Lösningar vektorfält

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Problemen tagna från häftet

"En första kurs i matematisk fysik"

andra upplagan

av Martin Cederwall

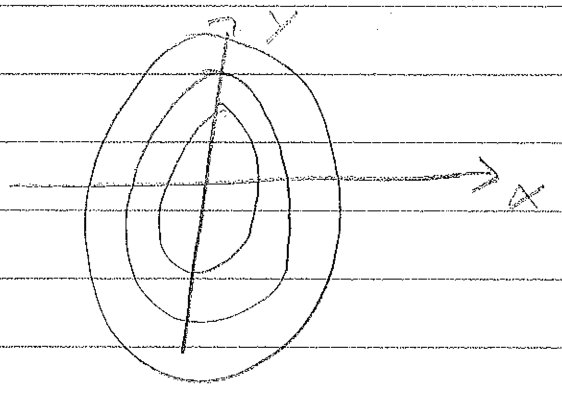
1.12

$$h(x,y) = \frac{k}{\left(\frac{x}{a}\right)^2 + \left(\frac{y}{\sqrt{2}a}\right)^2 + 1}$$

a) Högst i $(x,y) = (0,0)$

$$h(x,y) = c \Rightarrow k = \left[\left(\frac{x}{a}\right)^2 + \left(\frac{y}{\sqrt{2}a}\right)^2 + 1\right] c$$

$$\Rightarrow \frac{k-c}{c} = \left(\frac{x}{a}\right)^2 + \left(\frac{y}{\sqrt{2}a}\right)^2$$



b) $a = 1m,\ k = 10^3 m$

Borde egentligen undersöka $|\nabla h(x,y)|$ men givet i texten är att det är brantast på $y=0$. Räcker att titta på $\left|\frac{\partial}{\partial x} h(x,y)\right|$

$$\left|k \frac{\partial}{\partial x}\left(x^2 + \frac{y^2}{2} + 1\right)^{-1}\right| = +k\left(x^2 + \frac{y^2}{2} + 1\right)^{-2} \cdot 2x$$

Hitta max av $\left|\frac{\partial}{\partial x} h(x,y)\right|$

$$2k \frac{\partial}{\partial x}\left(\left(x^2+\frac{y^2}{2}+1\right)^{-2} \cdot x\right) = -8k\left(x^2+\frac{y^2}{2}+1\right)^{-3} x^2 + 2k\left(x^2+\frac{y^2}{2}+1\right)^{-2}$$

Sätt $=0$ och lös

$$\frac{-4x^2}{x^2+\frac{y^2}{2}+1} + 1 = 0$$

$$-4x^2 + x^2 + \frac{y^2}{2} + 1 = 0$$

$$x^2 - \frac{y^2}{6} - \frac{1}{3} = 0$$

$$x^2 = \frac{y^2}{6} + \frac{1}{3}$$

$$x = \pm\sqrt{\frac{y^2}{6} + \frac{1}{3}} \quad \text{och } y = 0$$

$\Rightarrow$ Brantast i punkterna $(\sqrt{1/3}, 0), (-\sqrt{1/3}, 0)$

$$h(1/3, 0) = \frac{1000}{(1/\sqrt{3})^2+1} = \frac{1000 \cdot 3}{1+3} = 750 \text{ m}$$

$$\text{Stigning } \frac{2000}{\sqrt{3}} \frac{1}{(1/3+1)^2} = \frac{2000}{\sqrt{3}} \frac{9}{16} \approx 650$$

1.13 a) $T = T_0(x^2 + 2yz - z^2)$

$\vec{x}_0 = (1,1,2)$

$\nabla T = T_0(2x, 2z, 2y-2z)$

$\nabla T(\vec{x}_0) = T_0(2, 4, -2)$

$N = \sqrt{2^2+4^2+2^2} = \sqrt{4+16+4} = \sqrt{24}$

I riktningen $\hat{n} = (2,4,-2)\frac{1}{\sqrt{24}}$

b) $\hat{n} = \frac{1}{3}(-2,2,1)$

$\hat{n}\cdot\nabla T = \frac{1}{3}(-2,2,1)\cdot(2,4,-2)T_0 =$

$= \frac{T_0}{3}(-4+8-2) = \frac{2T_0}{3}$

$$\boxed{\frac{2T_0}{3}\cdot v_0 = \frac{T_0}{s}}$$

$[T_0] = \frac{°C}{m^2}$, $[v_0] = \frac{m}{s}$ och $[x] = m$

$\Rightarrow \hat{n}\cdot\nabla T = \frac{°C}{m^2}\frac{m}{s}m \Rightarrow \frac{°C}{s}$

2.5

∇ i cylindriska koordinater

Kartesiska koord. skrivna i cyl.: $\vec{r} = (\rho\cos\theta, \rho\sin\theta, z)$

$$\frac{\partial \vec{r}}{\partial \rho} = (\cos\theta, \sin\theta, 0)$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-\rho\sin\theta, \rho\cos\theta, 0)$$

$$\frac{\partial \vec{r}}{\partial z} = (0, 0, 1)$$

$$h_\rho = 1$$
$$h_\theta = \rho$$
$$h_z = 1$$

och $\quad h_i \vec{e}_i \cdot \nabla\phi = \frac{\partial \phi}{\partial u_i} \Rightarrow \nabla\phi = \sum_i \vec{e}_i \frac{1}{h_i} \frac{\partial \phi}{\partial u_i}$

$$\Rightarrow \nabla\phi = \left(\frac{\partial \phi}{\partial \rho}, \frac{1}{\rho}\frac{\partial \phi}{\partial \theta}, \frac{\partial \phi}{\partial z} \right)$$

2.22

$$x = uv \qquad u \geq 0$$
$$y = u^2 + \lambda v^2$$
$$z = z$$

$$\frac{d\vec{r}}{\partial u} = (v, 2u, 0)$$

$$\frac{d\vec{r}}{dv} = (u, 2\lambda v, 0)$$

$$\frac{d\vec{r}}{dz} = (0, 0, 1)$$

$$\frac{d\vec{r}}{du} \cdot \frac{d\vec{r}}{dv} = uv + 4\lambda uv \overset{!}{=} 0$$

$$\Rightarrow \lambda = -1/4$$

$$\begin{cases} h_u = \sqrt{4u^2 + v^2} \\ h_v = \sqrt{u^2 + \frac{v^2}{4}} \\ h_z = 1 \end{cases}$$

$$\frac{d\vec{r}}{dz} \times \frac{d\vec{r}}{du} = \begin{matrix} x & y & z \\ 0 & 0 & 1 \\ v & 2u & 0 \end{matrix} = (-2u, v, 0) =$$

$$= -2\frac{d\vec{r}}{dv}$$ Vänsterorienterat

2.8

$$F = \frac{m}{4\pi r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$\frac{dr(\tau)}{d\tau} = \frac{m}{4\pi r^3}\,2\cos\theta \cdot C$$

$$\frac{d\theta(\tau)}{d\tau} = C\,\frac{m}{4\pi r^3}\,\frac{1}{r}\sin\theta$$

h_θ

$$\Rightarrow \frac{dr}{d\theta} = \frac{\frac{dr}{d\tau}}{\frac{d\theta}{d\tau}} = \frac{2r\cos\theta}{\sin\theta}$$

$$\Rightarrow \frac{dr}{r} = \frac{2\,d(\sin\theta)}{\sin\theta}$$

$$\Rightarrow d(\ln r) = d(\ln(\sin^2\theta))$$

Integrera ger $\ln r = \ln\sin^2\theta + \ln A$

$$\Rightarrow r(\theta) = A\sin^2\theta$$

Givet: $2 = A\sin^2\pi/4 = A/2 \Rightarrow A = 4$

$$\Rightarrow \begin{cases} r(\theta) = 4\sin^2\theta \\ \theta = \pi/6 \end{cases}$$

2.11 $\phi(r,\theta,\varphi) = r^2\cos 2\theta - 2ar\sin\theta\cos\varphi$

Hitta nivåytor

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$$

$$\Rightarrow \quad \phi = r^2(2\cos^2\theta - 1) - 2ar\sin\theta\cos\varphi$$

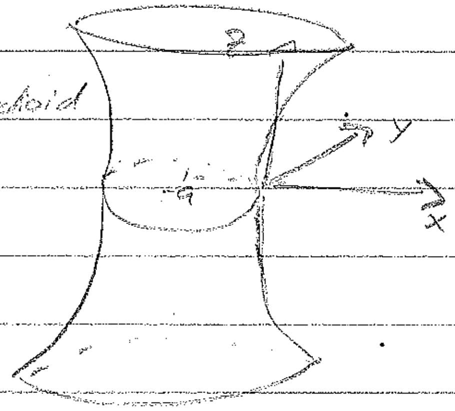
$$\phi = 2z^2 - x^2 - y^2 - z^2 - 2ax$$

$$z^2 - y^2 - x^2 - 2ax = c$$

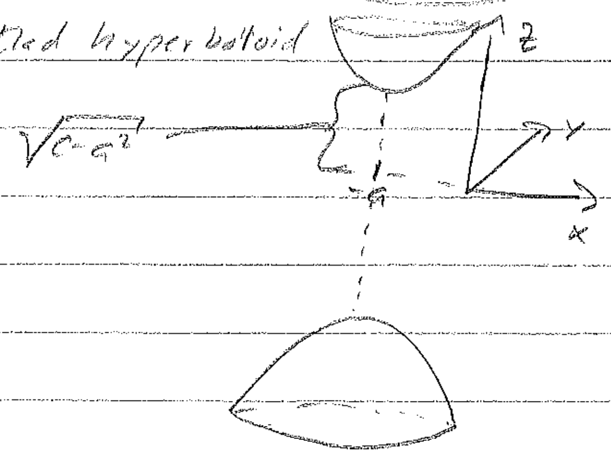
$$z^2 - y^2 - (x+a)^2 = c - a^2$$

$$(x+a)^2 + y^2 - z^2 = a^2 - c$$

$a^2 - c > 0$ Enmantlad hyperboloid



$a^2 - c < 0$ Tvåmantlad hyperboloid



2.11 forts.

Riktningsderivatans def: $\hat{n} \cdot \nabla\phi$

$$\Rightarrow \quad \hat{\theta}\cdot\nabla\phi = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -2r\sin 2\theta - 2a\cos\theta\cos\varphi =$$

$$= -2(r\sin 2\theta + a\cos\theta\cos\varphi)$$

$$P: (a, \pi/4, \pi) \Rightarrow -2\left(a\sin 2\frac{\pi}{4} + a\cos\frac{\pi}{4}\cos\pi\right) =$$

$$= -2\left(a\sin\pi/2 - a\frac{1}{\sqrt{2}}\right) = -2\left(a - a\frac{1}{\sqrt{2}}\right) = -\sqrt{2}(\sqrt{2}-1)a$$

$$= (\sqrt{2}-2)a$$

Fältlinjer enklast i kartesiska coord.

Från ovan $\phi(x,y,z) = z^2 - y^2 - (x+a)^2$

$$\nabla\phi = 2z - 2y - 2(x+a)$$

$c = -1/2$

$$\frac{dx}{d\tau} = +\cancel{2}(x+a) \Rightarrow x(\tau) = e^{\tau}(x_0+a) - a$$

$$\frac{dy}{d\tau} = +\cancel{2}y \qquad y(\tau) = e^{\tau}y_0$$

$$\frac{dz}{d\tau} = -\cancel{2}z \qquad z(\tau) = e^{-\tau}z_0$$

2.14

$$u = r(1-\cos\theta)$$
$$v = r(1+\cos\theta)$$
$$w = \varphi$$

– Invertera sambandet ovan

$$r = \frac{u}{1-\cos\theta} \Rightarrow v = \frac{u}{1-\cos\theta}(1+\cos\theta)$$

$$\Rightarrow v(1-\cos\theta) = u(1+\cos\theta)$$
$$v-u = (v+u)\cos\theta$$
$$\boxed{\cos\theta = \frac{v-u}{v+u}}$$

$$\Rightarrow r = \frac{u}{1-\frac{v-u}{v+u}} = \frac{(v+u)u}{\cancel{v}+u-\cancel{v}+u} = \frac{v+u}{2}$$

$$\boxed{r = \frac{v+u}{2}}$$

$$\sin^2\theta = 1 - \frac{(v-u)^2}{(v+u)^2} = \frac{v^2+u^2+2uv-v^2-u^2+2uv}{(v+u)^2}$$

$$\boxed{\sin\theta = \frac{2\sqrt{uv}}{u+v}}$$

Ortsvektorn för sfäriska koord $\vec{r} = (r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)$

blir då $\vec{r} = \left(\frac{\cancel{v+u}}{2}\frac{2\sqrt{uv}}{\cancel{u+v}}\cos w, \frac{\cancel{v+u}}{2}\frac{2\sqrt{uv}}{\cancel{u+v}}\sin w, \frac{\cancel{v+u}}{2}\frac{v-u}{\cancel{v+u}}\right)$

$$= \left(\sqrt{uv}\cos w, \sqrt{uv}\sin w, \frac{v-u}{2}\right)$$

$$\frac{\partial \vec{r}}{\partial u} = \left(\frac{1}{2}\sqrt{\frac{v}{u}}\cos w, \frac{1}{2}\sqrt{\frac{v}{u}}\sin w, -\frac{1}{2}\right)$$

$$h_u = \sqrt{\frac{1}{4}\frac{v}{u}\cos^2 w + \frac{1}{4}\frac{v}{u}\sin^2 w + \frac{1}{4}} =$$

$$= \sqrt{\frac{1}{4}\frac{v}{u} + \frac{1}{4}} = \frac{1}{2}\sqrt{\frac{v+u}{u}}$$

$$\frac{\partial \vec{r}}{\partial v} = \left(\frac{1}{2}\sqrt{\frac{u}{v}}\cos w, \frac{1}{2}\sqrt{\frac{u}{v}}\sin w, \frac{1}{2}\right)$$

Ortogonalitet $\frac{\partial \vec{r}}{\partial u} \cdot \frac{\partial \vec{r}}{\partial v} = 0$ osv teste $\underline{\text{en}}$

$$\left(\frac{1}{2}\sqrt{\frac{v}{u}}\cos w, \frac{1}{2}\sqrt{\frac{v}{u}}\sin w, -\frac{1}{2}\right)\cdot\left(\frac{1}{2}\sqrt{\frac{u}{v}}\cos w, \frac{1}{2}\sqrt{\frac{u}{v}}\sin w, \frac{1}{2}\right) =$$

$$= \frac{1}{4}\cos^2 w + \frac{1}{4}\sin^2 w - \frac{1}{4} = \underline{0}$$

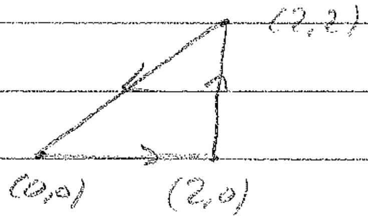
$$\nabla = \sum_i \hat{e}_i \frac{1}{h_i}\frac{\partial}{\partial u_i} = \hat{u}\, 2\sqrt{\frac{u}{u+v}}\frac{\partial}{\partial u} \ldots$$

$$\underline{\vec{r} = (u h_u \hat{u}, v h_v \hat{v}, h_w w \hat{w})} =$$

$$= \left(\frac{1}{2}\sqrt{v+u}\sqrt{u}\,\hat{u}, \ldots\right)$$

$$\int fg = Fg| - \int F g'$$

3.1 $\vec{F} = (ye^x, xe^y)$



$$\int_C = \int_0^2 dx\Big|_{y=0} + \int_0^2 dy\Big|_{x=2} + \int_2^0 \substack{x=t\\y=t}\ \frac{d\vec{r}}{dt}\,dt$$

$$\int_0^2 0\,dx = 0$$

$$\int_0^2 2e^y\,dy = 2\Big[e^y\Big]_0^2 = 2(e^2-1)$$

$$\int_2^0 (te^t, te^t)\cdot\frac{d\vec{r}}{dt}\,dt = \int_2^0 (te^t, te^t)\cdot(1,1)\,dt =$$

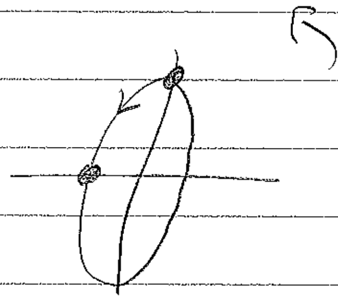
$$= 2\int_2^0 te^t\,dt = 2\Big[e^t t\Big]_2^0 - 2\int_2^0 e^t\,dt =$$

$$= -2e^2\cdot 2 - 2\Big[e^t\Big]_2^0 = -4e^2 - 2e^0 + 2e^2 =$$

$$= -2 - 2e^2$$

$$2(e^2-1) - 2 - 2e^2 = -4$$

3.3 $\vec{F} = (y^3, x^3)$



Ellipsen $x^2 + y^2/4 = 1$

$x = \cos\tau$

$y = 2\sin\tau$

$C: \frac{\pi}{2} \xrightarrow{\tau} 2\pi$

$$\int_C d\tau\, \vec{F}\cdot\frac{d\vec{r}}{d\tau} = \int_{\pi/2}^{\pi} d\tau\, (8\sin^3\tau, \cos^3\tau)\cdot$$

$$\cdot(-\sin\tau, 2\cos\tau) =$$

$$= \int_{\pi/2}^{2\pi} d\tau\,(-8\sin^4\tau + 2\cos^4\tau) =$$

$$= \int_{\pi/2}^{2\pi} d\tau\,(-8\sin^2\tau(1-\cos^2\tau) + 2\cos^2\tau(1-\sin^2\tau)) =$$

$$= \int_{\pi/2}^{2\pi} d\tau\,(-8\sin^2\tau + \cancel{8}^{6}\sin^2\tau\cos^2\tau + 2\cos^2\tau - \cancel{2\sin^2\tau\cos^2\tau}) =$$

$$= \int_{\pi/2}^{2\pi} d\tau\,(-8\sin^2\tau + 2\cos^2\tau + 6\sin^2\tau\cos^2\tau) =$$

$$= \int_{\pi/2}^{2\pi} d\tau\,(-8 + 10\cos^2\tau + 6\sin^2\tau\cos^2\tau) =$$

$$= \int_{\pi/2}^{2\pi} d\sigma \left(-8 + 5(1+\cos 2\sigma) + 6\sin^2\sigma \cos^2\sigma \right) =$$

...

3.12

$$\int_S \left(\frac{A}{r^2} \hat{r} + B\hat{z} \right) \cdot d\vec{S}$$

S är sfären $|\vec{r}| = a$

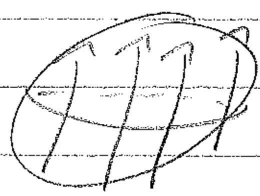
För sfären är $d\vec{S} = \hat{r}$

så första delen ger

$$\int_S \frac{A}{a^2} \, dS \quad \text{över sfärens yta}$$

$$\Rightarrow \quad 4\pi A$$

Andra termen är rotationsantisymmetrisk så integralen över övre halvsfären tar ut den nedre halvsfären.



$4\pi A$

3.4

$d\vec{r} \times \vec{B}$ $\quad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ dr_x & dr_y & dr_z \\ B_x & B_y & B_z \end{vmatrix}$

$\hat{x}: \quad (B_z\, dr_y - B_y\, dr_z)$

$$\int_0^{2\pi} \left(B_z \frac{dr_y}{de} de - B_y \frac{dr_z}{de} de \right) =$$

$$= \int_0^{2\pi} \left(\frac{B_0}{\cancel{a^2}} \frac{1}{\cancel{a^2}} \cancel{a^2} \frac{\cancel{a}e}{\pi} a\cos e - \frac{B_0}{\cancel{a}} \frac{1}{\cancel{a^2}} \cancel{a^3}(\cos e + \sin e) \frac{a}{\pi} \right) de =$$

$$= \int_0^{2\pi} (\cos e \; e \cos e + \sin e) = 0$$

$\hat{y} \quad dr_z B_x - dr_x B_z = \frac{B_0}{a^3} \int_0^{2\pi} \left((a^3(\cancel{\cos e - \sin e}) \frac{dr_z}{de} de \right.$

$$\left. - \frac{a^3 e}{\pi} \frac{dr_x}{de} de \right) =$$

$$= \frac{B_0 a}{\pi} \int_0^{2\pi} e \sin e \, de = \frac{B_0 a}{\pi} \left(\left[-\cos(e) e \right]_0^{2\pi} - \int_0^{2\pi} \cancel{\cos e}\, de \right)$$

$$= -\frac{B_0 a 2}{\cancel{\pi}} \hat{y}$$

Samme for $\hat{z}$-led.

4.7 $\vec{F}(r,\theta,\varphi) = F_0\left(\frac{a}{r}\right)^3\left(2\sin\theta\cos\varphi\,\hat{r} - \cos\theta\cos\varphi\,\hat{\theta} + \sin\varphi\,\hat{\varphi}\right)$

ekv. 4.40 $\Rightarrow$ $\nabla\cdot\vec{F} = \frac{1}{h_r h_\theta h_\varphi}\sum_i \frac{\partial}{\partial u_i}\left(\frac{h_r h_\theta h_\varphi F_i}{h_i}\right)$

sfæriske koordinater $h_r = 1$, $h_\theta = r$, $h_\varphi = r\sin\theta$

$$\nabla\cdot\vec{F} = \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(\frac{r^2\sin\theta\, 2\sin\theta\cos\varphi}{r^3}\right)\right.$$

$$-\frac{\partial}{\partial\theta}\left(\frac{r^2\sin\theta}{r^4}\cos\theta\cos\varphi\right)$$

$$\left.+\frac{\partial}{\partial\varphi}\left(\frac{r^2\sin\theta}{r\sin\theta\, r^3}\sin\varphi\right)\right) =$$

$$= \frac{1}{r^2\sin\theta}\left(-\cancel{\frac{2\sin^2\theta\cos\varphi}{r^2}} - \frac{\cos\varphi}{r^2}(\cos^2\theta - \sin^2\theta)\right.$$

$1-\sin^2\theta$

$$\left.+\cancel{\frac{\cos\varphi}{r^2}}\right) = \underline{0}$$

4.13

$x = a\cosh u \cos v$

$y = a \sinh u \sin v$

$z = w$

$\Delta\phi = 0$ Hitta lösning sammanberoende bara på u

$\Delta\phi(u) = 0$

Enl. ekv 4.43 $\Delta\phi = \frac{1}{h_1h_2h_3}\sum_i \frac{\partial}{\partial u_i}\left(\frac{h_1h_2h_3}{h_i^2}\frac{\partial\phi}{\partial u_i}\right)$

$\phi = \phi(u)$

$\Rightarrow \Delta\phi(u) = \frac{1}{h_1h_2h_3}\frac{\partial}{\partial u}\left(\frac{h_1h_2h_3}{h_1^2}\frac{\partial\phi}{\partial u}\right)$

$\vec{r} = (a\cosh u \cos v, a\sinh u \sin v, w)$

$\frac{\partial\vec{r}}{\partial u} = (a\sinh u \cos v, a\cosh u \sin v, 0)$

$\left|\frac{\partial r}{\partial u}\right| = \sqrt{a^2\sinh^2 u\cos^2 v + a^2\cosh^2 u \sin^2 v}$

$\cosh^2 u - \sinh^2 u = 1$

$\Rightarrow \left|\frac{\partial r}{\partial u}\right| = a\sqrt{\cos^2 v(\cosh^2 u - 1) + \cosh^2 u \sin^2 v} =$

$= a\sqrt{\cosh^2 u - \cos^2 v}$

Forts. längre fram

4.13 forts

$$\frac{\partial \vec{r}}{\partial v} = (-a\cosh u \sin v, a\sinh u \cos v, 0)$$

$$\left|\frac{\partial \vec{r}}{\partial v}\right| = a\sqrt{\cosh^2 u \sin^2 v + \sinh^2 u \cos^2 v} =$$

$$= a\sqrt{\cosh^2 u(1-\cos^2 v) + (\cosh^2 u - 1)\cos^2 v} =$$

$$= a\sqrt{\cosh^2 u - \cos^2 v}$$

$$\frac{\partial \vec{r}}{\partial w} = (0,0,1) \qquad \left|\frac{\partial \vec{r}}{\partial w}\right| = 1$$

$$\Delta \phi(u) = \frac{1}{\cosh^2 u - \cos^2 v}\frac{\partial}{\partial u}\left(\frac{\partial \phi}{\partial u}\right) =$$

$$= \frac{1}{\cosh^2 u - \cos^2 v}\frac{\partial^2}{\partial u^2}\phi(u)$$

$$\phi(u) = Au + C$$

$$\frac{a^2}{25}\cosh^2 u \cos^2 v + \frac{a^2}{9}\sinh^2 u \sin^2 v = \frac{a^2}{16}$$

$\phi(u)$ oberoende av v. Sätt $v = 0$

$$\frac{1}{25}\cosh^2 u = \frac{1}{16}$$

andra ellipsen $\Rightarrow$ $\frac{1}{25}\cosh^2 u = \frac{1}{9}$

$$\begin{cases} \cosh u_7 = \frac{5}{4} & \text{ellips } 7 \\ \cosh u_{10} = \frac{5}{3} & \text{ellips } \frac{?}{?} \end{cases}$$

$$e^{u} + e^{-u} = \frac{5}{2} \qquad 7$$

$$\frac{e^{2u}+1}{e^{u}} = 5/2$$

$$e^{2u}+1 = \frac{5}{2}e^{u}$$

$$(e^{u})^2 - \frac{5}{2}e^{u} = -1$$

$$\left(e^{u} - \frac{5}{4}\right)^2 = -1 + \frac{25}{16}$$

$$e^{u} = +\frac{5}{4} \pm \sqrt{\frac{9}{16}}$$

$$e^{u} = \frac{5}{4} \pm \frac{3}{4} = 1/2,\ 2$$

$$\Rightarrow \quad u = \begin{cases} \ln 1/2 \\ \ln 2 \end{cases} \quad \underline{\text{men}} \quad u \geq 0$$

$$\Rightarrow u = \ln 2$$

$$\Rightarrow \quad \left.\begin{aligned} A\ln 2 + C &= 0 \\ A\ln 3 + C &= 2 \end{aligned}\right\} \quad A\ln 2 = -C$$

$$A\ln 3 - A\ln 2 = 2$$

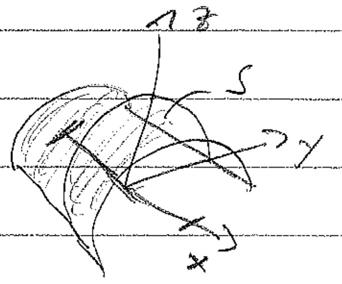
$$A = \frac{2}{\ln 3 - \ln 2}$$

$$C = -\frac{2\ln 2}{\ln 3 - \ln 2}$$

$$\phi(u) = \frac{2}{\ln 3 - \ln 2}(u - \ln 2)$$

4.17 $\quad S: \; y^2+z^2=1 \qquad -1 \le x \le 1, \; z \ge 0$

$$\vec{F} = (x, x^2yz^2, x^2y^2z)$$



Halvcylinder

Gauss $\quad \underbrace{\int_S \vec{F}\cdot d\vec{S} + \int_{sidor} + \int_{botten}}_{randen} = \int_V \nabla\cdot\vec{F}\, dV$

$$\int \frac{\partial f}{\partial x}\, dx = [f]_{x=x_1}^{x=x_2}$$

$$\int \nabla\cdot f\, dV = \int_{\partial S} f$$

Lett? $\quad \int_V \nabla\cdot\vec{F}\, dV = \int_V (1 + x^2z^2 + x^2y^2)\, dV =$

$=$ Cylindriska $= \int_V (1 + x^2\rho^2(\cos^2 e + \sin^2 e))\, dV =$

$z = \rho\cos e$

$y = \rho\sin e$

$x = x$

$$= \int_V (1 + x^2\rho^2)\, dV = \int_{-1}^{1} dx \int_0^1 d\rho \int_0^{\pi} de\, (1 + x^2\rho^2)\rho =$$

$$= \pi \int_{-1}^{1} dx \left[\frac{\rho^2}{2} + \frac{x^2\rho^4}{4}\right]_0^1 = \pi \int_{-1}^{1} dx \left(\frac{1}{2} + \frac{x^2}{4}\right) =$$

$$= \pi \left[\frac{x}{2} + \frac{x^3}{12}\right]_{-1}^{1} = 2\pi\left(\frac{1}{2} + \frac{1}{12}\right) = \frac{2\pi\cdot 7}{6\cdot 2} = \boxed{\frac{7\pi}{6}}$$

Ok

$\longrightarrow$

Siderna & botten ?

höger platta: $\hat{n} = (1,0,0)$

$$\int_S \vec{F}\cdot d\vec{S} = \int_S x\, dS$$ och x konstant på plattan

$$\Rightarrow \boxed{\frac{\pi}{2}}$$

vänster platta $\hat{n} = (-1,0,0)$

$$\int_S \vec{F}\cdot d\vec{S} = \int_S -x\, dS \overset{(x=-1)}{=} \frac{\pi}{2}$$

botten $\hat{n} = (0,0,-1)$

$$\int_S \vec{F}\cdot d\vec{S} = \int_S -x^2y^2z\, dS \quad \& \quad z=0 \Rightarrow 0$$

$$\Rightarrow \int_S \vec{F}\cdot d\vec{S} = \overbrace{\int_V \nabla\cdot\vec{F}\, dV}^{7\pi/6} - \overbrace{\int_{sider}}^{\pi} - \overbrace{\int_{botten}}^{0}.$$

$$\Rightarrow \int_S \vec{F}\cdot d\vec{S} = \frac{7\pi}{6} - \pi = \boxed{\frac{\pi}{6}}$$

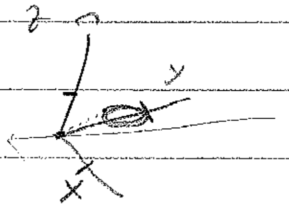
4.18 $\vec{F} = F_0\left(\left(\frac{\pi y}{a} + \sin\frac{\pi z}{a}\right)\hat{x} + \frac{x}{a}\hat{y} + \frac{\pi x}{a}\cos\frac{\pi z}{a}\,\hat{z} \right)$

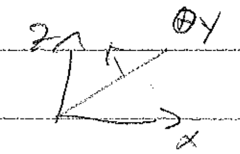
$$\int_C \vec{F}\cdot d\vec{r} = \int_S \nabla\times\vec{F}\, d\vec{S} \quad \text{Stoke}$$

Cirkel $x^2+y^2+z^2=a^2$

$x=z$

$2x^2+y^2=a^2$ ellips ? (projiseret på x-y planet jo)





$$\hat{n} = \frac{1}{\sqrt{2}}(1,0,-1)$$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix} = \left(\partial_y F_z - \partial_z F_y \,,\ ? \,,\ \partial_x F_y - \partial_y F_x\right)$$

$$= F_0\left(0 \,,\ ? \,,\ -\frac{1}{a} + \frac{\pi}{a} \right) = \frac{F_0}{a}\left(0 \,,\ ? \,,\ -1+\pi \right)$$

$$\int_S \nabla\times\vec{F}\cdot d\vec{S} = \frac{F_0}{\sqrt{2}a}\int_S \left(+\pi - 1 \right) dS$$

cirkel med radie a

$$\boxed{\frac{F_0}{a\sqrt{2}}(\pi-1)\,\pi a^2 = \frac{F_0 \pi a}{\sqrt{2}}(\pi-1)}$$

4.21

Gauss: $\int_V \nabla \cdot \vec{F}\, dV = \int_{\partial V} \vec{F} \cdot dS$

Velj $\vec{F} = (x, y, z)$

$$\Rightarrow \int_V 3\, dV = \int_{\partial V} (x, y, z) \cdot dS$$

$$\Rightarrow V = \frac{1}{3} \int_{\partial V} \vec{r} \cdot dS$$

Størens volum $\frac{4\pi r^3}{3}$

$$\frac{1}{3} \int_{\partial V} \vec{r} \cdot dS = \frac{1}{3} \int_{\partial V} \vec{r} \cdot \hat{r}\, r^2 \sin\theta\, d\theta\, d\varphi$$

$$\Rightarrow \frac{1}{3} r^3 \int_0^{\pi} d\theta \int_0^{2\pi} d\varphi \sin\theta = -\frac{2\pi r^3}{3} \left[\cos\theta\right]_0^{\pi} =$$

$$= -\frac{2\pi r^3}{3} \left[-1 - 1\right] = \frac{4\pi r^3}{3}$$

$\mathcal{S}.10$ Devise $\oint_{\partial S} d\vec{r} \times \vec{V} = \int_S (d\vec{S} \times \nabla) \times \vec{V}$

Stokes $\oint_{\partial S} \vec{F} \cdot d\vec{r} = \int_S (\nabla \times \vec{F}) \cdot d\vec{S}$

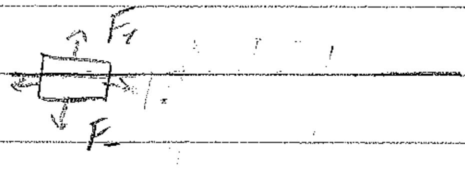
$$\int_S \varepsilon_{ijk} \partial_j F_k \, dS_i = \oint_{\partial S} F_i \, dr_i$$

$$\int_S \varepsilon_{ijk} \, dS_j \partial_k \left(\varepsilon_{lim} V_m \right) = -\int_S \varepsilon_{jik} \, dS_j \partial_k \left(\varepsilon_{lim} V_m \right) =$$

$$= \int_S \varepsilon_{jki} \, dS_j \partial_k \left(\varepsilon_{lim} V_m \right) \overset{\text{Stokes}}{=} \oint_{\partial S} \varepsilon_{lim} V_m \, dr_i =$$

$$= \oint_{\partial S} d\vec{r} \times \vec{V}$$

6.1



ytladningstetthet

Gauss: $\oint_S \vec{F}\cdot d\vec{S} = \int_V \nabla\cdot\vec{F}\,dV$

$$\int_V \nabla\cdot\vec{F}\,dV = \sigma\cdot A \quad \text{innesluttet ladning}$$

$$(F_1 A - F_2 A) = \sigma A$$

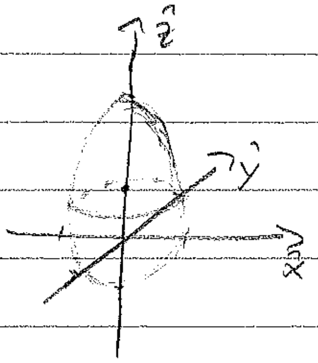
$$\hat{n}(F_1 - F_2) = \sigma$$

6.2 $\vec{F} = \overbrace{4x\hat{y}}^{\vec{F}_I} + \overbrace{\frac{\vec{r}-\hat{z}}{|\vec{r}-\hat{z}|^3}}^{\vec{F}_{II}}$

$S: 4x^2+4y^2+(z-1)^2=1$

Beräkna $\oint_S \vec{F}\times d\vec{S}$

S



centrerad kring $z=1$

$\boxed{\text{Gauss-analog sats} \quad \int_V \nabla\times\vec{F}\,dV = \oint_{\partial V} d\vec{S}\times\vec{F}}$

$\nabla\times\vec{F}_I = \cancel{\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ 0 & 4x & 0 \end{vmatrix}} = \hat{z}\,4$

och volymen på ellipsoiden är $\frac{4\pi}{3}\frac{1}{2}\frac{1}{2}\cdot 1 = \frac{\pi}{3}$

och minus från byte på av ordning på kryssprodukten

$\Rightarrow \oint_S \vec{F}_I\times dS = -\frac{4\pi}{3}\hat{z}$

$\oint_S \vec{F}_{II}\times d\vec{S} = \oint_S \left(\frac{\vec{r}-\hat{z}}{|\vec{r}-\hat{z}|^3}\right)\times d\vec{S} = \left\{ \vec{r}\,' = \vec{r}-\hat{z} \right\} =$

$= \oint_{S'} \frac{\vec{r}\,'}{|\vec{r}\,'|^3}\times d\vec{S}\,' = \oint_{S'} \frac{\vec{r}\,'}{|\vec{r}\,'|^2}\times\hat{r}'\,dS' = 0$

$\Rightarrow \oint_S \vec{F}\times d\vec{S} = \underline{-\frac{4\pi}{3}\hat{z}}$

6.3

$$\int_S \vec{F}\cdot d\vec{S} \qquad \vec{F} = \frac{q}{4\pi}\frac{\vec{r}-\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}$$

$$\vec{r}_0 = \frac{3a}{5}(\hat{x}+\hat{y}-\hat{z})$$

S är sfären $|r| = a$

$$|\vec{r}_0| = \frac{3a}{5}\cdot\sqrt{3} = \frac{3\sqrt{3}a}{5} = \sqrt{\frac{27}{25}}\,a$$

Singularitet utanför S.

Gauss sats gäller omodifierad inuti S

$$\int_S \vec{F}\cdot dS = \frac{q}{4\pi}\int_S\left(\frac{\vec{r}}{|\vec{r}-\vec{r}_0|^3} - \frac{\vec{r}_0}{|\vec{r}-\vec{r}_0|^3}\right)\cdot d\vec{S} =$$

$$= \left\{\vec{r}' = \vec{r}-\vec{r}_0\right\} = \frac{q}{4\pi}\int_{S'}\frac{\vec{r}'}{|\vec{r}'|^3}\cdot d\vec{S}' =$$

$$= \frac{q}{4\pi}\int_{S'}\frac{\hat{r}'}{|\vec{r}'|^2}\cdot d\vec{S}'$$

och S' är en yta som inte omsluter origo

$$\Rightarrow \quad \underline{\underline{0}}$$

6.4 $\int_S \vec{F} \cdot d\vec{S}$

S kub med sida 4, center i origo

$$\vec{F} = \frac{\vec{r} - \vec{a}}{|\vec{r} - \vec{a}|^3}$$

$$\left(\text{Gauss} \quad \int_S \vec{F} \cdot d\vec{S} = \int_V dV \, \nabla \cdot \vec{F} \right)$$

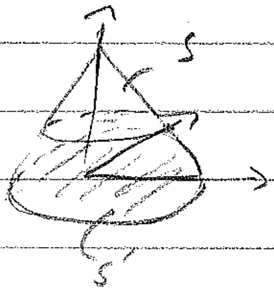
runt $(1,0,0)$ $\vec{F} = \frac{1}{r^2}$ punktkälla med styrka 4π

$$\Rightarrow \int_S \vec{F} \cdot d\vec{S} = \underline{4\pi}$$

6.6 $\vec{F} = F_0 \frac{a^2}{(x^2+y^2+z^2)^{3/2}} \left(x, y, z + \frac{z}{a}\frac{(x^2+y^2+z^2)^{3/2}}{a^2}\right)$

normalytintegralen över $S: x^2+y^2 = (z-3a)^2$

$0 \le z \le 3a$



$\vec{F} = F_0 a^2 \frac{\vec{r}}{(\vec{r}^2)^{3/2}} + F_0 \frac{z}{a}\hat{z}$

$\vec{F} = F_0 a^2 \frac{\hat{r}}{|\vec{r}|^2} + \frac{F_0 z}{a}\hat{z}$

Slut volymen med en en bottencirkel

Singulära delen av fältet ger bidraget

$4\pi F_0 a^2/2 = 2\pi F_0 a^2$ till ytintegralen

halva rymdvinkeln upptas

Divergensen av ickesingulära delen:

$$\nabla \cdot \underbrace{\frac{F_0 z}{a}\hat{z}}_{F_1} = \frac{F_0}{a}$$

=> Gauss för den ickesingulära delen

$$\int_S \vec{F}_1 \cdot dS + \int_{S'} \vec{F}_1 \cdot dS' = \int_V \nabla \cdot \vec{F}_1 \, dV$$

där S är konen och S' cirkeln

$$\int_V \nabla \cdot \vec{F}_1 \, dV = \int_V \frac{F_0}{a} dV = \frac{1}{\cancel{3a}} \pi (3a)^2 \cdot \cancel{3a} = \pi 9 a^2 F_0$$

och $\int_{S'} \vec{F}_1 \cdot dS' = \int_{S'} -\frac{F_0 z}{a} dS'$ men $z=0$ på S'

$\Rightarrow \int_{S'} \vec{F}_1 \cdot dS' = 0$

$\Rightarrow \int_S \vec{F}_1 \cdot dS = \pi 9 a^2$

och det singulära bidraget till ytintegralen var $2\pi F_0 a^2$

$\Rightarrow \int_S \vec{F} \cdot d\vec{S} = 11 \pi F_0 a^2$

6.7

Beräkna $\int_C \vec{F}\cdot d\vec{r}$ $\quad C: (\rho, \varphi, z) = (2+\frac{1}{2}\cos\frac{t}{2}, t, 0)$

$\qquad\qquad\qquad\qquad\qquad\qquad t: 0 \to 4\pi$

$$\vec{F}(\vec{r}) = \underbrace{\frac{(x-1)\hat{y} - y\hat{x}}{(x-1)^2+y^2}}_{\vec{F}_I} + \hat{\rho}(1-\rho)e^{-\rho}\sin\varphi$$

$$+\hat{\varphi}(e^{-\rho}\cos\varphi - z\sin\varphi) + \hat{z}\cos\varphi$$

Stokes sats på $\vec{F}_{II}$ $\quad \int_S (\nabla\times\vec{F})\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$

$$\nabla\times\vec{F}_2 = \left(\frac{1}{\rho}\frac{\partial \vec{F}_z}{\partial\varphi} - \frac{\partial \vec{F}_\varphi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial \vec{F}_\rho}{\partial z} - \frac{\partial \vec{F}_z}{\partial \rho}\right)\hat{\varphi}$$

$$+\frac{1}{\rho}\left(\frac{\partial(\rho \vec{F}_\varphi)}{\partial\rho} - \frac{\partial \vec{F}_\rho}{\partial\varphi}\right)\hat{z} =$$

$$= \left(-\frac{\sin\varphi}{\rho} + \sin\varphi\right)\hat{\rho} + \frac{1}{\rho}\Big(e^{-\rho}\cos\varphi - z\sin\varphi$$

$$+\rho(-e^{-\rho}\cos\varphi) - (1-\rho)e^{-\rho}\cos\varphi\Big)\hat{z}$$

$$= \left(-\frac{\sin\varphi}{\rho} + \sin\varphi\right)\hat{\rho} + \frac{1}{\rho}\Big(\cancel{e^{-\rho}\cos\varphi} - z\sin\varphi$$

$$-\cancel{\rho e^{-\rho}\cos\varphi} - \cancel{e^{-\rho}\cos\varphi} + \cancel{\rho e^{-\rho}\cos\varphi}\Big)\hat{z}$$

$$= \left(-\frac{\sin\varphi}{\rho} + \sin\varphi\right)\hat{\rho} - \frac{z}{\rho}\sin\varphi\,\hat{z}$$

$$\int_S (\nabla\times\vec{F}_{II})\cdot \underset{\hat{z}}{d\vec{S}} = \int_S -\frac{z}{\rho}\sin\varphi\, dS\Big|_{\underset{\text{på ytan}}{z=0}} = 0$$

$$F_I(\vec r) = \frac{(x-1)\hat y - y\hat x}{(x-1)^2 + y^2} \qquad \begin{aligned} x' &= x-1 \\ y' &= y \end{aligned}$$

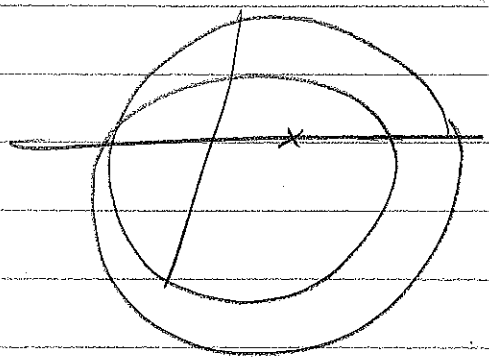
$$\Rightarrow \quad F_I(\vec r\,') = \frac{x'\hat y' - y'\hat x'}{x'^2 + y'^2}$$

täljaren pekar i $\hat e$-led
med längden $\sqrt{x'^2 + y'^2} = \rho'$

$$\Rightarrow \quad F_I(\vec r\,') = \frac{\rho'\hat e}{\rho'^2} = \frac{\hat e}{\rho}$$

Detta är en virveltråd i punkten $x=1,\ y=0$

Kurvans utseende:



Cirklar runt virveltråden 2 ggr

$$\Rightarrow \quad \int_C \vec F_I \cdot d\vec r = 2 \cdot (2\pi) = 4\pi$$

8.2 $\quad \vec{F} = \frac{a\cos\theta}{r^2}\hat{r} + \frac{b\sin\theta}{r^3}\hat{\theta}$

Virvelfritt: $\nabla\times\vec{F} = 0$

$$\nabla\times\vec{F} = \frac{1}{r}\left(\frac{\partial}{\partial r}\left(\frac{b\sin\theta}{r^2}\right) - \frac{\partial}{\partial\theta}\left(\frac{a\cos\theta}{r^2}\right)\right) =$$

$$= \frac{1}{r}\left(-\frac{2b\sin\theta}{r^3} + \frac{a\sin\theta}{r^3}\right)$$

$$\Rightarrow \underline{a - 2b = 0}$$

Hitta potentialen: $-\nabla\phi(r,\theta) = \vec{F}$

$$\frac{\partial\phi}{\partial r} = -\frac{a\cos\theta}{r^3} \Rightarrow \phi(r,\theta) = +\frac{a\cos\theta}{2r^2}$$

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{b\sin\theta}{r^3} \Rightarrow \phi(r,\theta) = +\frac{b}{r^2}\cos\theta$$

och $a = 2b$ för rotationsfritt fält.
Existerar bara en potential i det fallet.

$$\Rightarrow \phi(r,\theta) = \frac{b\cos\theta}{r^2} + \phi_0$$

8.3 $\vec{F}$ har den skalära potentialen ϕ

$\vec{F}$ kan beskrivas med en vektorpotential $\vec{A}$ också

$$\left.\begin{array}{l}\vec{F} = -\nabla\phi \\ \vec{F} = \nabla\times\vec{A}\end{array}\right\} \Rightarrow -\nabla\phi = \nabla\times\vec{A}$$

$$\phi = \frac{\cos\theta}{r^2}$$

$$-\nabla\phi = \frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta}$$

$$\Rightarrow \nabla\times\vec{A} = \frac{2\cos\theta}{r^3}\hat{r} + \frac{\sin\theta}{r^3}\hat{\theta} + 0\,\hat{e}$$

Antag $A_\theta = 0$, $A_r = 0$

$$\Rightarrow \hat{r}: \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_e\sin\theta) = \frac{2\cos\theta}{r^3}$$

$$\frac{\partial}{\partial\theta}(A_e\sin\theta) = \frac{2\sin\theta\cos\theta}{r^2}$$

$$\Rightarrow A_e = \frac{\sin\theta}{r^2}$$

Se om det uppfyller $\hat{\theta}$ villkoret också.

$$\hat{\theta}: -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sin\theta}{r}\right) = +\frac{1}{r}\frac{\sin\theta}{r^2} = \frac{\sin\theta}{r^3} \quad \text{Ja!}$$

$$\Rightarrow \vec{A} = \frac{\sin\theta}{r^2}\hat{e} + \nabla f$$

f är en godtycklig skalär funktion.

9.1 $\Delta\phi(x) = -q\delta(x)$

Homogen løsn. $\phi_H(x) = Ax + B$

Partikulær løsning $\Delta\phi_p(x) = -q\delta(x)$ $\quad \partial_x\phi_p(x) = -\frac{q}{2}(\Theta(x) - \Theta(-x))$

$$\phi_p(x) = -\frac{q}{2}|x|$$

$$\phi(x) = \phi_H + \phi_p = -\frac{q}{2}|x| + Ax + B$$

9.2

$$\phi(\vec{r})\Big|_{z=0} = \int_V G(\vec{r},\vec{r}')\rho(\vec{r}')dV\Big|_{z=0} = 0 \quad \text{Dirichlet (homogent)}$$

$$G(\vec{r},\vec{r}') = \frac{1}{4\pi}\frac{1}{|(x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}|} - \frac{1}{4\pi}\frac{1}{|(x-x')\hat{x} + (y-y')\hat{y} + (z+z')\hat{z}|}$$

$$= \frac{1}{4\pi}\frac{1}{|\vec{r}-\vec{r}'|} - \frac{1}{4\pi}\frac{1}{|(x-x')\hat{x} + (y-y')\hat{y} + (z+z')\hat{z}|}$$

Speglad i z planet för z' koord

9.3

Homogent Neumann

$$\partial_z \Phi(\vec r)\Big|_{z=0} = \partial_z \int_V G(\vec r, \vec r\,')\rho(\vec r\,')\,dV\Big|_{z=0} = 0$$

$$\partial_z G(\vec r, \vec r\,') = \partial_z\left[\frac{1}{4\pi|\vec r - \vec r\,'|} + \eta \frac{1}{4\pi}\frac{1}{|(x-x')\hat x + (y-y')\hat y + (z+z')\hat z|}\right] =$$

$$= \partial_z\left[\frac{1}{4\pi}\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} + \eta\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right]_{z=0}$$

$$= \left[-\frac{1}{2}\frac{1}{4\pi}\frac{2(z-z')}{((x-x')^2+(y-y')^2+(z-z')^2)^{3/2}} + \frac{\eta}{4\pi}\left(-\frac{1}{2}\right)\frac{2(z+z')}{((x-x')^2+(y-y')^2+(z+z')^2)^{3/2}}\right]_{z=0} =$$

$$= \frac{1}{4\pi}\frac{z' - \eta z'}{((x-x')^2+(y-y')^2+z'^2)^{3/2}}$$

Si $\eta = +1 \Rightarrow$ Homogent Neumann

$$G(\vec r, \vec r\,') = \frac{1}{4\pi}\frac{1}{|\vec r - \vec r\,'|} + \frac{1}{4\pi}\frac{1}{|(x-x')\hat x + (y-y')\hat y + (z+z')\hat z|}$$

9.13 $\rho < a$ $\qquad \phi = \phi_0 \cos m\theta \quad \rho = a$

$$\Delta \phi = 0$$

I polära koord. $\dfrac{1}{\rho}\dfrac{\partial}{\partial \rho}\left(\rho \dfrac{\partial \phi}{\partial \rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2}{\partial \theta^2}\phi = 0$

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \phi}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Antag ett vinkelberoende $\propto \cos m\theta$

$\Rightarrow \phi(\rho, \theta) = f(\rho)\cos m\theta$

$$\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f(\rho)}{\partial \rho}\right)\cos m\theta - \frac{m^2}{\rho}\cos m\theta f(\rho) = 0$$

$$\Rightarrow \frac{\partial}{\partial \rho}\left(\rho \frac{\partial f(\rho)}{\partial \rho}\right) = \frac{m^2}{\rho} f(\rho)$$

$$\rho \frac{\partial}{\partial \rho}\left(\rho \frac{\partial f(\rho)}{\partial \rho}\right) = m^2 f(\rho)$$

Standard i ansätt en potensfkt $f(\rho) = A\rho^p$

$$\Rightarrow \rho \frac{\partial}{\partial \rho}\left(\rho p \rho^{p-1}\right)\cancel{A} = m^2 \cancel{A} \rho^p$$

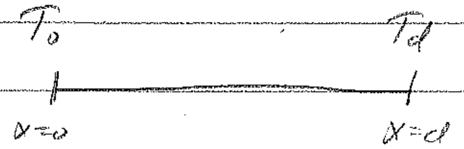
$$p \rho \frac{\partial}{\partial \rho}(\rho^p) = m^2 \rho^p$$

$$p^2 \rho^p = m^2 \rho^p \Rightarrow p = \pm m \qquad \sum_{m=0}^{\infty}$$

$$\Rightarrow \phi(\rho = a, \theta) = \phi_0 \cos m\theta \Rightarrow \phi(\rho, \theta) = \phi_0 \left(\frac{\rho}{a}\right)^m \cos m\theta$$

10.1

$$\Delta T = 0$$



$$\partial_x^2 T = 0 \quad \Rightarrow \quad T(x) = Ax + B$$

$$T(0) = T_0 \quad \Rightarrow \quad B = T_0$$

$$T(d) = T_d \quad \Rightarrow \quad T_d = Ad + T_0$$

$$\Rightarrow \quad A = \frac{T_d - T_0}{d}$$

$$\Rightarrow \quad T(x) = (T_d - T_0)\frac{x}{d} + T_0$$

10.2

Poisson $\Delta T(r,\theta,e) = -\lambda \rho_0$

Dirichlet $T(a,\theta,e) = T_0(1+\frac{1}{2}\cos\theta)$

Homogenlösning Ansats $T(r,\theta) = f(r)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = 0$$

$$\Rightarrow \quad r^2\frac{\partial f}{\partial r} = A$$

$$\frac{\partial f}{\partial r} = \frac{A}{r^2} \quad \text{singulär}$$

Partikulärlösn: $\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = -r^2\rho_0\lambda$

$$r^2\frac{\partial f}{\partial r} = -\frac{r^3\rho_0\lambda}{3} + A$$

$$\frac{\partial f}{\partial r} = -\frac{r\rho_0\lambda}{3} + \cancel{\frac{A}{r^2}} \quad \text{singulär}$$

$$f(r) = -\frac{r^2\rho_0\lambda}{6} + B$$

$$f(a) = 0 \quad \Rightarrow \quad B = \frac{a^2\rho_0\lambda}{6}$$

$$\Rightarrow \quad f(r) = (a^2 - r^2)\frac{\rho_0}{6}$$

11.1 ME : 1, $\nabla \cdot \vec{E} = 0$

2, $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

3, $\nabla \cdot \vec{B} = 0$

4, $\nabla \times \vec{B} - \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = 0$

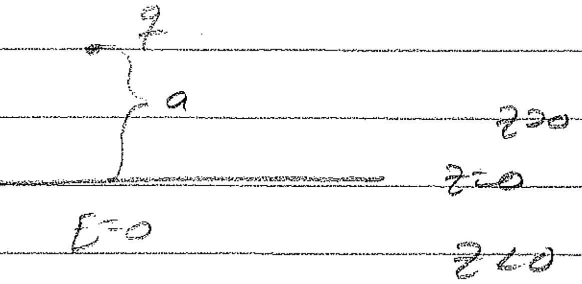
$\nabla \times (4) \Rightarrow \quad \nabla \times (\nabla \times \vec{B}) - \varepsilon_0 \mu_0 \frac{\partial}{\partial t}(\nabla \times \vec{E}) = 0$

$$\nabla \underbrace{(\nabla \cdot \vec{B})}_{0 \text{ med } 3} - \nabla^2 \vec{B} - \varepsilon_0 \mu_0 \frac{\partial}{\partial t} \underbrace{(\nabla \times \vec{E})}_{-\frac{\partial \vec{B}}{\partial t} \text{ med } 2} = 0$$

$$\Rightarrow \quad -\nabla^2 \vec{B} + \varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} \vec{B} = 0$$

$$\nabla^2 \vec{B} - \frac{\partial^2}{\partial t^2} \frac{1}{c^2} \vec{B} = 0 \quad , \quad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

11.2



Fältet i $z<0$ är $\vec{E}=0$ (metall är en bra ledare, leder bort laddningar som skapar fält)

$z>0$ $$\phi_>(\vec{r}) = \frac{q}{4\pi}\frac{1}{|\vec{r}-a\hat{z}|} - \frac{q}{4\pi}\frac{1}{|\vec{r}+a\hat{z}|} \quad \text{(spegling)}$$

$E_>(\vec{r}) = -\nabla\phi_>(\vec{r})$ Vi är endast intresserade av $\hat{z}$-komponenten av fältet.

$$E_>^z(\vec{r}) = -\partial_z\phi_>(\vec{r}) = -\partial_z\left[\frac{q}{4\pi}\frac{1}{\sqrt{x^2+y^2+(z-a)^2}} - \frac{q}{4\pi}\frac{1}{\sqrt{x^2+y^2+(z+a)^2}}\right]$$

$$= \frac{q}{4\pi}\left[\frac{(z-a)}{(x^2+y^2+(z-a)^2)^{3/2}} - \frac{z+a}{(x^2+y^2+(z+a)^2)^{3/2}}\right]$$

Undersök diskontinuiteten i fältet $\Rightarrow$ ytladdningstätheten

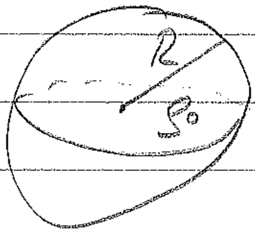
$$\sigma = E_> - \overset{0}{E_<}\Big|_{z=0} = -\frac{qa\cdot 2}{4\pi}\frac{1}{(x^2+y^2+a^2)^{3/2}} = -\frac{qa}{2\pi}\frac{1}{(x^2+y^2+a^2)^{3/2}}$$

$$Q_{yta} = -\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\frac{qa}{2\pi}\frac{1}{(x^2+y^2+a^2)^{3/2}} = -\frac{qa}{2\pi}\cdot\frac{2\pi}{a} = \underline{-q}$$

$$Q_{yta} = -\frac{qa}{2\pi}\int_0^{\infty}\frac{\rho}{(\rho^2+a^2)^{3/2}}\cdot 2\pi\,d\rho = -qa\left[\frac{-1}{\sqrt{\rho^2+a^2}}\right]_0^{\infty} = \underline{-q}$$

11.19 $\vec{g} = -\nabla\phi \qquad \nabla^2\phi = \gamma\rho$

ρ massetetthet



ρ_0 konstant

Gauss $\displaystyle\int_S \vec{g}\cdot d\vec{S} = \int_V \nabla\cdot\vec{g}\, dV$

$$\int_V \nabla\cdot\vec{g}\, dV = -\int_V \nabla\cdot(\nabla\phi)\, dV =$$

$$= -\int_V \nabla^2\phi\, dV = -\gamma\int_V \rho_0\, dV =$$

$$= -\gamma\rho_0 V_r$$

$\Rightarrow \displaystyle\int_S \vec{g}\cdot d\vec{S} = -\gamma\rho_0 V_r$

$\Rightarrow g_r \cdot A_r = -\gamma\rho_0 V_r$ (sfærisk symmetrisk problem)

$g_r = -\gamma\rho_0 \dfrac{V_r}{A_r}$

$r<R \quad V_r = \dfrac{4\pi r^3}{3}$

$\qquad\quad A_r = 4\pi r^2$

$r>R \quad V_r = \dfrac{4\pi R^3}{3}$

$\qquad\quad A_r = 4\pi r^2$

$\Rightarrow \quad g_r = -\gamma\rho_0 \dfrac{4\pi r^3}{3} \Big/ 4\pi r^2 = -\dfrac{\gamma\rho_0}{3} r \qquad r<R$

$g_r = -\gamma\rho_0 \dfrac{4\pi R^3}{3} \Big/ 4\pi r^2 = -\dfrac{\gamma\rho_0 R^3}{3} \dfrac{1}{r^2}$

12.1

T_{ij} tensor $\Rightarrow T'_{ij} = L_{ik}L_{jl}T_{kl}$

Titta på hur en diagonal komponent transformeras

$T'_{ij} = (L_{ik}L_{jl}T_{kl})_{k=l} = L_{ik}L_{jk}T_{kk} = \delta_{ij}T_{kk}$

komponent ii i T' $\quad (\delta_{ij}T_{kk})_{i=j} = T_{kk}$

alltså: $T_{kk} \to T_{kk}$ skalär

12.3

$\delta_{ij} \to \delta'_{ij}$

Antag att T_{ij} är en tensor, vi vet att

T_{ii} är en skalär från 12.1

$T'_{ij} = L_{ik}L_{jl}T_{kl}$

och $T_{ii} = \delta_{ij}T_{ij} \to \delta_{ij}T_{ij} = T_{ii}$ (skalär).

$\delta_{ij}T_{ij} \to \delta'_{ij}L_{ik}L_{jl}T_{kl} \overset{?}{=} \delta_{ij}T_{ij} = T_{ii}$

$\delta_{mn}\underbrace{R_{mnij}L_{ik}L_{jl}}_{=\delta_{mk}\delta_{ln}}T_{kl} \overset{?}{=} \delta_{ij}T_{ij}$ om ovanstående ska gälla

Vi vet att $L_{im}L_{ik} = \delta_{mk}$
$L_{jn}L_{jl} = \delta_{ln}$

Det betyder att $R_{mnij} = L_{im} L_{jn}$

och $\sigma'_{ij} = R_{mnij} \sigma_{mn}$

så $\sigma'_{ij} = L_{im} L_{jn} \sigma_{mn}$ tensor

12.8

$$E = \frac{mv^2}{2} \qquad v = \vec{\omega} \times \vec{r}$$

$$\Rightarrow \quad T = \int_V dV \frac{\rho}{2} (\vec{\omega} \times \vec{r})^2 = \int_V dV \frac{\rho}{2} \varepsilon_{ijk} \omega_j x_k \varepsilon_{ilm} \omega_l x_m$$

$$= \frac{1}{2} \int dV \rho \underbrace{\varepsilon_{ijk} \varepsilon_{ilm}}_{} \omega_j \omega_l x_k x_m$$

$$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$$

$$= \frac{1}{2} \int dV \rho (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) \omega_j \omega_l x_k x_m =$$

$$= \frac{1}{2} \underbrace{\int dV \rho (\delta_{jl} \;-\; x_j x_l)}_{I_{jl}} \omega_j \omega_l = \frac{1}{2} I_{jl} \omega_j \omega_l$$