}(4)$ as highest weight modules.

2.39. Which is the highest weight in the adjoint representation of $\mathfrak{sp}(4)$?

2.40. How many conjugacy classes of representations are there for $Spin(5)$?

2.41. A construction of the 14-dimensional Lie algebra G_2. Inspecting the root space of G_2, one finds that all roots of G_2 are weights of $\mathfrak{sl}(3)$, and that it consists of the weights for the adjoint (the roots) of $\mathfrak{sl}(3)$ together with the weight for the two 3-dimensional representations, which we can call $\mathbf{3}$ and $\bar{\mathbf{3}}$. There should be a formulation of G_2 with manifest $\mathfrak{sl}(3)$ and generators $J_m{}^n$, K_m and L^m. Construct the brackets by making some Ansatz and checking the Jacobi identities. (Hint: Only some of the Jacobi identities are "non-trivial", in the sense that they do not follow from the $\mathfrak{sl}(3)$ covariance.)

2.42. The *Weyl group* of a semi-simple Lie algebra is the discrete group generated by reflections in hyperplanes orthogonal to the simple roots. It is a symmetry of the root system. Reflection in the hyperplane orthogonal to α_i maps a vector β to $w_i(\beta) = \beta - \frac{2(\beta,\alpha_i)}{(\alpha_i,\alpha_i)}\alpha_i$. Describe the Weyl groups of $\mathfrak{sl}(2)$ and $\mathfrak{sl}(3)$ (number of elements, multiplication table).

2.43. Describe the Weyl groups of $\mathfrak{sp}(4)$ and G_2.

2.44. Find a set of simple roots for $\mathfrak{so}(2n)$. Show that the Dynkin diagram has the form above.

2.45. Show that the real Lie algebra $\mathfrak{so}(4)$ of rotations in 4 euclidean dimensions is $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.

2.46. Consider a Lie algebra of rank 3, defined by a triangular Dynkin diagram, so that each node is connected with the other two with single lines. Show that such root vectors can not form a basis of 3-dimensional Euclidean space. (The algebra still exists, but it is not finite-dimensional, but a so called affine Kac–Moody algebra.)

2.13. (Real forms)

The split real form. The compact real form. $\mathfrak{su}(n)$ as the compact real form of $\mathfrak{sl}(n, \mathbb{C})$. ...

2.14. Tensor products of representations and tensor formalism

In Section 1.2, the general framework for tensors was outlined, and the important issue was to keep manifest covariance with respect to the general linear group (or algebra) associated with the freedom of choice of basis for any vector space. Now, suppose that we want a tensor formalism based on another Lie algebra. Each of the classical matrix algebras is defined in terms of a "fundamental" representation, that of the matrices themselves, where the module is a "vector" that the matrices act on. The Lie algebras in question are subalgebras of $\mathfrak{gl}(\dim V)$, where V is this fundamental module. All rules from Section 1.2 still apply. The only difference is the introduction of some invariant tensors, which can then be used, following the same rules. These invariant tensors are the following: For $\mathfrak{so}(n)$, there is an invariant metric η_{mn}, For $\mathfrak{sp}(2n)$, there is an invariant antisymmetric tensor ϵ_{mn}. For $\mathfrak{sl}(n)$, and for its subalgebra $\mathfrak{so}(n)$, there is an antisymmetric n-index tensor $\epsilon^{m_1\ldots m_n}$ (and with lower indices). For $\mathfrak{so}(n)$ the tensorial framework can be extended to include spinors by the introduction of γ-matrices.

Examples: symmetric and antisymmetric tensor product of fundamentals in $\mathfrak{sl}$, $\mathfrak{so}$ and $\mathfrak{sp}$. Under $\mathfrak{sl}(n)$, the tensor product of a fundamental representation with itself simply splits into the direct sum of the symmetric and the antisymmetric part (and the same statement for the anti-fundamental). In tensor language, this is written $t_{mn} = t_{(mn)} + t_{[mn]}$. Under $\mathfrak{so}(n)$, the metric can be used to contract the indices, and the symmetric part of the tensor product consists of the direct sum of a singlet and a traceless symmetric tensor, $t_{mn} = \frac{1}{n} g_{mn} g^{pq} t_{pq} + (t_{(mn)} - \frac{1}{n} g_{mn} g^{pq} t_{pq}) + t_{[mn]}$. Under $\mathfrak{sp}(2n)$, it is instead the symmetric part of the tensor product that is irreducible, while the antisymmetric one splits into a singlet and an ϵ-traceless antisymmetric representation.

(Also the exceptional Lie algebras can be given a tensorial framework by the introduction of certain invariant tensors. G_2 has a "fundamental" 7-dimensional representation, and an invariant completely antisymmetric tensor σ_{ijk}. If we consider *e.g.* the antisymmetric tensor product of two 7-dimensional modules, it follows that it will consist of the 14-dimensional adjoint module and a 7-dimensional one.)

2.15. (The Weyl group)

3. Space-time symmetries—The Lorentz, Poincaré and conformal algebras

3.1. The Lorentz algebra

The Lorentz group in d dimensions is $SO(1,d-1)$, with Lie algebra $\mathfrak{so}(1,d-1)$. It preserves a metric with signature $(1,d-1)$, which can *e.g.* be taken to be $v^2 = \eta_{mn}v^m v^n$ with $\eta = \mathrm{diag}(-1,1,\ldots 1)$. If $\delta_J v = Jv$ one has the condition $0 = \delta(v^t\eta v) = v^t(\eta J + J^t\eta)v$, and thus that ηJ is antisymmetric.

One is most often interested not in the full group $SO(1,d-1)$ but in the subgroup $SO^+(1,d-1)$ of orthocronous Lorentz transformations preserving the time direction, equivalently mapping the forward light-cone to itself.

3.2. The Poincaré algebra

The Poincaré group/algebra also contains translations in Minkowski space-time. Call the generators P_m. Translations on flat space commute among themselves, and behave as vectors with respect to the Lorentz transformations.

The commutators between generators of the Poincaré algebra are (in a suitable normalisation)[1]

$$\begin{aligned}[J_{mn},J_{pq}] &= 2\eta_{[p[n}J_{m]q]}\ ,\\ [J_{mn},P_p] &= \eta_{p[n}P_{m]}\ ,\\ [P_m,P_n] &= 0\ .\end{aligned} \tag{3.1}$$

The Poincaré algebra is not semi-simple, since the translations form an abelian ideal. The structure is an example of a *semi-direct product*. The general structure of a semidirect product is $\mathfrak{g} \ltimes R$ where R is a $\mathfrak{g}$-module, with the bracket

$$\begin{aligned}[T_\alpha,T_\beta] &= f_{\alpha\beta}{}^\gamma T_\gamma\ ,\\ [T_\alpha,v_i] &= \varrho(T_\alpha)_i{}^j v_j\ ,\\ [v_i,v_j] &= 0\ ,\end{aligned} \tag{3.2}$$

where $v \in R$ and ϱ are representation matrices.

The Poincaré algebra is $\mathfrak{so}(1,d-1) \ltimes \mathbb{R}^d$. It consists of *all* coordinate transformations preserving the Minkowski metric. Given a manifold equipped with a metric, one can always ask the question if there are coordinate transformations that do not change the metric. Such transformations are called isometries. Flat Euclidean space and Minkowski d-dimensional spaces are examples of *maximally symmetric spaces*, for which the number of isometries takes the maximal value $\frac{1}{2}d(d+1)$, but they are not unique in this sense.

---

[1] The antisymmetrisations in the first of these equations is to be read as antisymmetrisations in the pairs $[nm]$ and $[pq]$, *i.e.*, "$[p[n\,m]q] = \frac{1}{4}(pn\,mq - pm\,nq - qn\,mp + qm\,np)$". A stricter notation uses vertical bars to "pause" and "resume" antisymmetrisation when needed, and one then writes "$[p|[n\,m]|q]$".

3.3. The $d = 4$ Lorentz algebra

$d = 4$ is in some aspects the trickiest of dimensions. This has to do with the fact that, as a complex Lie algebra, $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, but in Minkowski signature one has the Lie algebra $\mathfrak{so}(1,3)$, which is simple as a real Lie algebra (*i.e.*, it is *not* the sum of real forms of $\mathfrak{su}(2)$). For other signatures, the factorisation persists: $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $so(2,2) \simeq \mathfrak{so}(1,2) \oplus \mathfrak{so}(1,2)$.

A very useful way of presenting the 4-dimensional Lorentz algebra is to observe that it in fact is $\mathfrak{sl}(2,\mathbb{C})$, seen as a real Lie algebra. Note that the number of complex parameters in $\mathfrak{sl}(2,\mathbb{C})$ is 3, which gives 6 real parameters, the same as in $\mathfrak{so}(1,3)$. We will now demonstrate that the algebras are the same. Let the generators be $\{T_{mn}\}$, $m = 0,\ldots,3$. The algebra takes the form

$$[T_{mn}, T_{pq}] = 2\eta_{[p[n}T_{m]q]} \ . \qquad (3{\cdot}3)$$

Split the 4-dimensional vector index as $m = (\mu, 3)$, and let $j_\mu = \frac{1}{2}\epsilon_\mu{}^{\nu\lambda}T_{\nu\lambda}$, $k_\mu = T_{\mu 3}$. A short calculation then leads to

$$\begin{aligned} [j_\mu, j_\nu] &= -\tfrac{1}{2}\epsilon_{\mu\nu}{}^\lambda j_\lambda \ , \\ [j_\mu, k_\nu] &= -\tfrac{1}{2}\epsilon_{\mu\nu}{}^\lambda k_\lambda \ , \\ [k_\mu, k_\nu] &= \tfrac{1}{2}\epsilon_{\mu\nu}{}^\lambda j_\lambda \end{aligned} \qquad (3{\cdot}4)$$

(the first two brackets state that the j's generate $\mathfrak{sl}(2,\mathbb{R}) \simeq \mathfrak{so}(1,2)$ and that the k's transform in the 3-dimensional (adjoint) module). If we now form the complex generators $J_\mu = j_\mu + ik_\mu$, they satisfy

$$[J_\mu, J_\nu] = -\epsilon_{\mu\nu}{}^\lambda J_\lambda \ , \qquad (3{\cdot}5)$$

which is the algebra $\mathfrak{sl}(2,\mathbb{C})$.

(In the calculation above, one uses the 3-dimensional ϵ tensor, defined to be completely antisymmetric with $\epsilon^{012} = 1$. Note that this leads *e.g.* to $\epsilon_0{}^{12} = -1$ and $\epsilon_{\mu_1\mu_2\mu_3}\epsilon^{\nu_1\nu_2\nu_3} = -6\delta^{\nu_1\nu_2\nu_3}_{\mu_1\mu_2\mu_3}$ etc.)

The language of $SL(2,\mathbb{C})$ is useful, especially for dealing with spinors in $d = 4$. Note that it follows directly that $\mathfrak{so}(1,3)$ has a 2-dimensional complex representation, which turns out to be a spinor representation. More about this in Section 7.4.

3.4. The conformal algebra

Sometimes, field theories turn out to have a (space-time) symmetry which contains the Poincaré group but is larger. This typically happens when one encounters some kind of scale invariance, as we will see for $d = 4$ gauge theory.

Instead of asking which transformations leave the Minkowski metric invariant, we can ask for those that change it up to a scaling. Such a transformation is called a conformal transformation. So, we are looking for changes of coordinates $\delta_\xi x^m = \xi^m(x)$ which lead to $\delta\xi(\eta_{mn}dx^m dx^n) = 2\varphi(x)\eta_{mn}dx^m dx^n$. Using $\delta_\xi(dx^m) = d\xi^m = \partial_n\xi^m dx^n$, we get the condition $\partial_{(m}\xi_{n)} = \varphi(x)\eta_{mn}$.

The transformations by the generators of the Poincaré algebra are still of course solutions (with $\varphi = 0$), but there is more.

One solution is given by $\xi^m(x) = \varphi x^m$, leading to $\varphi(x) = \varphi$ (constant). This is called a dilatation, or a scaling. Another set of solutions is parametrised by a vector, as $\xi^m(x) = (v \cdot x)x^m - \frac{1}{2}x^2 v^m$. Then $\delta_\xi(dx^m) = (v\cdot dx)x^m + (v\cdot x)dx^m - (x\cdot dx)v^m$, and $\delta(dx\cdot dx) = (v\cdot dx)(x\cdot dx) + (v\cdot x)(dx\cdot dx) - (x\cdot dx)(v\cdot dx) = (v\cdot x)(dx\cdot dx)$. Such a transformation is called a special conformal transformation. In $d \geq 3$ there are no further conformal transformations.

So, in addition to the rotations and translations, we have dilatations and special conformal transformations. Call the corresponding generators Δ and K_m. It is straightforward to check that $[K_m, K_n] = 0$. Let us do one commutator in the conformal algebra, namely $[P_m, K_n]$. Translations and special conformal transformations with parameters a^m and v^m, respectively, are given by

$$\begin{aligned} \delta_a x^m &= a^m \ , \\ \delta_v x^m &= (v \cdot x)x^m - \tfrac{1}{2}x^2 v^m \ . \end{aligned} \tag{3.6}$$

Thus, $\delta_v(\delta_a x^m) = 0$ and

$$\delta_a(\delta_v x^m) = (v\cdot a)x^m + (v\cdot x)a^m - (a\cdot x)v^m = (a\cdot v)x^m + (a^m v_n - v^m a_n)x^n \ . \tag{3.7}$$

This is recognised as a dilatation (the first term) and a Lorentz transformation (the second term).

The conformal algebra is $\mathfrak{so}(2,d)$, with the generators J_{mn}, P_m, K_m, Δ. They can be arranged in a $(d+2)\times(d+2)$ antisymmetric matrix as

$$M_{MN} = \begin{pmatrix} 0 & \Delta & P_n \\ -\Delta & 0 & K_n \\ -P_m & -K_m & J_{mn} \end{pmatrix} \ . \tag{3.8}$$

The invariant matrix is

$$H_{MN} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \eta_{mn} \end{pmatrix} \ , \tag{3.9}$$

with signature $(2,d)$, which gives

$$M_M{}^N = M_{MP}H^{PN} = \begin{pmatrix} \Delta & 0 & P^n \\ 0 & -\Delta & K^n \\ -K_m & -P_m & J_m{}^n \end{pmatrix} \ . \tag{3.10}$$

(In $d = 2$ Minkowski space, the conformal algebra contains $\mathfrak{so}(2,2) \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, but turns out to be infinite-dimensional.)

3.5. The Galilei algebra

...

4. Space-time symmetries and transformations of fields

Physical fields depend on the space-time coordinates. In addition they can come in some module of *e.g.* the Lorentz group, such as vector, spinor etc. In the present Section, we consider symmetries that are defined as transformations of the coordinates. There are then two sources of how fields transform. One is the change of the coordinates which are the argument of the field, the other is the the effect on the module. We will now make this precise, for future use.

4.1. The Lie derivative

Let us begin with a scalar field ϕ. Consider a coordinate transformation of any kind, $x^m \mapsto x'^m(x)$. The condition that ϕ is a scalar amounts to $\phi \mapsto \phi'$, where

$$\phi'(x') = \phi(x) \ . \tag{4.1}$$

For an infinitesimal transformation $x'^m = x^m - \xi^m(x)$, one has $\phi'(x') = \phi'(x) - \xi^m \partial_m \phi$, and the infinitesimal version of the transformation is

$$\delta_\xi \phi(x) = \xi^m \partial_m \phi \ . \tag{4.2}$$

How does a vector field $V^m(x)$ transform? In addition to the dependence on the space-time point, it carries a vector index, which will be "rotated" by a change of basis. The vector index behaves as the index on dx^m: $dx'^m = \frac{\partial x'^m}{\partial x^n} dx^n$. The transformation is

$$V'^m(x') = (M^{-1})_n{}^m V^n(x) \ , \tag{4.3}$$

where $(M^{-1})_n{}^m = \frac{\partial x'^m}{\partial x^n}$. The infinitesimal version is

$$\delta_\xi V^m = \xi^n \partial_n V^m - \partial_n \xi^m V^n \equiv L_\xi V^m \ . \tag{4.4}$$

This is called the Lie derivative.

Note that $M_m{}^n$ is a $GL(d)$ group element, and that the corresponding $\mathfrak{gl}(d)$ element for an infinitesimal transformation is $\partial_m\xi^n$. The transformation of any tensor follows. It reads

$$T'^{m_1\ldots m_p}_{n_1\ldots n_q}(x') = (M^{-1})_{r_1}{}^{m_1}\ldots(M^{-1})_{r_p}{}^{m_p}M_{n_1}{}^{s_1}\ldots M_{n_q}{}^{s_q}T^{r_1\ldots r_p}_{s_1\ldots s_q}(x)\ . \tag{4.5}$$

The product of M's and M^{-1}'s is simply the representation matrix for the group element acting in the tensor product representation. The corresponding holds for the infinitesimal transformation, which for a field in any $\mathfrak{gl}(d)$ module reads

$$\delta_\xi T = L_\xi T = (\xi\cdot\partial)T + \varrho(\partial.\xi^{\cdot})T\ . \tag{4.6}$$

(This is very compact notation. It should be "expanded", so that one convinces oneself how it works, see exercise 4.10).

Consider the commutator of two Lie derivatives. When acting on a scalar, only the first term in eq. (4.6) contributes, and we will have

$$[L_\xi, L_\eta]\phi = [\xi^m\partial_m, \eta^n\partial_n]\phi = L_{[\xi,\eta]}\phi\ , \tag{4.7}$$

where

$$[\xi,\eta]^m = \xi^n\partial_n\eta^m - \eta^n\partial_n\xi^m = (L_\xi\eta)^m = -(L_\eta\xi)^m\ . \tag{4.8}$$

This is simply the commutator between the vector fields $\xi = \xi^m\partial_m$ and $\eta = \eta^m\partial_m$. We can verify that the same commutator is obtained when acting on an arbitrary tensor:

Not surprisingly, general coordinate transformations form a Lie algebra, which is infinite-dimensional.

The transformations derived here hold for general coordinate transformations. We are often interested in the transformations of field under a small subgroup containing global symmetries, such as Poincaré transformations.

4.2. Discrete space-time symmetries: Parity and time reversal

Exercises

4.1. Exponentiate the infinitesimal Lorentz boosts of the Lie algebra $\mathfrak{so}(1,d-1)$ and derive the standard finite Lorentz boosts ($SO(1,d-1)$ group elements) on the form "$x' = \gamma(v)(x-vt)$" etc.

4.2. Show that the algebra of rotations in 4-dimensional Euclidean space is $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$.

4.3. Show that the full Lorentz group contains two disjoint components consisting of the transformation preserving time direction and those reversing it, and that the orthocronous subgroup is the component connected to the identity.

4.4. Show that Poincaré transformations generate the most general transformations preserving the metric on Minkowski space, *i.e.*, that it is the algebra of isometries of Minkowski space.

4.5. Count the isometries of a sphere S^n. Is it a maximally symmetric space?

4.6. Show that there are no more conformal transformations than the ones given above for $d \geq 3$.

4.7. Check that two special conformal transformations commute.

4.8. Consider the d-dimensional surface $\eta_{MN}x^Mx^N = -R^2$ embedded in flat space with metric $\eta = \mathrm{diag}(-1,-1,1,\ldots,1)$. What is the signature of the metric on the surface? Show that the isometry algebra is $\mathfrak{so}(2,d-1)$. (Such a maximally symmetric space is called anti-de Sitter space.)

4.9. Calculate the commutator between two Lie derivatives L_ξ and L_η, *i.e.*, between two coordinate transformations.

4.10. Derive the infinitesimal form of the transformation (4.5), and verify that it produces eq. (4.6).

4.11. In 2-dimensional Minkowski space, let $u = \frac{1}{\sqrt{2}}(x^0 + x^1)$, $v = \frac{1}{\sqrt{2}}(x^0 - x^1)$. What is the metric in this coordinate system? Show that any transformation to new coordinates $u' = f(u)$, $v' = g(v)$ is a conformal transformation.

4.12. A non-linear realisation. Consider the quotient space $SL(2,\mathbb{R})/U(1)$, defined as equivalence classes of elements in $SL(2,\mathbb{R})$ modulo the right action of a $U(1)$. Two elements g and g' in $SL(2,\mathbb{R})$ (2×2 real matrices with unit determinant) are considered equivalent if they are related by a $U(1)$ transformation as $g' = gh$, where

$$h = e^{\theta j} \ , \quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ .$$

Show that all elements in $SL(2,\mathbb{R})$ are in the same equivalence class as an element of the form

$$g = \frac{1}{\sqrt{y}} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \ .$$

Use this parametrisation to derive the transformation of the complex number $z = x + iy$ for such a representative of the equivalence class under the left action $g \mapsto Mg$ with

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \ , \quad ad - bc = 1 \ .$$

Show that the metric

$$ds^2 = \frac{dzd\bar{z}}{(\mathrm{Im}\, z)^2}$$

is invariant under $SL(2,\mathbb{R})$. (This is the so called Poincaré upper half plane, describing a 2-dimensional hyperbolic space with constant curvature.) Discuss what the $SL(2,\mathbb{R})$ isometry means. Is this a maximally symmetric space?

4.3. Differential forms, wedge product, dualisation, integration

Coordinates x^m. $x'^m = x'^m(x)$.

$$\begin{aligned} dx'^m &= dx^n \frac{\partial x'^m}{\partial x^n} = dx^n (M^{-1})_n{}^m \;, \\ \partial'_m &= \frac{\partial x^n}{\partial x'^m}\partial_n = M_m{}^n \partial_n \;. \end{aligned} \tag{4.9}$$

M is an element in $GL(d)$. Scalar (0-form): $\omega'(x') = \omega(x)$. 1-form: $\omega = ds^m \omega_m$, $dx'^m \omega'_m(x') = dx^m \omega_m(x)$, which gives $\omega'_m(x') = M_m{}^n \omega_n(x)$. p-form: $\omega = \frac{1}{p!} dx^{m_1} \wedge dx^{m_2} \wedge \ldots \wedge dx^{m_p} \omega_{m_1 m_2 \ldots m_p}$. $\omega'_{m_1 \ldots m_p}(x') = M_{m_1}{}^{n_1} \ldots M_{m_p}{}^{n_p} \omega_{n_1 \ldots n_p}(x)$. Forms are (locally) elements in completely antisymmetric modules of $GL(d)$.

$d = dx^m \partial_m$, "$d\omega = d \wedge \omega$". $d^2 = 0$.

$$(d\omega)_{m_1 \ldots m_{p+1}} = (p+1)\partial_{[m_1} \omega_{m_2 \ldots m_{p+1}]} \;. \tag{4.10}$$

Transformation:

$$\begin{aligned} (d\omega)'_{m_1 \ldots m_{p+1}}(x') &= (p+1) M_{[m_1}{}^{n_1} \partial_{|n_1|} (M_{m_2}{}^{n_2} \ldots M_{m_{p+1}]}{}^{n_{p+1}} \omega_{n_2 \ldots n_{p+1}}(x)) \\ &= (p+1) M_{[m_1}{}^{n_1} \ldots M_{m_{p+1}]}{}^{n_{p+1}} \partial_{n_1} \omega_{n_2 \ldots n_{p+1}}(x) \\ &\quad + p(p+1) M_{[m_1}{}^{n_1} \partial_{|n_1|} M_{m_2}{}^{n_2} \ldots M_{m_{p+1}]}{}^{n_{p+1}} \omega_{n_2 \ldots n_{p+1}}(x) \end{aligned} \tag{4.11}$$

The second term vanishes thanks to antisymmetrisation, since

$$M_{[m_1}{}^{n_1} \partial_{|n_1|} M_{m_2]}{}^{n_2} = \frac{\partial x^{n_1}}{\partial x'^{[m_1}} \frac{\partial}{\partial x^{|n_1|}} \frac{\partial x^{n_2}}{\partial x'^{m_2]}} = \frac{\partial^2 x^{n_2}}{\partial x'^{[m_1} \partial x'^{m_2]}} = 0 \;. \tag{4.12}$$

Therefore, the exterior derivative of a p-form is (meaning, transforms as) a $(p+1)$-form.

A form ω such that $d\omega = 0$ is called closed. A form $\omega = d\alpha$ is called exact.

Wedge product: $\alpha \wedge \beta$ gives

$$(\alpha \wedge \beta)_{m_1 \ldots m_{p+q}} = \frac{(p+q)!}{p!q!} \alpha_{[m_1 \ldots m_p} \beta_{m_{p+1} \ldots m_{p+q}]} \ . \tag{4.13}$$

Dualisation: $GL(d)$ is "broken" to $SO(d)$ (or, some real form of $SO(d)$) by the introduction of a metric g_{mn}. Then, a p-form ω can be converted into a $(d-p)$-form $\star\omega$:

$$\star\omega_{m_1 \ldots m_{d-p}} = \frac{1}{p!\sqrt{|g|}} g_{m_1 n_1} \cdots g_{m_{d-p} n_{d-p}} \epsilon^{n_1 \ldots n_d} \omega_{n_{d-p+1} \ldots n_d} \ . \tag{4.14}$$

Integration: A d-form is unique, up to multiplication by a scalar.

$$\begin{aligned} \Omega &= \frac{1}{d!} dx^{m_1} \wedge \ldots \wedge dx^{m_d} \Omega_{m_1 \ldots m_d} \\ &= dx^1 \wedge \ldots \wedge dx^d \Omega_{1 \ldots d} \ . \end{aligned} \tag{4.15}$$

$$\int_{\mathscr{M}} \Omega = \int_{\mathscr{M}} dx^1 \ldots dx^d \Omega_{1 \ldots d} = \int_{\mathscr{M}} d^d x \sqrt{|g|} \star \Omega \ . \tag{4.16}$$

The integral of an exact form vanishes over a manifold without boundary: $\int_{\mathscr{M}} d\omega = 0$ if $\partial\mathscr{M} = 0$. If $\mathscr{M}$ has a boundary $\partial\mathscr{M}$, then

$$\int_{\mathscr{M}} d\omega = \int_{\partial\mathscr{M}} \omega \ . \tag{4.17}$$

A 2-form Maxwell field strength F is defined from the connection 1-form A by $F = dA$. The gauge symmetry is $\delta_\Lambda A = d\Lambda$, which implies $\delta_\Lambda F = 0$. The Bianchi identity $dF = 0$ is automatically satisfied. The rest of Maxwell's equations read $d\star F = j$, where j: $dj = 0$ is a $(d-1)$-form.

<u>Exercises</u>

4.13. Interpret Gauss' law in terms of eq. (4.17).

4.14. In which dimensions can a form be both real and self-dual ($\star\omega = \pm\omega$) in Minkowski signature? In Euclidean signature? What does this have to do with exercise 4.2?

4.15. Define the contraction of a p-form Ω with a vector, resulting in a $(p-1)$-form, as $\imath_v \Omega_{m_1 \ldots m_{p-1}} = v^m \Omega_{m m_1 \ldots m_{p-1}}$. Let ω be a 1-form. Show that $(\imath_v \omega \wedge + \omega \wedge \imath_v)\Omega = (\imath_v \omega)\Omega = (v^m \omega_m)\Omega$.

5. The action principle for classical fields

5.1. The action principle

In mechanics, the action principle is a convenient way to encode the dynamics as an optimisation problem: Solutions to the equations of motion correspond precisely to stationary points of an action functional. The corresponding holds for field theory. In addition, it provides a natural basis for quantum field theory.

Recall the equations of motion for a particle in Newtonian dynamics, in any dimension, with a conservative force. One forms the Lagrangian $L = T - V$, where T is kinetic energy and V is potential energy.

$$L = \tfrac{1}{2}m\dot{x}^2 - V(x)\ . \tag{5.1}$$

The action S is formally the integral of L over time,

$$S = \int dt(T - V) = \int dt\left(\tfrac{1}{2}m\dot{x}^2 - V(x)\right)\ . \tag{5.2}$$

The action is a functional, *i.e.*, a function from a function space to the real numbers. Finding a stationary point of S means that a small change $x(t) \mapsto x(t) + \delta x(t)$ leaves the S unchanged to linear order in δx. In the Newtonian particle example,

$$\delta S = \int dt\,(m\delta\dot{x}\cdot\dot{x} - \delta x\cdot\nabla V)\ . \tag{5.3}$$

Partial integration gives (we are intentionally sloppy about boundary terms)

$$\delta S = -\int dt\,\delta x\cdot(m\ddot{x} + \nabla V) = \int dt\,\delta x(t)\cdot\frac{\delta S}{\delta x(t)}\ . \tag{5.4}$$

$\frac{\delta S}{\delta x(t)}$ is the *functional derivative* of S with respect to $x(t)$. The equations of motion are obtained as $\frac{\delta S}{\delta x(t)} = 0$, just as the stationary points of a function $f(y)$ are the solutions to $\frac{df}{dy} = 0$.

Particle mechanics can be thought of as a 1-dimensional field theory. We want to extend the principle to fields, for example scalar fields, spinor fields and gauge fields. The guiding principle is that the action should be invariant under any symmetry at hand. For field theories on Minkowski space, this means Poincaré symmetry, and possibly other "internal" symmetries, such as the gauge symmetry of Yang–Mills theory. We would also like the action to be "local", *i.e.*, possible to express as the integral over space-time of a Lagrangian density, in order to avoid non-local interactions.

We would normally like the field equations (equations of motion) to contain two derivatives for bosonic fields and one for fermionic fields. Consider a collection of fields $\{\Phi^I\}$, where I is some index. The action should be some functional of $\Phi^I(x)$. It will contain derivatives $\partial_m\Phi$, but there should be no explicit dependence on $\partial_m \partial_n \Phi$ (this is no absolute rule, but we have no reason to be more general).

$$S = \int dt\, L = \int d^d x\, \mathscr{L}(\Phi(x), \partial\Phi(x)) \;. \tag{5.5}$$

The field equations become

$$0 = -\frac{\delta S}{\delta \Phi^I(x)} = \partial_m \frac{\partial \mathscr{L}}{\partial_m \Phi^I} - \frac{\partial \mathscr{L}}{\partial \Phi^I} \;. \tag{5.6}$$

5.2. Integration and invariance

Let us investigate how we can conclude that the action is invariant. The first issue is integration. It should be independent of how we choose to parametrise the space-time, *i.e.*, invariant under general coordinate transformations (also if we are working on flat Minkowski space).

The integration $\int d^d x$ is not invariant under general coordinate transformations. When $x^m \mapsto x'^m(x)$, it picks up a Jacobian of the change of coordinates:

$$\int d^d x' = \int d^d x \left| \frac{\partial x'}{\partial x} \right| = \int d^d x |M|^{-1} \;, \tag{5.7}$$

where $|\cdot|$ denotes the determinant. Therefore, the transformation of the integrand must compensate for this scaling. It is not a scalar, but a *scalar density*. The *Lagrangian density* $\mathscr{L}$ must transform as

$$\mathscr{L}'(x') = |M| \mathscr{L}(x) \;.$$

Normally, we have a metric g_{mn} at hand. Its determinant transforms as $|g'(x')| = |M|^2 |g(x)|$, so $\sqrt{|g|}$ has the desired property. (Here, we let $|g| = |\det g|$ to avoid a minus sign under the square root.)

Invariance under coordinate transformations (reparametrisations) is thus ensured by letting $\mathscr{L} = \sqrt{|g|}\bar{\mathscr{L}}$, where $\bar{\mathscr{L}}$ is a scalar, and

$$S = \int d^d x \sqrt{|g|} \bar{\mathscr{L}} \;. \tag{5.8}$$

If we now specialise to Minkowski space, and use coordinates where $\det \eta = -1$, the explicit square root in the integration measure disappears. If one wants to use coordinates where the metric takes another form (such as *e.g.* spherical coordinates for the spatial directions), it needs to be reinserted. In Minkowski space, we will have

$$S = \int d^d x\, \mathscr{L} \;, \tag{5.9}$$

where $\mathscr{L}$ is a Lorentz scalar.

5.3. Hamiltonian formalism

In any relativistic field theory, the Lagrangian density $\mathscr{L}$ is a Lorentz scalar. One advantage of the formalism is that it manifests the symmetries. If we go to a Hamiltonian framework, a choice of time coordinate must be made, and Lorentz symmetry is not manifest. The derivation of the Hamiltonian formalism below is formally identical to the one in mechanics.

Define the momentum conjugate to a field Φ^I as

$$\Pi_I(x) = \frac{\partial\mathscr{L}}{\partial\dot{\Phi}^I} \ . \tag{5.10}$$

This is completely analogous to the definition of (generalised) momentum in mechanics. The field equations can then be written

$$0 = \dot{\Pi}_I + \partial_i \frac{\partial\mathscr{L}}{\partial_i\Phi^I} - \frac{\partial\mathscr{L}}{\partial\Phi^I} = \dot{\Pi}_I - \frac{\delta L}{\delta\Phi^I} \ , \tag{5.11}$$

where $i = 1, \ldots, d-1$ is a spatial vector index. Define the Hamiltonian as

$$H = \int d^{d-1}x\, \mathscr{H} = \int d^{d-1}x \left(\dot{\Phi}^I \Pi_I - \mathscr{L} \right) \ . \tag{5.12}$$

Eq. (5.12) is called a Legendre transformation. Its form implies that it is natural to see the Hamiltonian density as a function of Φ (and $\partial_i\Phi$, but not $\dot{\Phi}$) and Π. Namely, consider a variation $d\Phi$ and the ensuing variation in the momentum. Then

$$\begin{aligned} dH &= \int d^{d-1}x \left(d\dot{\Phi}^I \Pi_I + \dot{\Phi}^I d\Pi_I - \frac{\delta L}{\delta\dot{\Phi}^I} d\dot{\Phi}^I - \frac{\delta L}{\delta\Phi^I} d\Phi^I \right) \\ &= \int d^{d-1}x \left(\dot{\Phi}^I d\Pi_I - \frac{\delta L}{\delta\Phi^I} d\Phi^I \right) \ . \end{aligned} \tag{5.13}$$

This implies that

$$\begin{aligned} \frac{\delta H}{\delta\Phi^I} &= -\frac{\delta L}{\delta\Phi^I} \\ \frac{\delta H}{\delta\Pi_I} &= \dot{\Phi}^I \ . \end{aligned} \tag{5.14}$$

Using the field equations (5.11) in the first of these equations, we have Hamilton's equations

$$\begin{aligned} \frac{\delta H}{\delta\Phi^I} &= -\dot{\Pi}_I \\ \frac{\delta H}{\delta\Pi_I} &= \dot{\Phi}^I \ . \end{aligned} \tag{5.15}$$

The field equations, containing terms which are second order in time derivatives, are replaced by twice as many first order equations.

Poisson bracket:

$$\{\Phi^I(x), \Pi_J(y)\} = \delta^I_J \delta^{d-1}(x-y) \ , \tag{5.16}$$

and, more generally, for any functions F and G of the phase space variables,

$$\{F, G\} = \int d^{d-1}x \left(\frac{\delta F}{\delta\Phi^I(x)} \frac{\delta G}{\delta\Pi_I(x)} - \frac{\delta F}{\delta\Pi_I(x)} \frac{\delta G}{\delta\Phi^I(x)} \right) \ . \tag{5.17}$$

Then, Hamilton's equation can be written as special cases of

$$\dot{F} = \{F, H\} \tag{5.18}$$

for any function F on phase space. If F in addition has some explicit time dependence, we have

$$\dot{F} = \frac{\partial F}{\partial t} + \{F, H\} \tag{5.19}$$

The Hamiltonian is the generator of time translations. Any conserved quantity must have vanishing Poisson bracket with H.

Exercises

5.1. Perform a coordinate transformation from orthonormal to spherical coordinates on flat $\mathbb{R}^3$. Derive the metric and integration measure in spherical coordinates.

6. Scalar fields

We are looking for some "natural" dynamics for scalar fields. The equations of motion are likely to be at most second order in time derivatives, which in the light of Lorentz invariance should apply to all derivatives. There is one Lorentz scalar combination of the derivatives, namely the d'Alembertian

$$\Box = \partial^m \partial_m = -(\partial_0)^2 + \partial_i \partial_i \ . \tag{6.1}$$

A simple equation for a scalar field $\phi(x)$ is

$$\Box\phi = 0 \ . \tag{6.2}$$

This is the wave equation. It has solutions which are waves propagating with speed 1 (the speed of light), which is readily seen in the Fourier transformed picture. Define

$$\phi(x) = \frac{1}{(2\pi)^{d/2}} \int d^d k\, e^{ik_m x^m} \tilde{\phi}(k) \ . \tag{6.3}$$

Then the wave equation translates to $k^2\tilde{\phi}(k) = 0$, so $\tilde{\phi}$ has support only on the light cone $k^2 = 0$. The d-dimensional wave vector k_m consists of $k_0 = \omega$ and $k_i = \frac{2\pi}{\lambda} n_i$, where ω is the angular frequency of the wave, λ the wave length and $\vec{n}$ a unit vector in the direction of propagation. This leads to the relativistic dispersion relation $\omega^2 + \vec{k}^2 = 0$, $\omega = (\pm)|\vec{k}|$. If we would pass to a quantum theory, ω is energy and $\vec{k}$ momentum, k_m becomes d-dimensional momentum, and the dispersion relation becomes the condition $p^2 = 0$ which applies to massless particles.

How is this generalised to propagation inside the light cone, to wave vectors k_m with $k^2 < 0$? If we introduce a real dimensionful parameter m with dimension (length)$^{-1}$, we can write the *Klein–Gordon equation*

$$(\Box - m^2)\phi = 0 \ . \tag{6.4}$$

In classical field theory, m defines a length scale In terms of the Fourier-transformed field the equation becomes $(k^2 + m^2)\tilde{\phi}(k) = 0$, so $\tilde{\phi}$ has support on the hyperboloid $k^2 = -m^2$. The wave vector now lies inside the light cone, and the dispersion relation is $\omega = (\pm)\sqrt{\vec{k}^2 + m^2}$. In quantum theory this translates to $p^2 = -m^2$, which applies to relativistic particles with mass m.

An action, which upon variation leads to the Klein–Gordon equation (for a real scalar ϕ), is

$$S = -\tfrac{1}{2} \int d^d x \left(\eta^{mn} \partial_m \phi \partial_n \phi + m^2 \phi^2\right) \ . \tag{6.5}$$

The overall factor in the action is of course not determined from the field equations. The choice made here corresponds to "canonical normalisation", which means that the time derivative enters in the Lagrangian density as $\frac{1}{2}\dot{\phi}^2$. The sign corresponds to positive kinetic energy.

Defining the conjugate momentum $\pi = \frac{\partial \mathscr{L}}{\partial \dot{\phi}} = \dot{\phi}$, and $H = \int d^{d-1}x\, \dot{\phi}\pi - L$, the Hamiltonian becomes

$$H = \tfrac{1}{2} \int d^{d-1} x \left(\pi^2 + (\nabla\phi)^2 + m^2 \phi^2\right) \ . \tag{6.6}$$

The Hamiltonian is positive definite, which is yet another argument for the choice of sign for the mass term in the Klein–Gordon equation. The other sign would correspond to a quadratic potential which has a maximum instead of a minimum. At the same time it would lead to "tachyonic" propagation, which violates causality. Propagation with speed above the speed of light signals instability.

Notice how the sign in "$L = T - V$" goes hand in hand with the sign in the Minkowski metric.

The action (6.5) contains the Minkowski metric explicitly. It has global Poincaré symmetry; the coordinate transformations preserving the Minkowski metric. Accordingly, the Klein–Gordon equation (6.4) does not behave as a scalar under general coordinate transformations, only under the Poincaré subgroup. What if we want to use other coordinates, say spherical coordinates for spatial directions, to describe a physical situation? Or if we want to describe the dynamics of scalar fields on some other space-time than (flat) Minkowski space? Then we do not have access to the Minkowski metric, but need a metric g_{mn} which transforms as a tensor under general coordinate transformations. An invariant action is

$$S=-\tfrac{1}{2}\int d^dx\sqrt{|g|}\left(g^{mn}\partial_m\phi\partial_n\phi+m^2\phi^2\right)\ , \tag{6.7}$$

where g^{mn} are the components of the inverse metric. The field equation arising on variation from this action is

$$\frac{1}{\sqrt{|g|}}\partial_m\left(\sqrt{|g|}g^{mn}\partial_n\phi\right)-m^2\phi=0\ . \tag{6.8}$$

The first term is the proper definition of the d'Alembertian $\Box$ in any coordinate system. It is probably not obvious that this term behaves as a scalar, but this should be the case since it has been derived from an invariant action (see exercise 6.1).

<u>Exercises</u>

6.1. Show that the divergence of a vector, defined as $\frac{1}{\sqrt{|g|}}\partial_m(\sqrt{|g|}A^m)$, is a scalar.

6.2. In spherical coordinates on $\mathbb{R}^3$, derive the Laplacian.

6.3. In $d=4$, determine how the static, spherically symmetric solution to the Klein–Gordon equation, corresponding to a point source at the origin, is affected by $m\neq 0$.

7. Spinors

7.1. Spinors in $d=3$

Recall the Pauli matrices,

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\ ,\qquad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\ ,\qquad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\ . \tag{7.1}$$

They provide a basis for the Hermitean (2×2)-matrices, and (multiplied by i) generate $\mathfrak{su}(2) \simeq \mathfrak{so}(3, \mathbb{R})$. Their existence shows that $\mathfrak{so}(3, \mathbb{R})$ has a 2-dimensional (complex) representation, the spinor representation. It is straightforward to verify that

$$\sigma_i \sigma_j = \delta_{ij} I + i \epsilon_{ijk} \sigma_k \ . \tag{7.2}$$

The second term is antisymmetric $[ij]$, so we have

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2 \delta_{ij} I \ . \tag{7.3}$$

This equation, rather than eq. (7.2) is the one we will generalise to arbitrary dimension. Note that eq. (7.2) is very specific to 3 dimensions, since it uses ε_{ijk}, while eq. (7.3) can be written in any dimension, *i.e.*, for any Lie algebra $\mathfrak{so}(d)$, as will be demonstrated in Section 7.3.

The Pauli σ-matrices provide a special case, for $\mathfrak{so}(3, \mathbb{R})$, of γ-matrices, which fulfil

$$\gamma_m \gamma_n + \gamma_n \gamma_m = 2 \eta_{mn} I \ , \tag{7.4}$$

Where η is the metric left invariant by some orthogonal algebra $\mathfrak{so}(p, q)$.

Another, also 3-dimensional, example is provided by $\mathfrak{so}(1,2) \simeq \mathfrak{sl}(2, \mathbb{R})$. Take the basis for the traceless matrices

$$(\gamma_0)_\alpha{}^\beta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \qquad (\gamma_1)_\alpha{}^\beta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \ , \qquad (\gamma_2)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ . \tag{7.5}$$

The second index can be lowered with ϵ to get a basis for the symmetric matrices,

$$(\gamma_0 \epsilon)_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \qquad (\gamma_1 \epsilon)_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \qquad (\gamma_2 \epsilon)_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ . \tag{7.6}$$

These γ-matrices behave very similarly to the σ-matrices, only with some (important) sign changes. Check that

$$\gamma_m \gamma_n + \gamma_n \gamma_m = 2 \eta_{mn} I \ , \tag{7.7}$$

where η is the Minkowski metric $\mathrm{diag}(-1, 1, 1)$, invariant under $\mathfrak{so}(1, 2)$.

7.2. Rotation generators in spinor representations

The significance of the identity (7.3) is not yet clear. We will now see that if we can find γ-matrices, satisfying this equation, this is all that is needed in order to have a spinor representation. Consider a

Lie algebra element T_{mn}, labelled by an antisymmetric pair of indices. Consider the representation matrices (this is what will be shown)

$$(\varrho(T_{mn}))_\alpha{}^\beta = \tfrac{1}{4}(\gamma_{mn})_\alpha{}^\beta \ , \tag{7.8}$$

where the expression in the right hand side is defined by

$$\gamma_{mn} = \gamma_{[m}\gamma_{n]} = \tfrac{1}{2}(\gamma_m\gamma_n - \gamma_n\gamma_m) \ . \tag{7.9}$$

The factor $\frac{1}{4}$ is just a standard normalisation, and depends on the normalisation of the basis elements T_{mn}.

Direct calculation gives (pulling $\gamma_p\gamma_q$ to the left, using the γ-matrix identity)

$$\begin{aligned}\gamma_m\gamma_n\gamma_p\gamma_q &= \gamma_m(2\eta_{np} - \gamma_p\gamma_n)\gamma_q \\ &= 2\eta_{np}\gamma_m\gamma_q - (2\eta_{mp} - \gamma_p\gamma_m)\gamma_n\gamma_q \\ &= 2\eta_{np}\gamma_m\gamma_q - 2\eta_{mp}\gamma_n\gamma_q + \gamma_p\gamma_m(2\eta_{nq} - \gamma_q\gamma_n) \\ &= 4\eta_{p[n}\gamma_{m]}\gamma_q + 4\eta_{q[n}\gamma_{|p|}\gamma_{m]} + \gamma_p\gamma_q\gamma_m\gamma_n \ .\end{aligned} \tag{7.10}$$

Antisymmetrising in $[mn]$ and $[pq]$ then gives

$$[\gamma_{mn}, \gamma_{pq}] = 2\eta_{p[n}(\gamma_{m]}\gamma_q - \gamma_{|q|}\gamma_{m]}) - (p \leftrightarrow q) = 8\eta_{[p[n}\gamma_{m]q]} \ .$$

This is the appropriate orthogonal algebra. Scaling to $\varrho(T_{mn}) = \frac{1}{4}\gamma_{mn}$ yields

$$[\varrho(T_{mn}), \varrho(T_{pq})] = 2\eta_{[p[n}\varrho(T_{m]q]}) \ .$$

Compare with eq. (3.3).

This shows that, if one finds γ-matrices, one has found a spinor representation.

7.3. Spinors in arbitrary number of dimensions. Gamma matrices

The spinor representations are particular to the orthogonal algebras $\mathfrak{so}(d)$. They are, strictly speaking, not representations of $SO(d)$, but of its double cover, $Spin(d)$.

Let us start in even dimensions, from the definition of the algebra $\mathfrak{so}(2n, \mathbb{C})$. We do not yet specialise to Minkowski or Euclidean or any other signature. We do not need to worry that spinors only exist for certain signatures and not for others. The existence of a representation is never a matter of the choice of real form. What the real form *does* affect is the possibility to choose a representation module as a *real* vector space. We have already seen (inspect the Pauli matrices and $\mathfrak{so}(1,2)$ γ-matrices!) that $\mathfrak{so}(3,\mathbb{R})$ has complex spinors, while they may be taken real for $\mathfrak{so}(1,2)$. Indeed, we may move between different signatures by multiplying some γ-matrices with i, thus converting a space direction to a time direction and *vice versa*.

We take the invariant metric as

$$\eta = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} . \tag{7.11}$$

(If we take the real Lie algebra defined by this metric, it has signature (n,n).) The generators $T_M{}^N$ are matrices such that $T\eta$ is antisymmetric, which gives

$$T_M{}^N = \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} , \tag{7.12}$$

where B and C are antisymmetric. The matrices A generate a $\mathfrak{gl}(n)$ subalgebra. We see that a vector of dimension $2n$ splits into a vector and a covector under $\mathfrak{gl}(n)$, $V^M = (v^m, w_m)$.

Consider forms in n dimensions, $\Omega = \sum_{p=0}^n \Omega^{(p)}$, where p is the form degree. (The term form here just refers to elements in modules of $\mathfrak{gl}(n)$ with p antisymmetric lower indices.) The dimension of this vector space is $\sum_{p=0}^n \binom{n}{p} = 2^n$. Consider the actions of wedge product and contraction (if you are not yet familiar with this language, from Section 4.3, the calculation is soon spelled out in components):

$$\begin{aligned} \gamma^m\Omega &= \sqrt{2}dx^m \wedge \Omega , \\ \gamma_m\Omega &= \sqrt{2}\imath_m\Omega . \end{aligned} \tag{7.13}$$

Then we have (see exercise 4.15) $(\gamma^m\gamma_n + \gamma_n\gamma^m)\Omega = 2\delta_n^m\Omega$. In addition, $\gamma^m\gamma^n + \gamma^n\gamma^m = 0 = \gamma_m\gamma_n + \gamma_n\gamma_m$. Let $\gamma^M = (\gamma^m, \gamma_m)$ be a $2n$-vector of matrices. Then,

$$\gamma^M\gamma^N + \gamma^N\gamma^M = 2\eta^{MN} . \tag{7.14}$$

Here comes the corresponding statements in a less streamlined, but maybe more explicit, component language. A spinor Ω in $d = 2n$ is a collection of all totally antisymmetric tensors ("forms") under $\mathfrak{gl}(n)$ with different number of indices ("form degree"), from 0 to n:

$$\Omega = \begin{pmatrix} \omega \\ \omega_m \\ \omega_{m_1m_2} \\ \vdots \\ \omega_{m_1\ldots m_n} \end{pmatrix} . \tag{7.15}$$

What γ^m and γ_m do is to step up or down one step, *i.e.*, decrease or increase the number of indices by 1:

$$\begin{aligned}(\gamma_m\Omega)_{m_1\ldots m_p} &= \sqrt{2}\omega_{mm_1\ldots m_p}\ ,\\ (\gamma^m\Omega)_{m_1\ldots m_p} &= \sqrt{2}p\delta^m_{[m_1}\omega_{m_2\ldots m_p]}\ .\end{aligned} \tag{7.16}$$

Obviously, because of antisymmetry, $\gamma^{(m}\gamma^{n)}=0$, $\gamma_{(m}\gamma_{n)}=0$, corresponding to the 0 blocks in the metric. One also has

$$\begin{aligned}(\gamma^m\gamma_n\Omega)_{m_1\ldots m_p} &= \sqrt{2}p\delta^m_{[m_1}(\gamma_{|n|}\Omega)_{m_2\ldots m_p]} = 2p\delta^m_{[m_1}\omega_{|n|m_2\ldots m_p}\ ,\\ (\gamma_n\gamma^m\Omega)_{m_1\ldots m_p} &= \sqrt{2}(\gamma^m\Omega)_{nm_1\ldots m_p} = 2(p+1)\delta^m_{[n}\omega_{m_1\ldots m_p]}\\ &= 2\delta^m_n\omega_{m_1\ldots m_p} - 2p\delta^m_{[m_1}\omega_{|n|m_2\ldots m_p]}\ .\end{aligned} \tag{7.17}$$

Thus $\gamma^m\gamma_n+\gamma_n\gamma^m=2\delta^m_n$, corresponding to the unit matrix entries in the metric.

Matrices satisfying eq. (7.14) are called γ matrices, and the objects they act on are called spinors. Here we have represented a spinor in $2n$ dimensions as a collection of forms. We should show that the spinor forms a module of the orthogonal algebra. Define the matrices $\gamma^{MN}=\gamma^{[M}\gamma^{N]}$. Then $J^{MN}=\frac{1}{4}\gamma^{MN}$ fulfil the same algebra (under commutation) as the $\mathfrak{so}(2n)$ generators. Thus $\mathfrak{so}(2n)$ has a 2^n-dimensional spinor representation. This spinor is called a Dirac spinor.

Consider how J^{MN} act on the spinor. The different components contain 2 wedge products (J^{mn}), 1 wedge product and 1 contraction (${J_m}^n$) or 2 contractions (J_{mn}). The form degree is shifted by -2, 0 or 2. Forms of even degree and odd degree do not mix under rotations, and form separate modules, each of dimension 2^{n-1}. These modules are called chiral spinors, or Weyl spinors. They are irreducible.

These considerations were performed in the complex Lie algebra, alternatively for the real form $\mathfrak{so}(n,n)$. The existence of Dirac spinors and chiral spinors is not affected by the choice of real form (Euclidean, Minkowski, etc.). The reality properties of the spinor modules depend on it, however. We will not go into details.

In odd dimensions $d=2n-1$, a similar construction can be performed. There are no chiral spinors, and the irreducible spinor module has dimension 2^{n-1}. (See exercise 7.2.)

A very concrete way to construct γ-matrices in increasing number of dimensions is the following. Suppose that we are in $d=2n$ dimensions, with γ-matrices

$$\gamma^m = \begin{pmatrix} 0 & \tilde{\sigma}^m \\ \sigma^m & 0 \end{pmatrix}\ . \tag{7.18}$$

In order to increase by 1 time (0) and 1 spatial dimension ($d+1$), one can let the gamma matrices in $d+2$ dimensions be ($M=0,m,d+1$)

$$\Gamma^M = \begin{pmatrix} 0 & \tilde{\Sigma}^m \\ \Sigma^m & 0 \end{pmatrix}\ , \tag{7.19}$$

where

$$\Sigma^0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\ , \qquad \Sigma^{d+1} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\ , \qquad \Sigma^m = \gamma^m\ ;$$
$$\tilde{\Sigma}^0 = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix}\ , \qquad \tilde{\Sigma}^{d+1} \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\ , \qquad \tilde{\Sigma}^m = \gamma^m\ .$$

If one only increases by one (spatial) direction, Σ^0 is left out. Then $\Sigma^{M'} = \tilde{\Sigma}^{M'} = \Gamma^{M'}$. Verify that these matrices indeed are γ-matrices.

This construction tells us directly that the dimension of a Dirac spinor is multiplied by 2 when the number of dimensions is increased from $2n$ to $2n+2$, and remains unchanged when dimension is increased from $2n$ to $2n+1$. A chiral (Weyl) spinor in $2n+2$ dimensions becomes a Dirac spinor in $2n+1$ dimensions, and the distinction between chiralities disappears. A Dirac spinor in $2n+1$ dimensions becomes a Dirac spinor (two Weyl spinors with opposite chiralities) in $2n$ dimensions.

7.4. Spinors in 4 dimensions

It is convenient to use the version of the $d=4$ Lorentz algebra given in Section 3.3, $\mathfrak{so}(1,3) \simeq \mathfrak{sl}(2,\mathbb{C})$, where the latter is seen as a Lie algebra over $\mathbb{R}$. We know that there is at least one 2-dimensional complex spinor representation. Note that the dimension 2 of this spinor representation matches the dimension of a Weyl spinor ($2^{\frac{4}{2}-1}=2$).

Let us denote a chiral spinor ψ_α, $\alpha=1,2$, and let it transform under $\mathfrak{sl}(2,\mathbb{C})$ as $\delta_\ell\psi_\alpha = \ell_\alpha{}^\beta\psi_\beta$.

Let us raise indices with ϵ as $\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta$. Then (since $\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = -\delta_\alpha^\beta$), $\psi_\alpha = -\epsilon_{\alpha\beta}\psi^\beta$.

The complex conjugate of a chiral spinor transforms as a spinor of the opposite chirality. With upper index, $\bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}$. Then $\bar\psi$ transforms as

$$\delta_\ell\bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\overline{(\ell\psi)}_{\dot\beta} = -\epsilon^{\dot\alpha\dot\beta}\bar\ell_{\dot\beta}{}^{\dot\gamma}\epsilon_{\dot\gamma\dot\delta}\bar\psi^{\dot\delta} = -\bar\psi^{\dot\beta}\bar\ell_{\dot\beta}{}^{\dot\alpha}\ , \tag{7.20}$$

where we have used $\epsilon\ell\epsilon = \ell^t$ for a traceless matrix ℓ (see eq. (2.30)).

The dotted indices are introduced in order to maintain a working tensor formalism. Otherwise one would have been allowed for example to contract one spinor index with another one on a complex conjugated spinor. This is not allowed, since it would demand that "$\bar\psi^\alpha\chi_\alpha$" be a scalar, which is is not, since $\bar\ell - \ell \neq 0$.

We will construct the γ matrices. If we let them act on a Dirac spinor

$$\Psi_A = \begin{pmatrix} \psi_\alpha \\ \chi^{\dot\alpha} \end{pmatrix}\ , \tag{7.21}$$

and use our knowledge that a γ matrix changes the chirality of a chiral spinor, we find that the γ matrices must take the form

$$(\gamma^m)_A{}^B = \begin{pmatrix} 0 & \tilde{\sigma}^m_{\alpha\dot{\beta}} \\ \sigma^{m\dot{\alpha}\beta} & 0 \end{pmatrix} , \tag{7.22}$$

where

$$\begin{aligned} (\tilde{\sigma}^m\sigma^n + \tilde{\sigma}^n\sigma^m)_\alpha{}^\beta &= 2\eta^{mn}\delta_\alpha^\beta , \\ (\sigma^m\tilde{\sigma}^n + \sigma^n\tilde{\sigma}^m)^{\dot{\alpha}}_{\dot{\beta}} &= 2\eta^{mn}\delta^{\dot{\beta}}_{\dot{\alpha}} . \end{aligned} \tag{7.23}$$

In addition, we would like to demand that $((\sigma^m\psi)^\dagger)_\alpha = (\tilde{\sigma}^m\psi^\dagger)_\alpha$, implying that $\tilde{\sigma}^m_{\alpha\dot{\beta}} = \epsilon_{\alpha\gamma}\bar{\sigma}^{\gamma\dot{\delta}}\epsilon_{\dot{\delta}\dot{\beta}}$. A suitable choice is

$$\begin{aligned} \sigma^0 = -\tilde{\sigma}^0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \sigma^1 = \tilde{\sigma}^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \\ \sigma^2 = \tilde{\sigma}^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma^3 = \tilde{\sigma}^3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} , \end{aligned} \tag{7.24}$$

Note that all the matrices are hermitean, so $\tilde{\sigma}^m_{\alpha\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\gamma}}\sigma^{\dot{\gamma}\delta}\epsilon_{\delta\alpha}$.

The vector space of hermitean matrices is spanned by the 4 matrices $\tilde{\sigma}^m_{\alpha\dot{\beta}}$. Hermitean matrices are identified with vectors, and we can instead of v^m write $v_{\alpha\dot{\beta}}$, where

$$v_{\alpha\dot{\beta}} = v_m\tilde{\sigma}^m_{\alpha\dot{\beta}} = \begin{pmatrix} v^0 + v^1 & v^2 + iv^3 \\ v^2 - iv^3 & v^0 - v^1 \end{pmatrix}_{\alpha\dot{\beta}} . \tag{7.25}$$

The transformation of a hermitean matrix v is $\delta_\ell v = \ell v + v\ell^\dagger$, which is hermitean, showing that hermitean matrices form an $\mathfrak{sl}(2,\mathbb{C})$-module.

Starting from a Dirac spinor, it is possible to impose a "reality constraint", a *Majorana condition.* This condition, on a Dirac spinor Ψ of eq. (7.21), reads $\chi = \bar{\psi}$. The Majorana condition is not compatible with a chirality condition, which would set *e.g.* $\chi = 0$.

7.5. The Dirac equation

With the γ matrices at hand, it is possible to form an operator $\partial\!\!\!/ = \gamma^m\partial_m$ acting on a Dirac spinor, so that the result also is a Dirac spinor. The square of this operator is $\partial\!\!\!/^2 = \gamma^m\gamma^n\partial_m\partial_n = \Box$.

We can introduce a mass parameter (again, in classical field theory, really inverse length) and write the Dirac equation

$$\partial\!\!\!/\psi - m\psi = 0 . \tag{7.26}$$

We will soon make a Fourier expansion and solve the Dirac equation for plane wave solutions in flat space. However, we can immediately conclude that any solution will satisfy $0 = (\partial\!\!\!/ + m)(\partial\!\!\!/ - m)\psi =$

$(\Box - m^2)\psi$. Any Fourier mode, carrying a functional dependence $e^{ik_m x^m}$, must have $k^2 = -m^2$, so we will have propagation inside the light-cone[2] .

Remember that if we are in even dimensions (*e.g.* 4), the gamma matrices exchange the two chiralities. They must have the form

$$\gamma^m = \begin{pmatrix} 0 & \tilde{\sigma}^m \\ \sigma^m & 0 \end{pmatrix} , \tag{7.27}$$

where $\sigma^m\tilde{\sigma}^n + \sigma^n\tilde{\sigma}^m = 2\eta^{mn}I$, $\tilde{\sigma}^m\sigma^n + \tilde{\sigma}^n\sigma^m = 2\eta^{mn}I$. Writing out the two Weyl components of the Dirac equation gives

$$0 = \begin{pmatrix} -m & \tilde{\sigma}^m\partial_m \\ \sigma^m\partial_m & -m \end{pmatrix} \begin{pmatrix} \lambda \\ \chi \end{pmatrix} = \begin{pmatrix} \tilde{\sigma}^m\partial_m\chi - m\lambda \\ \sigma^m\partial_m\lambda - m\chi \end{pmatrix} . \tag{7.28}$$

The two chiralities mix. In order to have a spinor satisfying the Dirac equation with $m \neq 0$, it must be a Dirac spinor. A Weyl spinor cannot have $m \neq 0$.

We will now consider the Dirac equation specifically in $d = 4$. Using the formalism of Section 7.4, the Dirac equation takes the form

$$\begin{cases} \partial_{\alpha\dot{\alpha}}\chi^{\dot{\alpha}} - m\lambda_\alpha = 0 , \\ \partial^{\dot{\alpha}\alpha}\lambda_\alpha - m\chi^{\dot{\alpha}} = 0 . \end{cases} \tag{7.29}$$

As long as $m \neq 0$, we can solve for χ in the second equation, $\chi^{\dot{\alpha}} = m^{-1}\partial^{\dot{\alpha}\alpha}\lambda_\alpha$. Inserting this in the first equation gives $m^{-1}\partial_{\alpha\dot{\alpha}}\partial^{\dot{\alpha}\beta}\lambda_\beta - m\lambda_\alpha = 0$, *i.e.*, $(\Box - m^2)\lambda_\alpha$. In a Fourier expansion, this translates to $(k^2 + m^2)\tilde{\lambda}_\alpha(k) = 0$, so the Fourier coefficients can have support only on the "mass hyperboloid" $k^2 = -m^2$. The Fourier modes of the other component are given as $\tilde{\chi}^{\dot{\alpha}}(k) = im^{-1}k^{\dot{\alpha}\alpha}\tilde{\lambda}_\alpha(k)$. Note that the on-shell degrees of freedom of a Dirac spinor in $d = 4$ are 2 complex, or 4 real "polarisations". They correspond to the 2 spin states (up/down) of an electron together with 2 spin states for its anti-particle, the positron. By finding a relativistic equation for the electron field, Dirac inadvertently made the theoretic discovery of the positron.

We learn, from solving the Dirac equation in terms of plane waves, that the number of local degrees of freedom *on shell*, *i.e.*, parametrising the solutions, amounts to half a spinor. This is a general statement, and holds also in odd dimensions.

If $m = 0$, the two Weyl component do not mix in the Dirac equation, and it is consistent to consider only one of them, say $\chi^{\dot{\alpha}}$. The field equation is simply $\partial_{\alpha\dot{\alpha}}\chi^{\dot{\alpha}} = 0$. It clearly implies $k^2\tilde{\chi}^{\dot{\alpha}}(k) = 0$,

[2] However, there is still an issue, not dealt with here, that waves may propagate inside the past as well as the future light-cone. In quantum field theory this has to do with the fact that the Dirac equation describes anti-particles as well as particles. The sign of the mass term is not important, and can in fact be reversed if one lets $\gamma^m \mapsto -\gamma^m$.

so the Fourier components $\tilde{\chi}$ have support only on the light-cone. For some light-like wave vector, choose a Lorentz frame where $k^m = \omega(1,1,0,0)$. Then,

$$k_{\alpha\dot{\alpha}} = \begin{pmatrix} 2\omega & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\alpha}} , \tag{7.30}$$

and the field equation implies $\chi^{\dot{1}} = 0$. Also in the massless case, the number of on-shell degrees of freedom is half the number of off-shell degrees of freedom.

We would like to write down an action for a spinor field. It must have the form

$$S = \int d^4x \left(\overline{\Psi} \gamma^m \partial_m \Psi - m \overline{\Psi} \Psi \right) . \tag{7.31}$$

Here it is not yet clear what $\overline{\Psi}$ is. If Ψ carries a spinor index as Ψ_A, $\overline{\Psi}$ must behave as $\overline{\Psi}^A$. In $d = 4$,

$$\begin{aligned} \Psi_A &= \begin{pmatrix} \psi_\alpha \\ \chi^{\dot{\alpha}} \end{pmatrix} , \\ \overline{\Psi}^A &= \begin{pmatrix} -\bar{\chi}^\alpha \\ \bar{\psi}_{\dot{\alpha}} \end{pmatrix} , \end{aligned} \tag{7.32}$$

The reason for the relative minus sign in $\overline{\Psi}$ is not obvious. Even without it, we would have had a Dirac spinor with the correct index structure[3] . One may argue for it based on unitarity, but we will not do that. Instead we will note that the sign is necessary for the action to make sense.

Now, consider the action (7.31) with the spinor decomposition (7.32). It becomes

$$S = \int d^4x \left(\bar{\psi}_{\dot{\alpha}} \partial^{\dot{\alpha}\alpha} \psi_\alpha - \bar{\chi}^\alpha \partial_{\alpha\dot{\alpha}} \chi^{\dot{\alpha}} + m(-\bar{\chi}^\alpha \psi_\alpha + \bar{\psi}_{\dot{\alpha}} \chi^{\dot{\alpha}}) \right) . \tag{7.33}$$

The kinetic terms (the ones with derivatives) are real, given that the fields are fermionic (anticommuting Grassmann numbers), modulo boundary terms. This observation also uses the hermiticity of ∂. In fact, we cannot even write an action for a bosonic spinor field, since it would become a total derivative. We see that the kinetic terms work and are well defined for each chiral component. The mass term is also real (and non-vanishing) given the fermionic property of the fields, and we see (again) that a mass term relies on both chiralities being present.

A comment concerning spinors on other spaces that flat space, or in non-orthonormal coordinate systems. Spinors are fields in modules of (some real form of) orthogonal algebras. Unlike scalars, and also forms, they can not be interpreted as transforming under some non-trivial representation of $\mathfrak{gl}(d)$, which simply does not have spinor representations. If one wants to formulate the Dirac

[3] Sometimes you will meet this "conjugation" written as "$\overline{\Psi} = \Psi^\dagger \gamma^0$". This notation is unfortunate due to its apparent non-covariance.

equation with curvilinear coordinates, or on some curved space, one needs to devise a way to contract the vector index on the γ matrices γ^a, which is an $\mathfrak{so}(d)$ vector index, with the index on a derivative ∂_m, which is a index in coordinate basis, *i.e.* one that behaves as in eq. (4.9) with $M \in GL(d)$. The necessary solution is to use a *vielbein* instead of a metric to describe the geometry. The vielbein is a matrix $e_m{}^a$, such that the metric is $g_{mn} = e_m{}^a e_n{}^b \eta_{ab}$. Note that this relation is invariant under local Lorentz transformations, acting on the indices a, b. The spinors also transforms under local Lorentz transformations. We will not write down the action, or even the Dirac equation. The formalism involves a "spin connection", analogous to the ones introduced in Section 8, needed for the local Lorentz invariance.

Exercises

7.1. Consider the matrix $\gamma = \frac{1}{(2n)!}\epsilon^{M_1 \ldots M_{2n}}\gamma_{M_1} \ldots \gamma_{M_{2n}} = \gamma_1 \ldots \gamma_{2n}$. What is γ^2? How does it anticommute with the γ matrices? Use γ to form projection operators on the two chiralities. If ψ is a spinor of definite chirality, what is the chirality of $v^M \gamma_M \psi$?

7.2. Construct the spinor module in odd dimensions. Hint: exercise 7.1.

7.3. Show that $(\varrho(T_{MN}))_A{}^B = \frac{1}{4}(\gamma_{MN})_A{}^B$ are representation matrices for the (Dirac) spinor representation.

7.4. Investigate the signs in the Killing forms of the real Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{sl}(1,2)$, as well as $\mathfrak{so}(1,3)$, and show that the latter can not be a sum of any of the former. (This can be done directly using the matrix realisations, there is no need to go to the Chevalley–Serre basis.)

7.5. Show that the γ matrices of eqs. (7.22,7.24) satisfy the relations (7.23).

7.6. In $d = 4$, form $\gamma = \frac{1}{4!}\epsilon^{mnpq}\gamma_m\gamma_n\gamma_p\gamma_q$, and check its action on chiral spinors. (γ is often called γ_5.)

7.7. In $d = 4$, show that the Majorana condition is consistent with the action of the γ matrices, *i.e.*, that $v_m \gamma^m \psi$ is Majorana if ψ is Majorana.

7.8. The tensor product of two spinor representations. Consider a bi-spinor $M_A{}^B$, where A is a Dirac spinor index. Compare the number of independent matrices with the number of matrices of the type $\gamma^{M_1 \ldots M_p} = \gamma^{[M_1} \ldots \gamma^{M_p]}$ (possibly also $\gamma^{M_1 \ldots M_p}\gamma$?) and try to make a conjecture about the tensor product of two spinors.

7.9. Chiral spinors under (some real form of) $\mathfrak{so}(8)$ are 8-dimensional. Call the spinor modules $\mathbf{8}_s$ and $\mathbf{8}_c$. They are both self-conjugate. Determine which antisymmetric tensors appear in $\mathbf{8}_s \otimes \mathbf{8}_s$, $\mathbf{8}_c \otimes \mathbf{8}_c$ and $\mathbf{8}_s \otimes \mathbf{8}_c$, and in the first two tensor products which of them belong to the symmetric and anti-symmetric part of the tensor product.

7.10. Chiral spinors under (some real form of) $\mathfrak{so}(10)$ are 16-dimensional. Call the spinor modules $\mathbf{16}$ and $\overline{\mathbf{16}}$. They are conjugate to each other. Determine which antisymmetric tensors appear in $\mathbf{16} \otimes \mathbf{16}$, $\overline{\mathbf{16}} \otimes \overline{\mathbf{16}}$ and $\mathbf{16} \otimes \overline{\mathbf{16}}$, and in the first two tensor products which of them belong to the symmetric and anti-symmetric part of the tensor product.

7.11. Construct a vector v as the square of a $d=4$ bosonic spinor λ as $v_{\alpha\dot{\alpha}} = \lambda_\alpha \bar{\lambda}_{\dot{\alpha}}$. Show that $v^2 = 0$. Perform the corresponding construction in $d=3$. (It works also in $d=6$ and $d=10$.)

7.12. Consider the tensor product of an irreducible spinor representation and the vector representation. Which irreducible representations should be contained in the tensor product? It may be helpful to start in $d=3$.

8. Spin 1 fields and gauge symmetry

Electromagnetism is formulated in terms of a connection (in physics often called a gauge potential) A_m, which is suitably seen as a 1-form $A = dx^m A_m(x)$. The field strength is $F = dA$ and the field equations $d{\star}F = 0$ (in the absence of sources). The field strength is invariant under the gauge transformations $\delta_\lambda A = -d\lambda$, which is a *local symmetry* of the theory, a *gauge symmetry*. The term "local" refers to the fact that $\lambda(x)$ is an arbitrary function of the coordinates, so the gauge symmetry removes local degrees of freedom from A.

Suppose there is some complex field $\phi(x)$ which transforms under gauge transformations as

$$\phi \mapsto e^{iq\lambda}\phi \, , \tag{8.1}$$

which for infinitesimal λ reads $\delta_\lambda \phi = iq\lambda\phi$. The field ϕ could in principle transform in any Lorentz module. One says that ϕ carries electromagnetic charge q. An immediate problem with charged fields is that the derivative of the field does not transform well under gauge transformations:

$$\partial_m \phi \mapsto e^{iq\lambda}(\partial_m \phi + iq\partial_m \lambda \phi) \, . \tag{8.2}$$

This is remedied by replacing the derivative with a *covariant derivative*:

$$D_m \phi = (\partial_m + iqA_m)\phi \, . \tag{8.3}$$

Then,

$$D_m \phi \mapsto e^{iq\lambda} D_m \phi \, , \tag{8.4}$$

so the covariant derivative also carries the same charge.

The Maxwell theory is an example of a much wider class of models, described by *non-abelian gauge theory* or *Yang–Mills theory*. Maxwell theory is a gauge theory with the group $U(1)$, which is abelian. The complex number $e^{i\lambda}$ is a $U(1)$ group element, with group multiplication being complex multiplication.

Consider, in the same spirit, some field ϕ transforming in some module of a Lie group G. G should be taken as the compact form of some Lie group (for reasons that will become clear later). Let $\phi \mapsto g\phi$, $g \in G$. (Here, we use a notation where we don't distinguish between the group element and the representation matrix.) Then,

$$D_m\phi = (\partial_m + A_m)\phi \qquad (8.5)$$

transforms the same way as ϕ, $D_m\phi \mapsto gD_M\phi$, if

$$A_m \mapsto g\partial_m g^{-1} + gAg^{-1} \ . \qquad (8.6)$$

The components of the connection A take values in $\mathfrak{g}$, the Lie algebra of G. In the covariant derivative (8.5), it acts on the field with the appropriate representation matrices.

The second term in the transformation (8.6) of A is the homogeneous one, which one expects of any field in the adjoint module. The first, inhomogeneous term is what characterises the transformation of a connection.

If we consider an infinitesimal gauge transformations, we take $g = 1 + \Lambda$, with $\Lambda \in \mathfrak{g}$. Then, $g^{-1} \approx 1 - \Lambda$. The transformations become

$$\begin{aligned} \delta_\Lambda\phi &= \Lambda\phi \ , \\ \delta_\Lambda A_m &= -\partial_m\Lambda - [A_m, \Lambda] = -D_m\Lambda \ . \end{aligned} \qquad (8.7)$$

For an abelian group, like $U(1)$, $D_m\Lambda = \partial_m\lambda$. The gauge field itself carries no charge and does not self-interact. This is no longer true for non-abelian gauge groups. The gluons of the strong interaction, for example, carry charge (*i.e.*, transform in a non-trivial representation) under the gauge group $SU(3)$.

How is a field strength formed? It should have the form $F = dA + \ldots$, so it is also an element in the adjoint module of $\mathfrak{g}$. We should not demand that it is invariant, but that it transforms homogeneously, *i.e.*, $F \mapsto gFg^{-1}$. Consider the commutator of two covariant derivatives, $[D_m, D_n]$, acting on a field ϕ. In form language,

$$D^2\phi = (d + A\wedge)(d + A)\phi = d^2\phi + d(A\phi) + A \wedge d\phi + A \wedge A\phi = (dA + A \wedge A)\phi \ . \qquad (8.8)$$

Since we know that this transforms homogeneously, we can conclude that the combination

$$F = dA + A \wedge A \qquad (8.9)$$

transforms as $F \mapsto gFg^{-1}$. In components,

$$F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n] \ . \tag{8.10}$$

Again, note the last term, which is absent in the $U(1)$ theory.

What happens to the Bianchi identity? Take a covariant exterior derivative of F. This gives

$$DF = d(dA + A \wedge A) + A \wedge (dA + A \wedge A) - (dA + A \wedge A) \wedge A = 0$$

This calculation really depends on the Jacobi identity, since the components containing A^3 are $[A_{[m}, [A_n, A_{p]}]]$.

Natural equations of motion are

$$D \star F = J \ . \tag{8.11}$$

This is the equation of motion for Yang–Mills theory with gauge group G. The presence of a local symmetry, governed by a Lie group, dictates the form of the interactions in the theory.

The mathematically precise formulation of Yang–Mills theory is somewhat beyond our scope. It relies on the concept of fibre bundles, especially principal bundles, and sections of these. The gauge connection is a connection on a principal bundle. What can be said, without mathematical rigour, is that the connection is not a "field" in the ordinary sense. It needs not be globally defined. This is due to the gauge symmetry (8.6). Instead, it is patched together by gauge transformations on overlaps in a chart of the manifold where the theory is defined. This means that also the global properties of the gauge group may be relevant, not only the Lie algebra.

An action for Yang–Mills theory (including the $U(1)$ Maxwell theory) is

$$S = -\frac{1}{4g^2} \int d^d x \sqrt{|g|} g^{mp} g^{nq} \mathrm{tr} F_{mn} F_{pq} \ .$$

Here, "tr" is the Killing metric, suitably normalised. Note that one needs to choose a compact real form of the gauge group; otherwise different components would have kinetic terms with different signs.

Suppose there is some charged matter which couples "minimally" to the gauge field through the covariant derivative. A typical example would be a gauge group $SU(n)$ and fermions in the fundamental module. An action would then be of the form (we specialise to Minkowski space)

$$S = S = -\frac{1}{4g^2} \int d^4 x \mathrm{tr} F^{mn} F_{mn} + \int d^4 x \left(\overline{\Psi} \gamma^m D_m \Psi - m \overline{\Psi} \Psi \right) \ . \tag{8.12}$$

If we now consider the field equations for the connection, they become

$$D_n F^{nm} = J^m \ , \tag{8.13}$$

where, schematically, $J^m = \bar{\Psi}\gamma^m\Psi$, and the product of $\bar{\Psi}$ and Ψ is taken to be in the adjoint of the gauge group.

Hamiltonian; generator of gauge symmetry.

A little about gauge fixing. Fourier expansion of free theory; polarisations.

Conformal in $d = 4$...

8.1. Chern–Simons theory

In $d = 3$, consider the action

$$S = \frac{k}{4\pi}\int \mathrm{tr}(A\wedge dA + \frac{2}{3}A\wedge A\wedge A) \ . \tag{8.14}$$

Note that this action is written, on any manifold, without reference to a metric. This is one criterion for a theory to be *topological*. The constant k is dimensionless (using $\hbar = 1$). The equations of motion following from this action is

$$\frac{\delta S}{\delta A} = \frac{k}{2\pi}F = 0 \ . \tag{8.15}$$

The solutions are flat connections.

Chern–Simons theory, in spite of its apparent simplicity, has many applications in mathematics (topology, knot theory,...) and physics (solid state theory, topological phases,...).

<u>Exercises</u>

8.1. Show that the conservation law $DJ = 0$ is consistent with the equation of motion (8.11) for the Yang–Mills field.

8.2. Construct the Hamiltonian for electromagnetism coupled to a spinor field. Construct the conserved electric charge.

8.3. An exercise with gauge fields minimally coupled to matter in the form of some spinor field...

8.4. Consider the Maxwell field strength 2-form

$$F = \frac{1}{4\pi r^3}(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy) \ ,$$

which is well defined outside the origin. What is the corresponding B-field? Show that F satisfies Maxwell's equations for $r > 0$. Calculate the surface integral $\int_S F = \int_S \vec{B} \cdot d\vec{S}$, where S is a surface enclosing $r = 0$, and conclude that there is a magnetic monopole at $r = 0$. Find a 1-form A such that $dA = F$. Is it well defined everywhere outside the origin?

9. Noether's theorem—Symmetries and conservation laws

9.1. From symmetry to current

What do we mean by a symmetry of a field theory? Given an action $S[\Phi]$, it is invariant (up to boundary terms) under an infinitesimal transformation $\Phi^I \mapsto \Phi^I + \delta\Phi^I$, if $S[\Phi + \delta\Phi] = S[\Phi]$ for *all* Φ.

$$\delta S[\Phi, \delta\Phi] = S[\Phi + \delta\Phi] - S[\Phi] = \int d^d x \, \partial_m K^m \ . \tag{9.1}$$

Note that it is important that this holds even for Φ not satisfying the field equations. We assume that S is given as the integral of a Lagrangian density as in eq. (5.5). Then

$$\begin{aligned} \delta S[\Phi, \delta\Phi] &= \int d^d x \left(\partial_m \delta\Phi^I \frac{\partial \mathscr{L}}{\partial(\partial_m \Phi^I)} + \delta\Phi^I \frac{\partial \mathscr{L}}{\partial \Phi^I} \right) \\ &= \int d^d x \, \delta\Phi^I \left(-\partial_m \frac{\partial \mathscr{L}}{\partial(\partial_m \Phi^I)} + \frac{\partial \mathscr{L}}{\partial \Phi^I} \right) + \int d^d x \, \partial_m \left(\delta\Phi^I \frac{\partial \mathscr{L}}{\partial(\partial_m \Phi^I)} \right) \ . \end{aligned} \tag{9.2}$$

Now let us evaluate this variation around a solution $\bar{\Phi}$ to the field equations. Then we get

$$\delta S[\bar{\Phi}, \delta\Phi] = \int d^d x \, \partial_m \left(\delta\Phi^I \frac{\partial \mathscr{L}}{\partial(\partial_m \Phi^I)} \right) \ . \tag{9.3}$$

If we now subtract (9.1), evaluated on shell, from (9.3), we find that there is a conserved current J^m:

$$\partial_m J^m = 0 \ , \tag{9.4}$$

where

$$J^m = \delta\Phi^I \frac{\partial \mathscr{L}}{\partial(\partial_m \Phi^I)} - K^m \ . \tag{9.5}$$

This is Noether's (first) theorem.

Let us take translation symmetry as an example. (We suppress the index on the fields.) When $\delta x^m = v^m$, $\delta\Phi = v^m\partial_m\Phi$, and $\delta\mathscr{L} = v^m\partial_m\mathscr{L}$, which gives $K^m = v^m\mathscr{L}$. The conserved current is

$$J^m = v^n\partial_n\Phi\frac{\partial\mathscr{L}}{\partial(\partial_m\Phi)} - v^m\mathscr{L}\ . \tag{9.6}$$

Let us check that J is conserved:

$$\begin{aligned}\partial_m J^m &= v^n\partial_m\partial_n\Phi\frac{\partial\mathscr{L}}{\partial(\partial_m\Phi)} + v^n\partial_n\Phi\partial_m\frac{\partial\mathscr{L}}{\partial(\partial_m\Phi)} - v^m\partial_m\mathscr{L}\\ &= v^n\partial_m\partial_n\Phi\frac{\partial\mathscr{L}}{\partial(\partial_m\Phi)} + v^n\partial_n\Phi\partial_m\frac{\partial\mathscr{L}}{\partial(\partial_m\Phi)} - v^m\left(\partial_m\Phi\frac{\partial\mathscr{L}}{\partial\Phi} + \partial_m\partial_n\Phi\frac{\partial\mathscr{L}}{\partial(\partial_n\Phi)}\right)\\ &= v^m\partial_m\Phi\left(\partial_n\frac{\partial\mathscr{L}}{\partial(\partial_n\Phi)} - \frac{\partial\mathscr{L}}{\partial\Phi}\right)\ ,\end{aligned} \tag{9.7}$$

which vanishes on-shell (and consists exactly of the terms in eq. (9.2) left out to reach eq. (9.3) in the derivation of Noether's theorem).

Take a simple example, a massless scalar field with $\mathscr{L} = -\frac{1}{2}\partial_m\phi\partial^m\phi$. Then,

$$\begin{aligned}J^m &= -v^n\partial_n\phi\partial^m\phi + \tfrac{1}{2}v^m\partial_n\phi\partial^n\phi = -v_n T^{mn}\ ,\\ \text{where } T^{mn} &= \partial^m\phi\partial^n\phi - \tfrac{1}{2}\eta^{mn}\partial_p\phi\partial^p\phi\ .\end{aligned} \tag{9.8}$$

Then,

$$\partial_m J^m = -v^n\partial_m\partial_n\phi\partial^m\phi - v^n\partial_n\phi\Box\phi + v^m\partial_m\partial_n\phi\partial^n\phi = -v^n\partial_n\phi\Box\phi\ . \tag{9.9}$$

9.2. Conserved charges

Given a divergence-free current J^m: $\partial_m J^m = 0$, we can interpret this equation as a continuity equation by decomposing it into time and space components. With $J^0 = \rho$, $J^i = j^i$ it reads $\dot\rho + \nabla\cdot j = 0$, which is the continuity equation for a density and the corresponding current. Integrating over a spatial volume V and applying Gauss' law we have

$$\frac{d}{dt}\int_V d^{d-1}x\,\rho = -\int_{\partial V} dS\cdot j\ . \tag{9.10}$$

The left hand side is the time derivative of the charge in V, the right hand side is minus the outflow of charge. This is conservation. If we extend the volume V to all of space, the right hand side should vanish and we get the result

$$\dot Q = 0\ ,\quad Q = \int d^{d-1}x\,J^0\ . \tag{9.11}$$

9.3. Charge conservation in electromagnetism

The situation for charge conservation in Maxwell theory may seem somewhat confusing. The conservation of charge follows from the field equations $\partial_n F^{mn} = j^m$, immediately leading to the continuity equation for electric charge, $\partial_m j^m = 0$. However, the $U(1)$ gauge symmetry in electromagnetism is a local symmetry, not a global one, for which Noether's theorem applies. The *global* symmetry responsible for charge conservation is the global subgroup of gauge transformations, *i.e.*, a transformation as in eq. (8.1) with *constant* λ.

9.4. The stress-energy tensor

In the example with translation symmetry, there is a vector of transformations, so we have $J^m(v) = -v_n T^{mn}$. T is the stress-energy tensor. T^{00} is the energy density, and T^{0i} the (spatial) momentum density.

Comment on gravity etc.

9.5. Noether's theorem in the Hamiltonian formalism—Charges as generators

In the Hamiltonian formalism, consider transformations generated by a function Q on phase space (possibly with explicit time dependence). We will think of Q as formed as the integral of some local expression,

$$Q = \int d^{d-1}x\, \mathscr{Q}(\Phi(x), \nabla\Phi(x), \Pi(x); t)\ . \tag{9.12}$$

The transformations generated by Q are defined by the Poisson brackets

$$\begin{aligned}
\delta\Phi^I &= \{\Phi^I, Q\} = \frac{\delta Q}{\delta \Pi_I} = \frac{\partial \mathscr{Q}}{\partial \Pi_I}\ , \\
\delta\Pi_I &= \{\Pi_I, Q\} = -\frac{\delta Q}{\delta \Phi^I} = -\frac{\partial \mathscr{Q}}{\partial \Phi^I} + \partial_i \frac{\partial \mathscr{Q}}{\partial(\partial_i \Phi^I)}\ .
\end{aligned} \tag{9.13}$$

Then the transformation of the action becomes

$$\begin{aligned}
\delta S &= \delta \int d^d x \left(\dot\Phi^I \Pi_I - \mathscr{H} \right) \\
&= \int d^d x \left(\frac{d}{dt}(\delta\Phi^I \Pi_I) - \delta\Phi^I \dot\Pi_I + \dot\Phi^I \delta\Pi_I \right) - \int dt\, \{H, Q\} \\
&= \int d^d x \left(\frac{d}{dt}(\delta\Phi^I \Pi_I) - \frac{\delta Q}{\delta \Pi_I}\dot\Pi_I - \frac{\delta Q}{\delta \Phi^I}\dot\Phi^I \right) + \int dt\, \{Q, H\} \\
&= \int dt \left(\int d^{d-1}x\, \frac{d}{dt}(\delta\Phi^I \Pi_I) - \dot Q + \frac{\partial Q}{\partial t} + \{Q, H\} \right)\ .
\end{aligned} \tag{9.14}$$

If Q is conserved on-shell, *i.e.*, if the equations of motion will imply $\dot{Q} = 0$, we have, using eq. (5.19) (but not the equations of motion!), $\frac{\partial Q}{\partial t} + \{Q, H\} = 0$. Thus,

$$\delta S = \int dt \, \frac{d}{dt} \left(\int d^{d-1}x \, \delta\Phi^I \Pi_I - Q \right) .$$

So, the action is invariant (up to a total derivative) under transformations generated by a conserved charge. In addition, the right hand side can be identified with $\int d^dx \, \dot{K}^0$, and thus

$$J^0 = \delta\Phi^I \frac{\partial \mathscr{L}}{\partial \dot{\Phi}^I} - K^0 = \delta\Phi^I \Pi_I - (\delta\Phi^I \Pi_I - \mathscr{Q}) = \mathscr{Q} .$$

This shows that a conserved charge is the generator of the corresponding transformations.

The conserved charges will form a Lie algebra (see exercise 9.7).

Exercises

9.1. Consider the action for a particle particle with mass m in Minkowski space,

$$S = -m \int d\tau \sqrt{-\eta_{mn} \frac{dx^m}{d\tau} \frac{dx^n}{d\tau}} .$$

Derive the currents and conserved charges corresponding to the Poincaré invariance.

9.2. In the example above with the current corresponding to translational symmetry for a scalar field, construct the charges, and show that they generate translations under the Poisson bracket.

9.3. Suppose the particle in exercise 9.1 is electrically charged. How can the action be modified to incorporate the Maxwell field? (Hint: the "pullback of A to the world-line, $a = d\tau \frac{dx^m}{d\tau} A_m$ can be integrated.) Check that the Lorentz force arises in the equation of motion for the particle.

9.4. The stress-energy tensor is symmetric. Why?

9.5. Verify that eq. (5.16) is a special case of eq. (5.17).

9.6. Show that the Poisson bracket is a Lie bracket, *i.e.*, that it satisfies the Jacobi identity.

9.7. Show that if Q and Q' are conserved charges, also $\{Q, Q'\}$ is a conserved charge.

9.8. Consider a (Newtonian) particle moving in an isotropic harmonic potential in n space dimensions, so that $V(x) = \frac{1}{2} k x^i x^i$. Show that the global symmetry is $U(n)$ rather than the expected $O(n)$. Find the conserved charges. Determine their Poisson brackets with the Hamiltonian. (Hint: it may be useful to change phase space variables to ones that in the corresponding quantum mechanical problem would be creation and annihilation operators.) Investigate how the $U(n)$ symmetry acts on the phase space variables x^i and p_i.

9.9. In Maxwell theory in the Hamiltonian formalism, which is the generator of gauge symmetry?

9.10. Symmetries of the Kepler problem. Consider the motion of a Newtonian particle with mass m in the central potential $V(\vec{r}) = -\frac{k}{r}$. Show that the components of the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ fulfil $\{L_i, H\} = 0$, and are conserved charges. Which is the Lie algebra generated by these charges? Consider the Runge–Lenz vector

$$\vec{A} = \vec{p} \times \vec{L} - km\hat{r} \ .$$

The dimensionless vector $\frac{\vec{A}}{km}$ is the so called eccentricity vector. Show that $\vec{A}$ is conserved. It is convenient to rescale the Runge–Lenz vector to

$$\vec{B} = \frac{\vec{A}}{\sqrt{2m|E|}} \ ,$$

where E is the energy, for $E \neq 0$. Investigate the algebra of conserved charges under the Poisson bracket. It may be different in the cases $E < 0$, $E = 0$ and $E > 0$. Such “hidden symmetries” may be used to relate solutions to the equations of motion with the same energy to each other.

10. SUPERSYMMETRY?