Idag ett nytt ämne:

# Variationskalkyl

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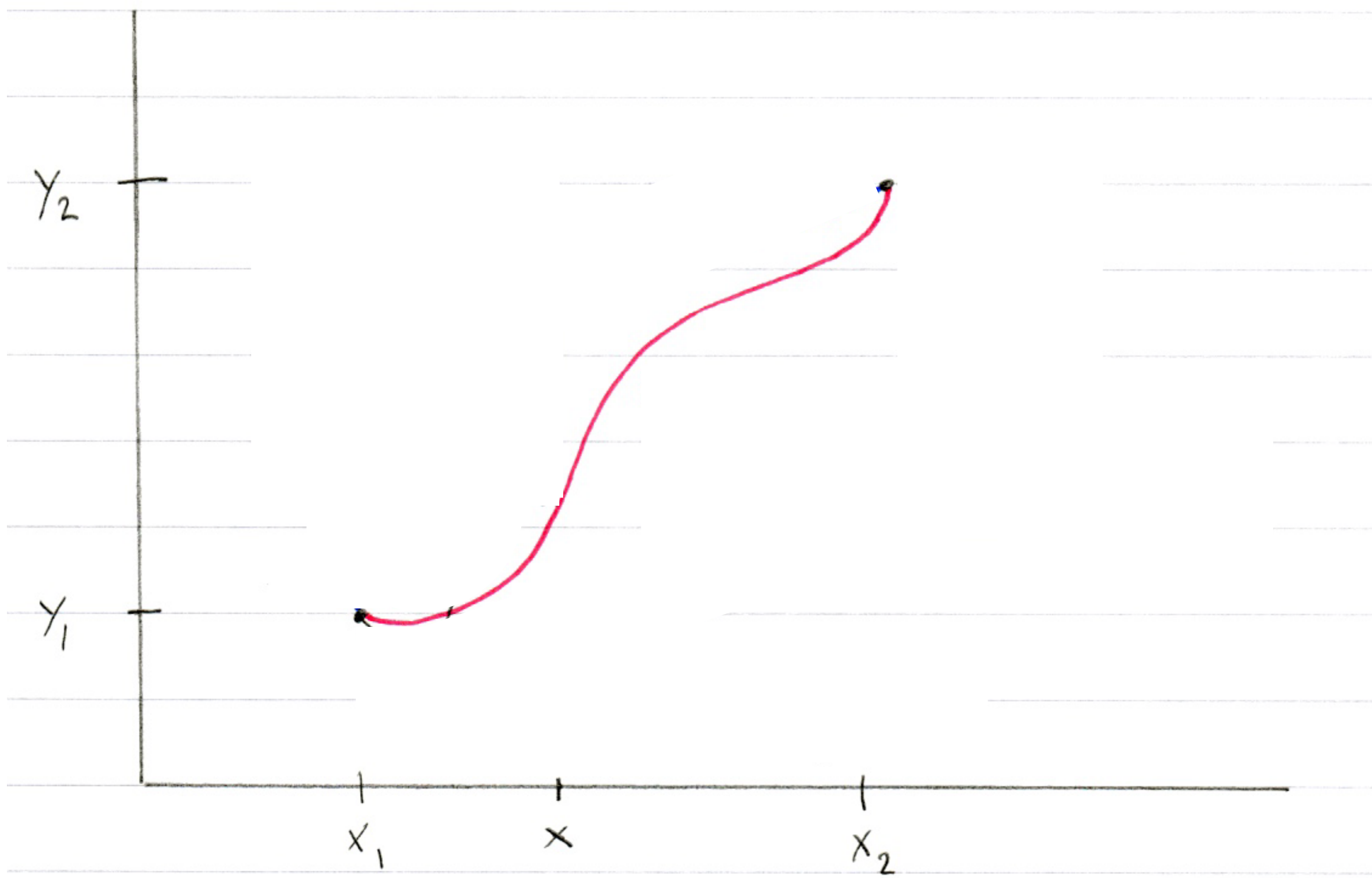
# *Variationskalkyl*

Rembrandt's "Dido Divides the Oxhide" (mid-1600s)

# Det ”arketypiska” problemet i variationskalkyl:



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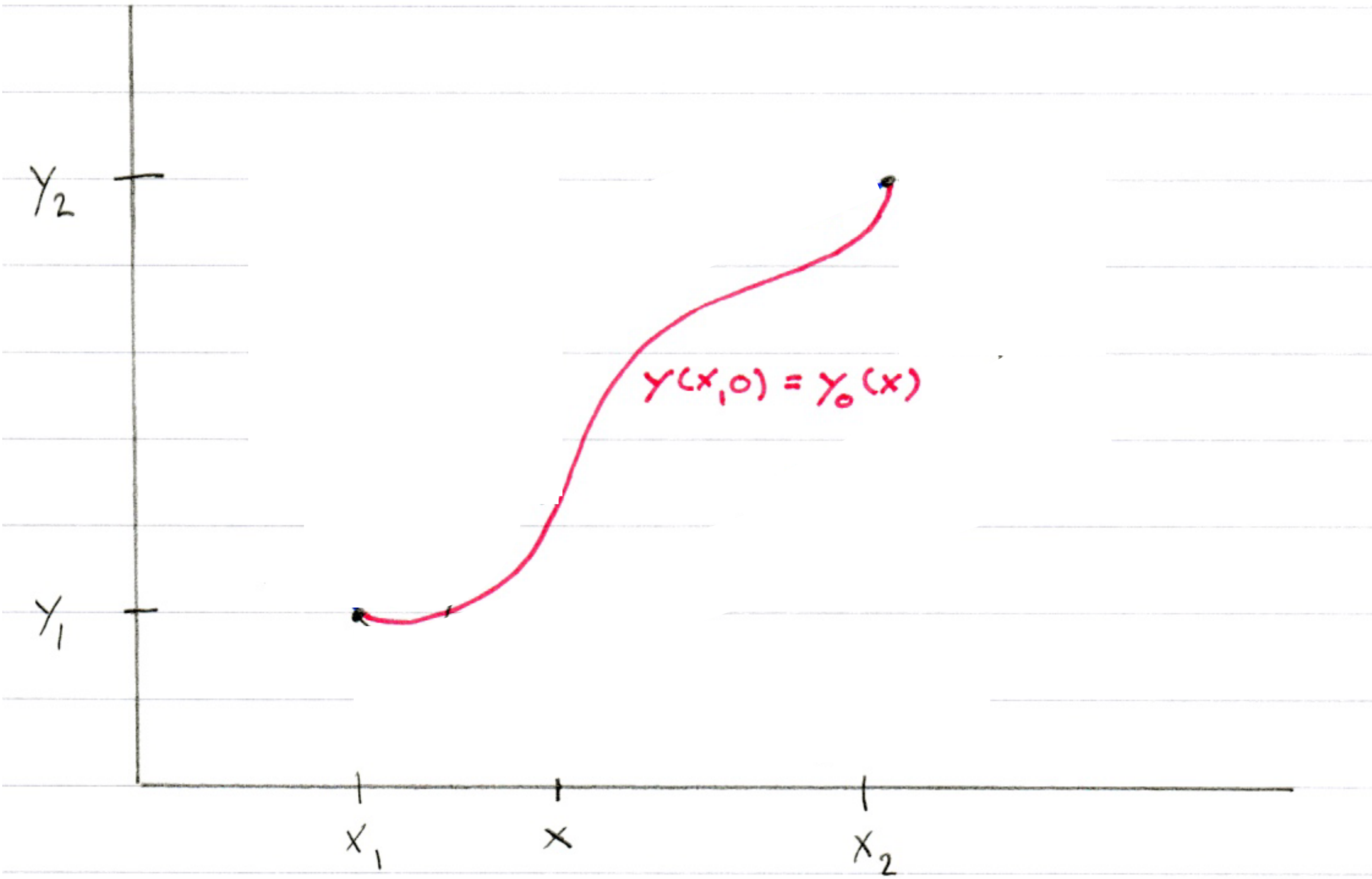

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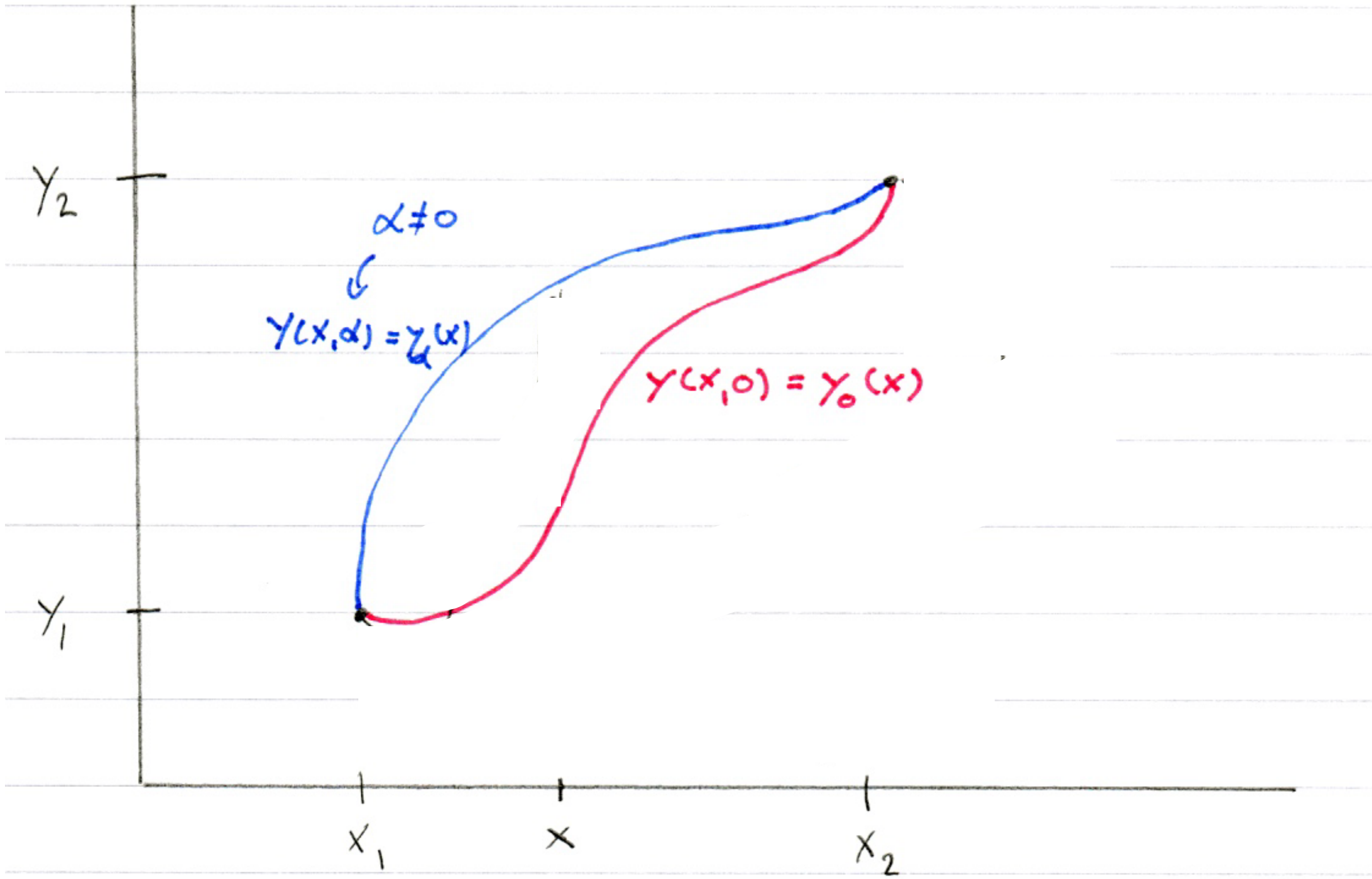

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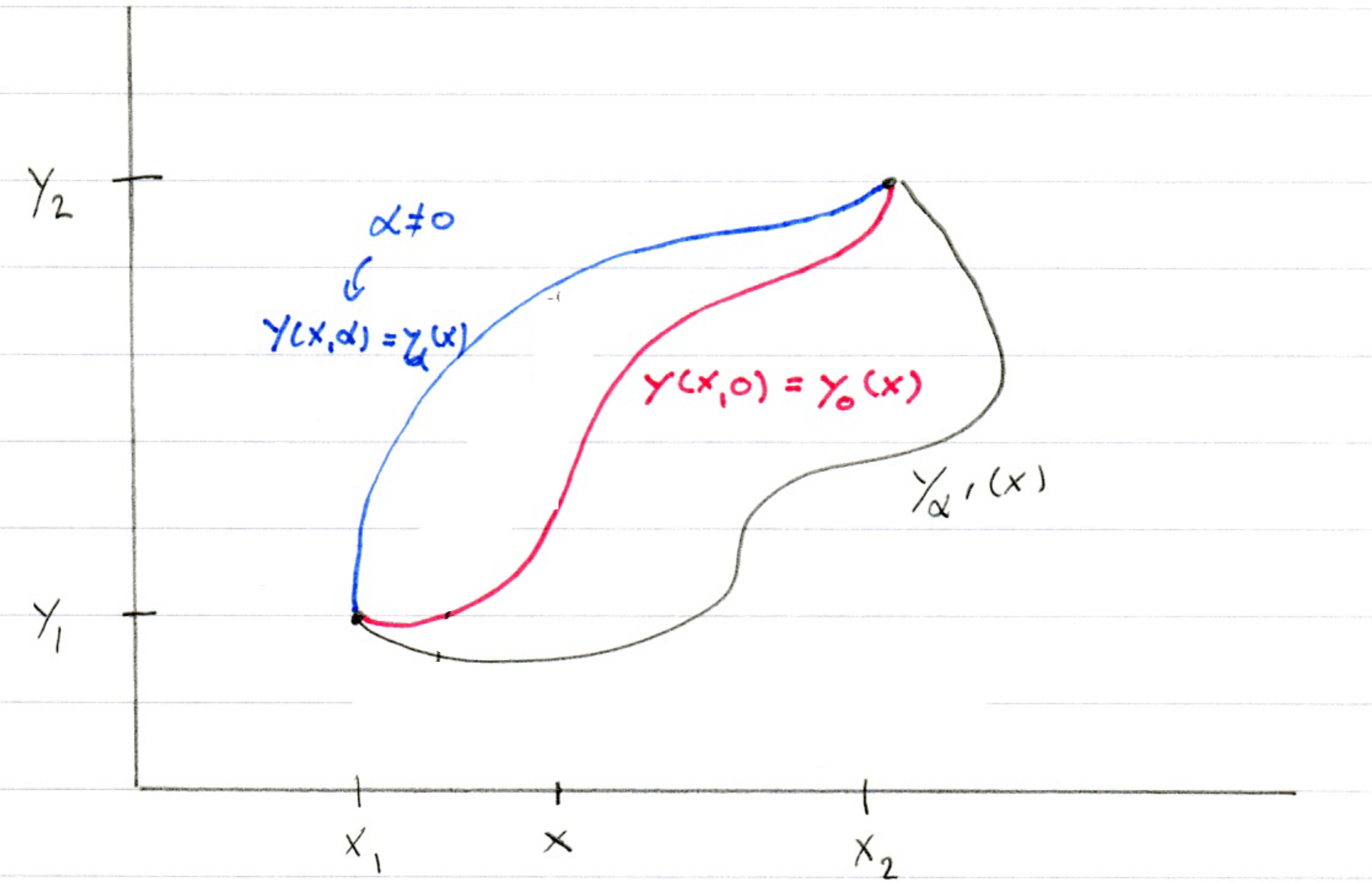

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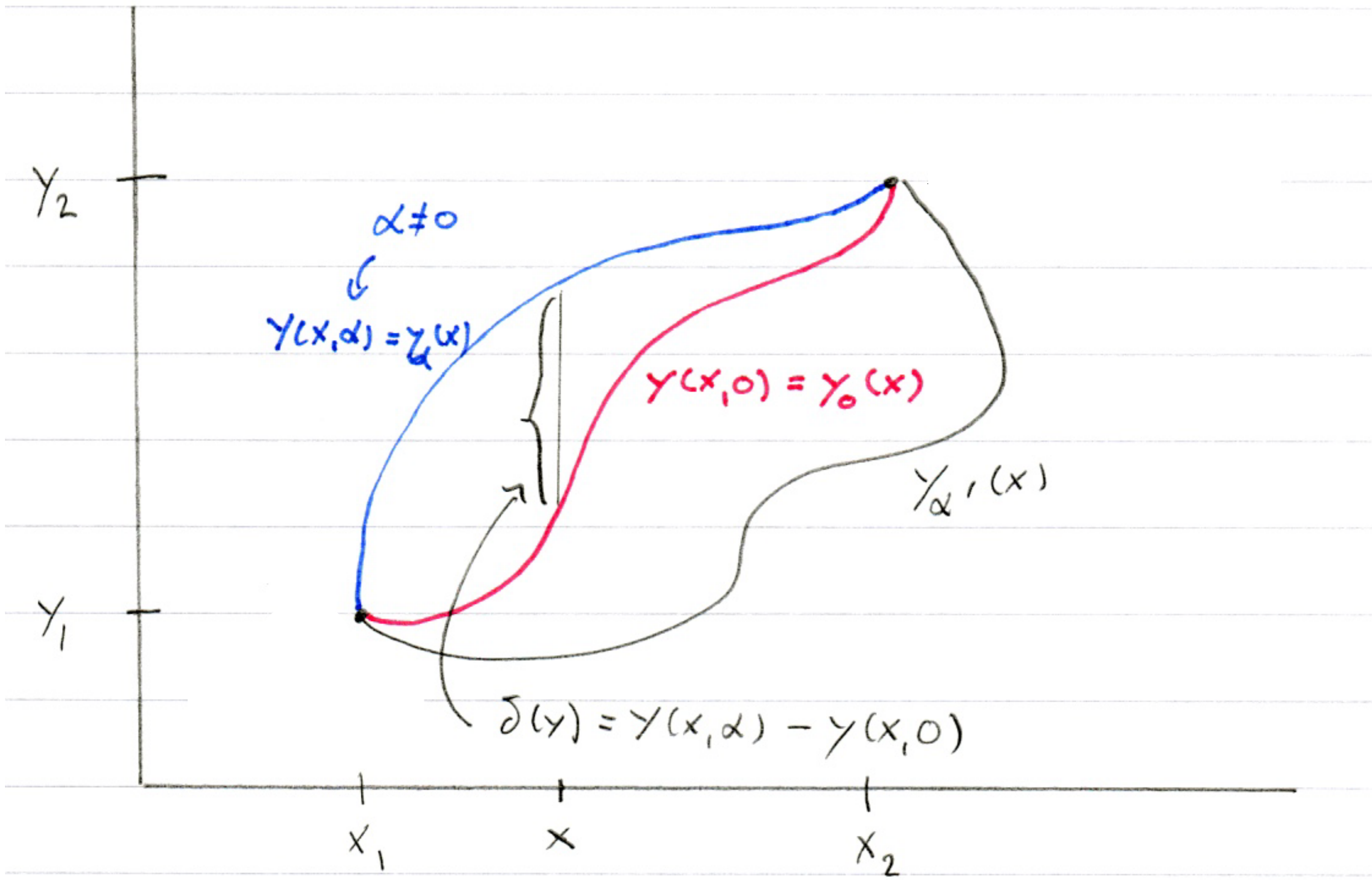

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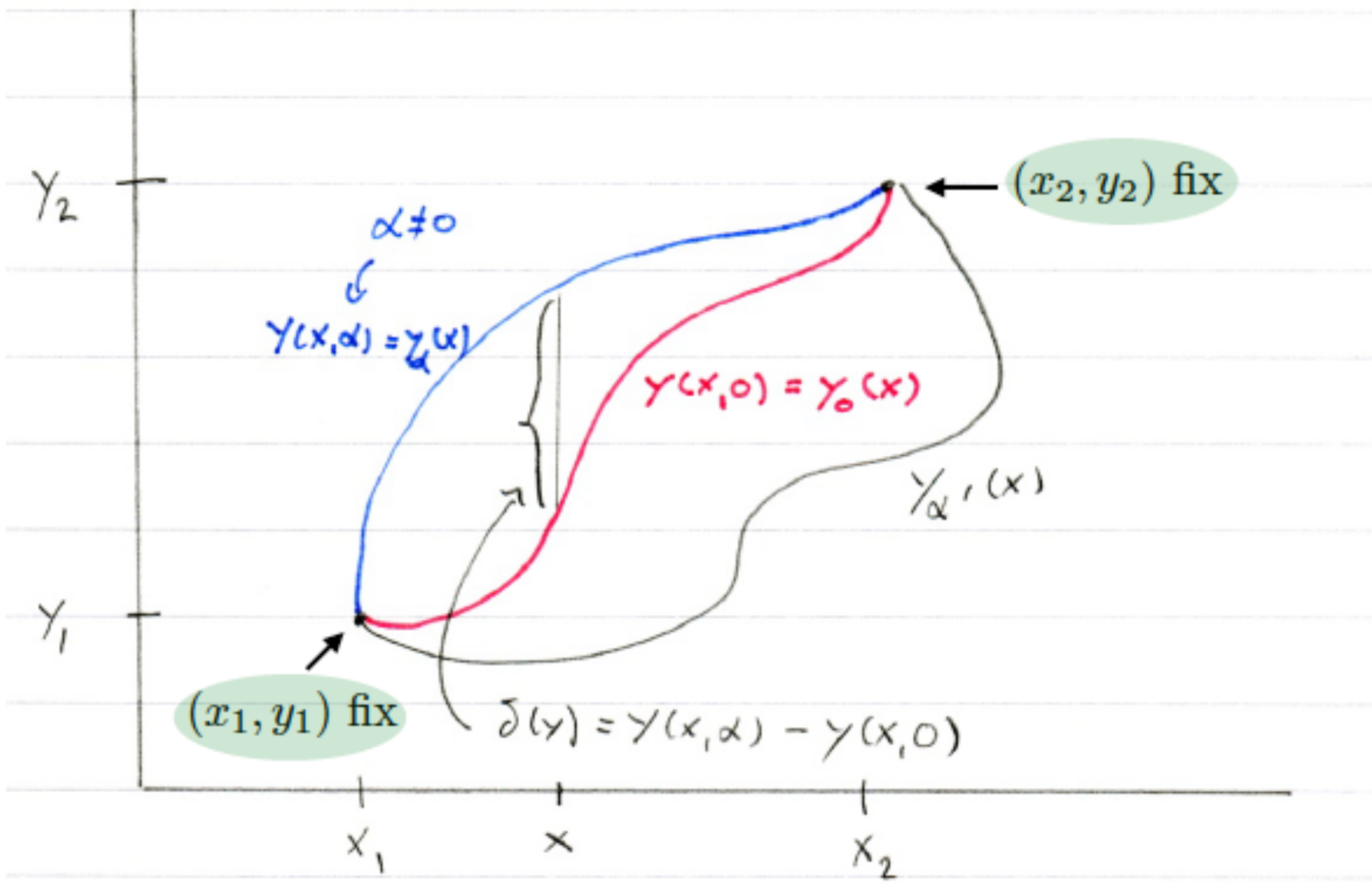

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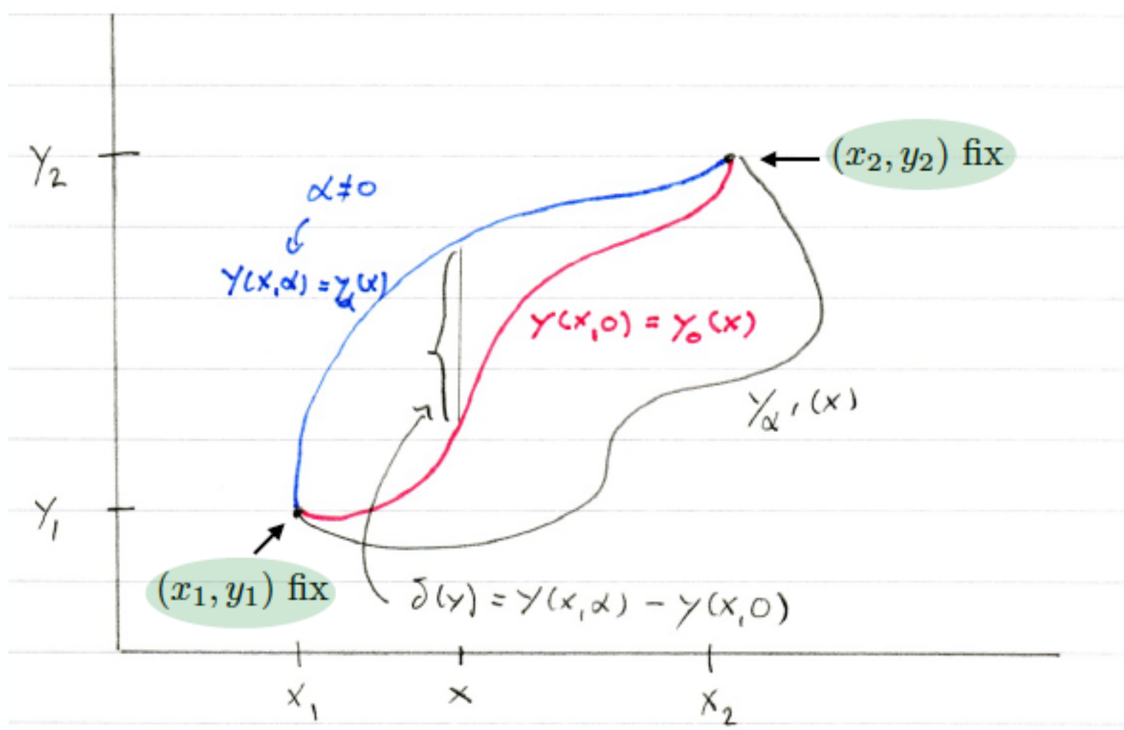


BILDA INTEGRALEN

FUNKTION AV $\alpha$

OBEROENDE VARIABEL

$$I(\alpha) = \int_{x_1}^{x_2} F\left[\, y(x,\alpha),\ y'(x,\alpha),\ x \,\right] dx$$

FUNKTIONAL AV $y_\alpha$

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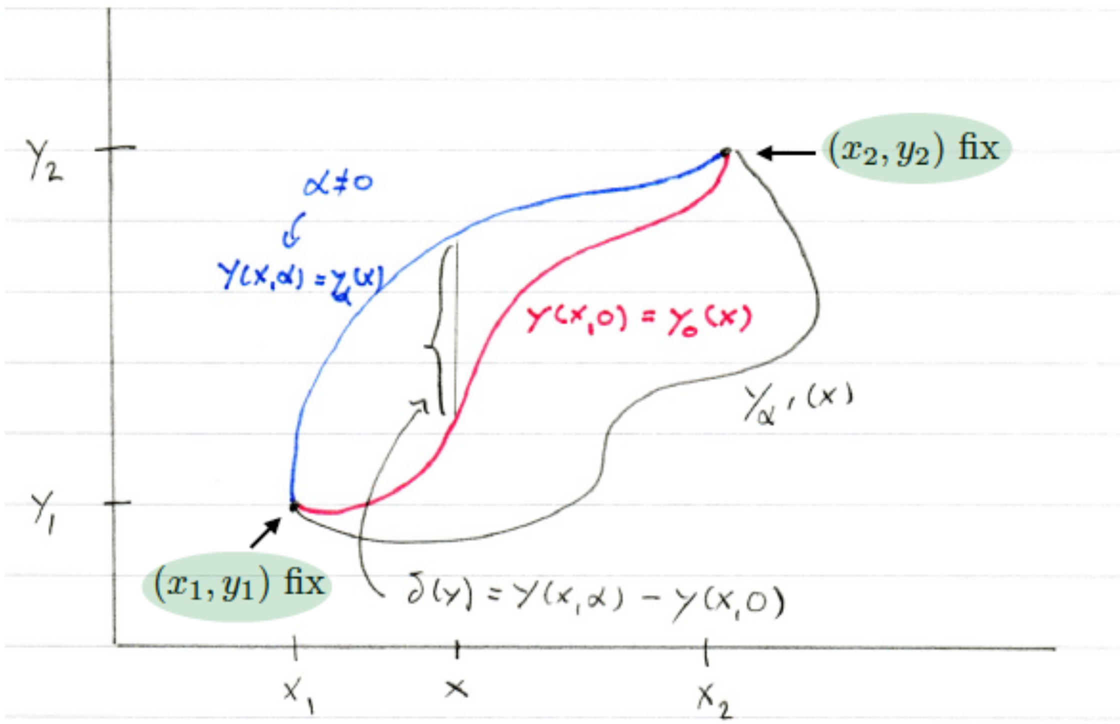


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PROBLEM: HITTA $y(x, \alpha_0)$ SÅ ATT $I(\alpha_0)$ ÄR STATIONÄR

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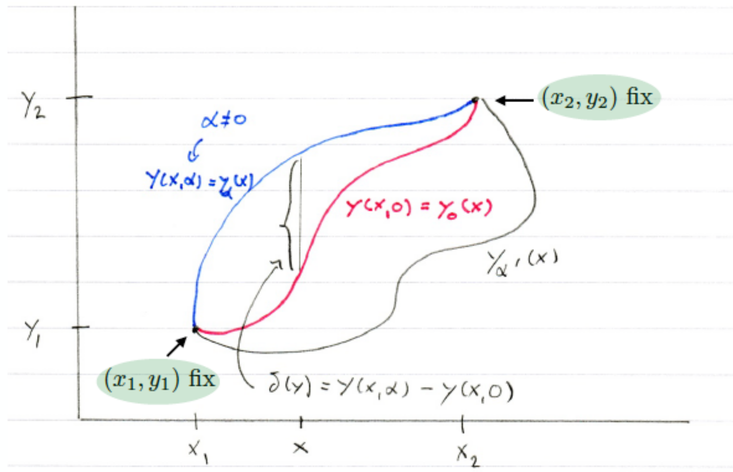


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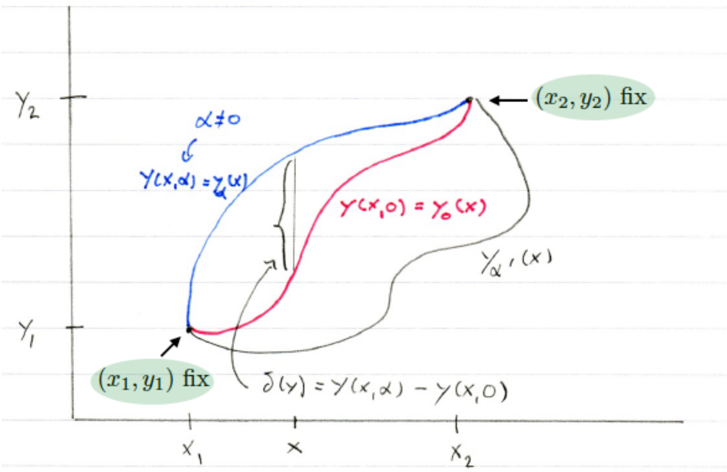
$$\left. \frac{dI(\alpha)}{d\alpha} \right|_{\alpha = \alpha_0 = 0} = 0$$

← välj "0" som benämning på den stationära lösningen

# Det ”arketypiska” problemet i variationskalkyl:




PROBLEM: HITTA $Y(x, \alpha_0)$ SÅ ATT $I(\alpha_0)$ ÄR STATIONÄR

$$\left.\frac{dI(\alpha)}{d\alpha}\right|_{\alpha = \alpha_0 = 0} = 0$$

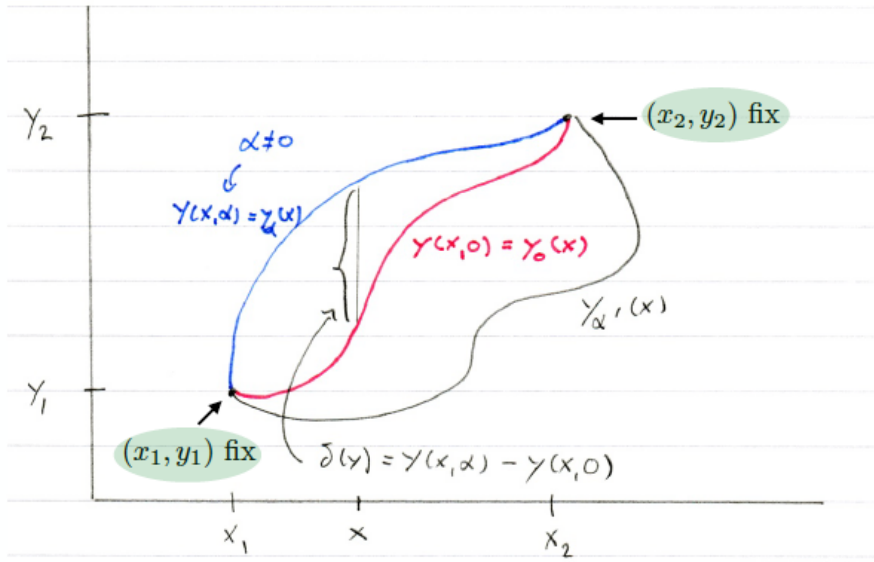
← välj "0" som benämning på den stationära lösningen

VAD BETYDER $\frac{dI}{d\alpha}$ ?

$$\frac{dI}{d\alpha} = \left.\frac{\delta I}{\delta Y_\alpha}\right|_{Y_\alpha = Y_0} = 0 \Rightarrow \delta I = 0$$

FUNKTIONALDERIVATA

dvs. $\frac{df}{dx} = 0 \Rightarrow df = 0$




FÖR ATT KUNNA RÄKNA PÅ DET HÄR, INFÖR

$$\delta y = y(x,\alpha) - y(x,0) = \alpha\,\eta(x)$$

godtycklig deformation

$$\Downarrow$$

$$y(x,\alpha) = y(x,0) + \alpha\,\eta(x) \qquad (1)$$

VILLKOR $\eta(x_1) = \eta(x_2) = 0$

$\eta(x)$ DERIVERBAR

BETRAKTA

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha} + \cancel{\frac{\partial F}{\partial x}\frac{\partial x}{\partial \alpha}} \right) dx \qquad (2)$$

derivering innanför integraltecknet OK

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derivering innanför integraltecknet OK

$$(1) \Rightarrow \frac{\partial y(x,\alpha)}{\partial \alpha} = \eta(x)$$

$$\frac{\partial y'(x,\alpha)}{\partial \alpha} = \frac{\partial}{\partial \alpha}\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}\frac{\partial y}{\partial \alpha} = \frac{d\eta(x)}{dx} \qquad (3)$$

$$(2)\,\&\,(3) \Rightarrow \frac{dI}{d\alpha} = \int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\frac{d\eta(x)}{dx} \right) dx \qquad (4)$$

$$\delta y = y(x,\alpha) - y(x,0) = \alpha\,\eta(x)$$

godtycklig deformation

VILLKOR $\eta(x_1) = \eta(x_2) = 0$

$\eta(x)$ DERIVERBAR

$$\Downarrow$$

$$y(x,\alpha) = y(x,0) + \alpha\,\eta(x) \qquad (1)$$

BETRAKTA

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial \alpha} + \cancel{\frac{\partial F}{\partial x}\frac{\partial x}{\partial \alpha}}\right)dx \qquad (2)$$

derivering innanför integraltecknet OK

$$\left.\begin{aligned}(1) \Rightarrow \frac{\partial y(x,\alpha)}{\partial \alpha} &= \eta(x)\\ \frac{\partial y'(x,\alpha)}{\partial \alpha} &= \frac{\partial}{\partial \alpha}\frac{\partial y}{\partial x} = \frac{\partial}{\partial x}\frac{\partial y}{\partial \alpha} = \frac{d\eta(x)}{dx}\end{aligned}\right\} \quad (3)$$

$$(2)\,\&\,(3) \Rightarrow \frac{dI}{d\alpha} = \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\frac{d\eta(x)}{dx}\right)dx \qquad (4)$$

PARTIELL INTEGRATION

$$\int_{x_1}^{x_2}\frac{\partial F}{\partial y'}\frac{d\eta(x)}{dx}\,dx = \left.\eta(x)\frac{\partial F}{\partial y'}\right|_{x_1}^{x_2} - \int_{x_1}^{x_2}\eta(x)\frac{d}{dx}\frac{\partial F}{\partial y'}\,dx \qquad (5)$$

$$(2) \,\&\, (3) \Rightarrow \frac{dI}{d\alpha} = \int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y} \eta(x) + \frac{\partial F}{\partial y'} \frac{d\eta(x)}{dx} \right) dx \qquad (4)$$

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'} \frac{d\eta(x)}{dx} dx \overset{\text{PARTIELL INTEGRATION}}{=} \eta(x) \frac{\partial F}{\partial y'} \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx} \frac{\partial F}{\partial y'} dx \qquad (5)$$

SÄTT IN (5) I (4)

$$\Rightarrow \frac{dI}{d\alpha}\Big|_{\alpha=0} = \int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \eta(x)\, dx = 0$$

$\eta(x)$ GODTYCKLIG !

$$\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0$$ EULERS EKVATION

$$(2) \,\&\, (3) \Rightarrow \frac{dI}{d\alpha} = \int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y} \eta(x) + \frac{\partial F}{\partial y'} \frac{d\eta(x)}{dx} \right) dx \quad (4)$$

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$\eta(x)$ GODTYCKLIG !

$$\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \qquad \text{EULERS EKVATION}$$

Leonhard Euler
1707-1783

# föreläsning 19/11 (I)

$$(1)\,\&\,(3) \Rightarrow \frac{dI}{d\alpha} = \int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y}\eta(x) + \frac{\partial F}{\partial y'}\frac{d\eta(x)}{dx} \right) dx \quad (4)$$

PARTIELL INTEGRATION

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y'}\frac{d\eta(x)}{dx}\,dx = \eta(x)\frac{\partial F}{\partial y'}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\frac{\partial F}{\partial y'}\,dx \quad (5)$$

SÄTT IN (5) I (4)

$$\Rightarrow \frac{dI}{d\alpha}\Big|_{\alpha=0} = \int_{x_1}^{x_2} \left( \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} \right) \eta(x)\,dx = 0$$

$\eta(x)$ GODTYCKLIG!

$$\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$ EULERS EKVATION

NÖDVÄNDIGT VILLKOR FÖR ATT $I(0)$ ÄR STATIONÄR, EJ TILLRÄCKLIGT VILLKOR! KOLLA LÖSNINGEN!

MAX ELLER MIN? MATEMATISKT TRIXIGT. ANVÄND FYSIKALISK INTUITION!

Leonhard Euler
1707-1783

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

EULERS EKVATION

Leonhard Euler
1707-1783

JFR. EULER-LAGRANGES EKVATIONER

$F \to L$
$x \to t$
$y \to x_i$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$

Joseph-Louis Lagrange
1736-1813

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 \qquad \text{EULERS EKVATION}$$

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Joseph-Louis Lagrange
1736-1813

$$\sim$$

mycket användbart specialfall

$$F = F[y, y', \cancel{x}] \Rightarrow \frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0$$

$$\left(\Rightarrow F - y'\frac{\partial F}{\partial y'} = \text{KONST.}\right)$$

<u>BEVIS</u>

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial y'}\frac{\partial y'}{\partial x} - \frac{\partial y'}{\partial x}\frac{\partial F}{\partial y'} - y'\frac{d}{dx}\frac{\partial F}{\partial y'}$$

$$= y'\underbrace{\left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)}_{\text{EULER}} = 0$$

Typexempel på variationskalkyl:

# ”Såpbubbleproblemet”

Typexempel på variationskalkyl:

# ”Såpbubbleproblemet”

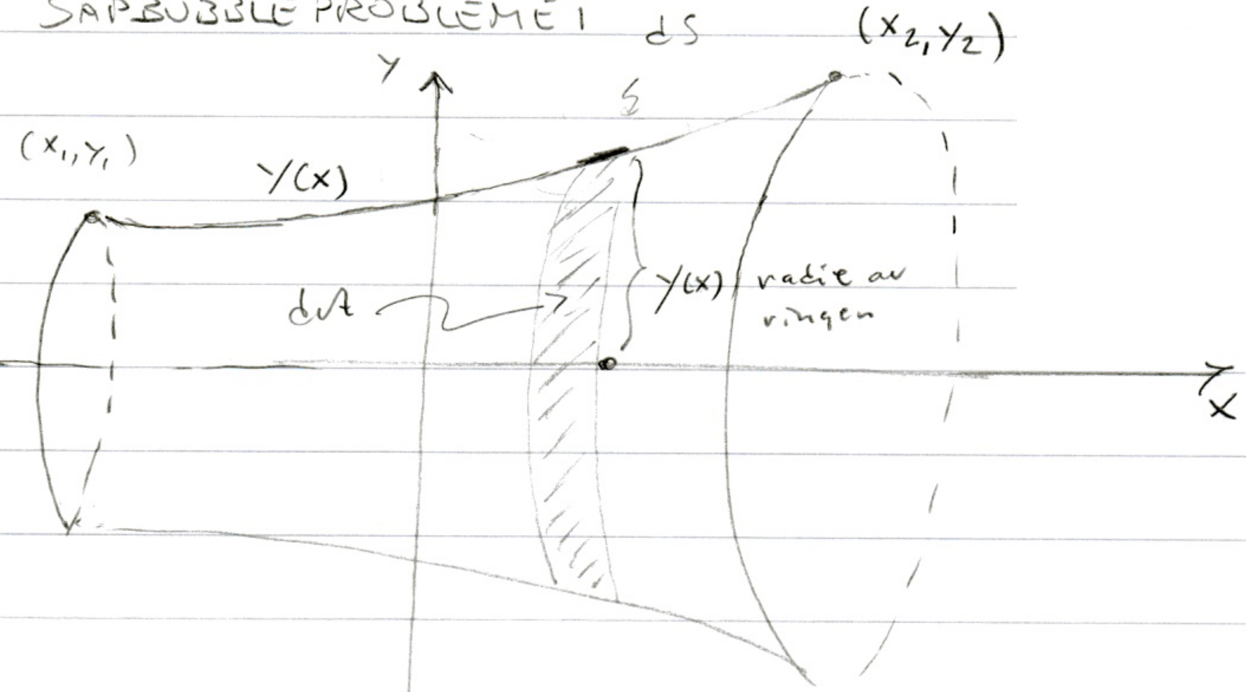
Einstein online

## From soap bubbles to Einstein

So what’s the connection with Einstein and his theories? In short: mathematics. The structure of the equations obeyed by minimal surfaces or soap bubbles is eerily similar to that of Einstein’s equations, the centerpiece of his general theory of relativity. To be sure, there are differences as well – for instance, the spacetimes of general relativity are intrinsically distorted, while minimal surfaces are distorted surfaces embedded in a higher-dimensional space. But there are strong analogies between the two geometrical situations, and many methods of finding solutions or studying properties of equations can be used in both contexts. Results from the theory of minimal surfaces give information that is important for analyzing solutions to Einstein’s equations, and vice versa.

föreläsning 19/11 (II)

EX → SÅPBUBBLEPROBLEMET



föreläsning 19/11 (II)

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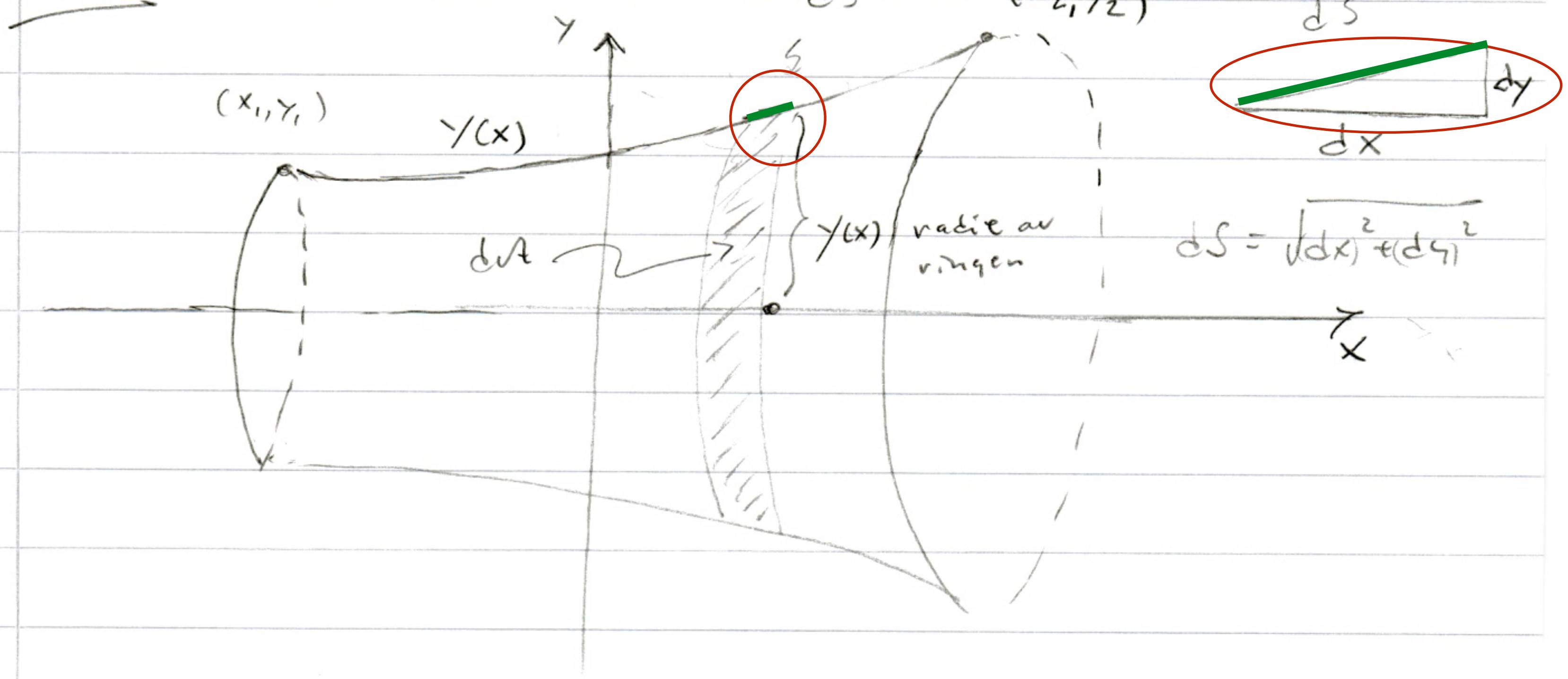

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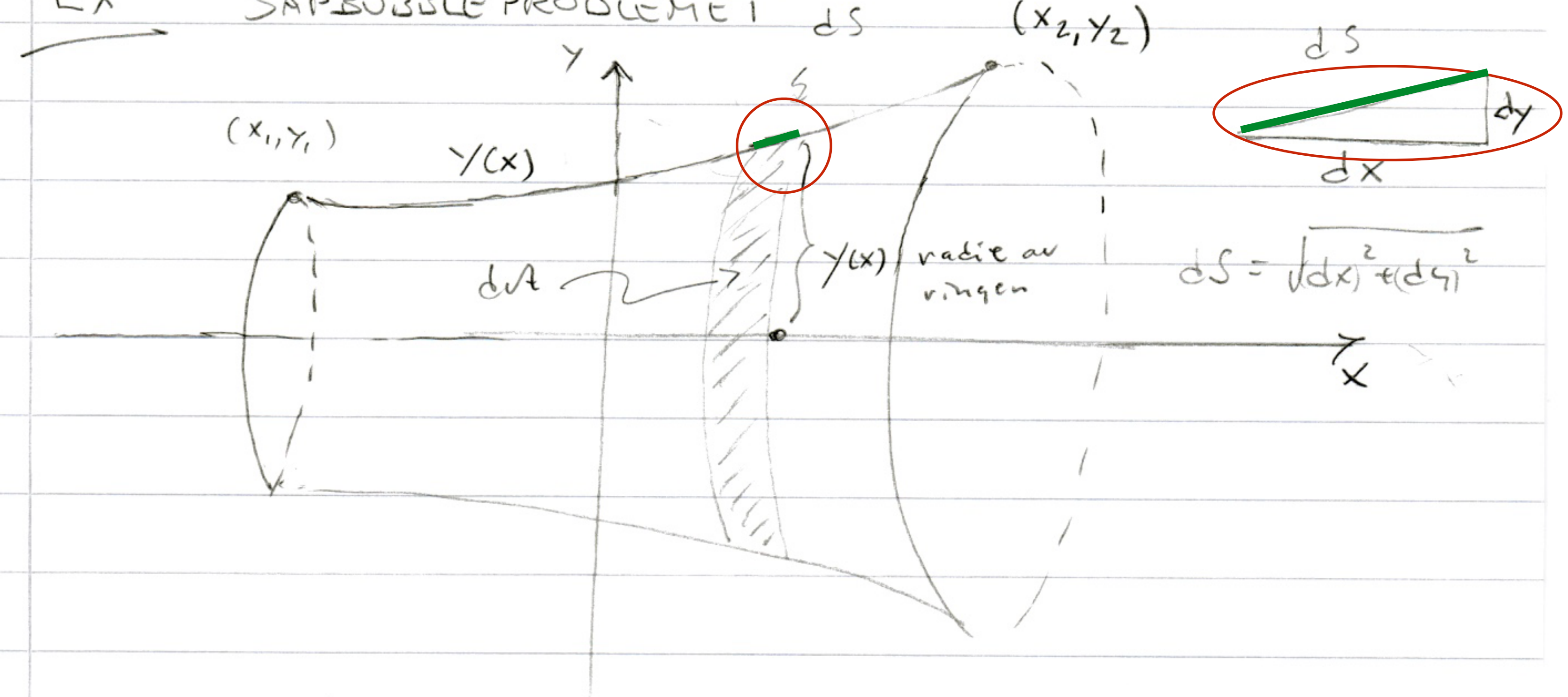


HUR SER YTAN UT? MINIMAL AREA $A$. VARFÖR?

$F = E - \cancel{TS}$ MINIMAL! $\Rightarrow$ $F = E = \sigma \cdot A$ MINIMAL

YTSPÄNNING AREA

föreläsning 19/11 (II)

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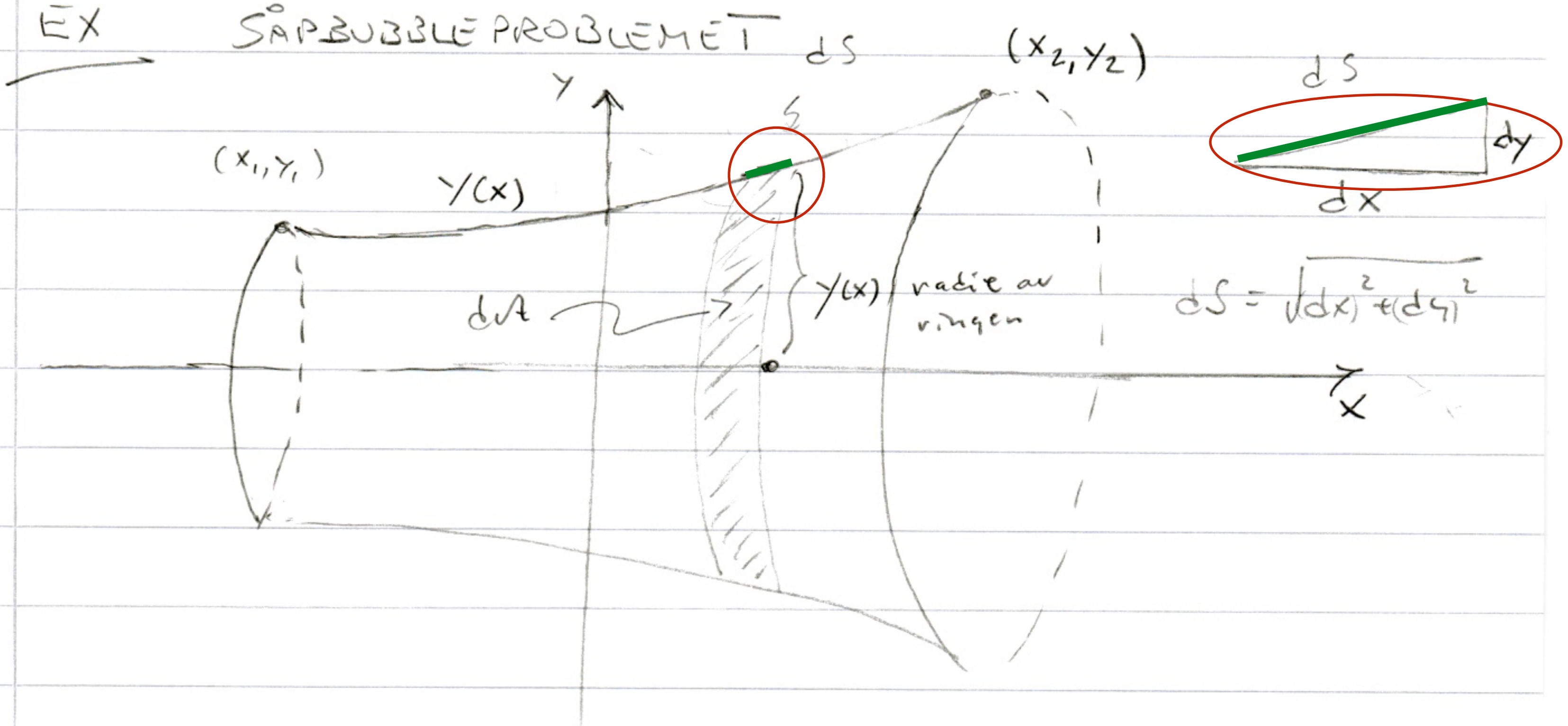


HUR SER YTAN UT? MINIMAL AREA $A$. VARFÖR?

$F = E - \cancel{TS}$ MINIMAL! $\Rightarrow$ $F = E = \underset{\text{YTSPÄNNING}}{\sigma} \cdot \underset{\text{AREA}}{A}$ MINIMAL

YTAN GENERERAS AV ATT ROTERA $y(x)$

RUNT $x$-AXELN (ROTATIONSSYMMETRI)

$$A = I = \int_{\text{ytan}} dA = \int \underbrace{2\pi y}_{\text{OMKRETS}} \, dS = 2\pi \int y \left( (dx)^2 + (dy)^2 \right)^{1/2}$$

$$= 2\pi \int_{x_1}^{x_2} \underbrace{y \left( 1 + y'^2 \right)^{1/2}}_{F(y(x),\, y'(x))} dx$$

för att knyta an till **variationskalkyl** vill vi beskriva ytan som en **integral**

Vi kan använda den förenklade Eulerekvationen ty inget explicit x-beorende i integranden

$F(y(x), y'(x)) = y(x)(1 + y'(x)^2)^{1/2}$

$$\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y'}\right) = 0 \Rightarrow F - y'\frac{\partial F}{\partial y'} = \overset{\text{KONST.}}{c} \tag{1}$$

$$\Rightarrow y(1+y'^2)^{1/2} - yy'^2\left(\frac{1}{1+y'^2}\right)^{1/2} = c \tag{2}$$

$$\Rightarrow \frac{y}{(1+y'^2)^{1/2}} = c \Rightarrow \frac{y^2}{(1+y'^2)} = c^2 \tag{3}$$

$$\Rightarrow \frac{1}{y'} = \frac{c}{(y^2-c^2)^{1/2}} \tag{4}$$

$$\Rightarrow \frac{1}{\left(\frac{dy}{dx}\right)} = \frac{dx}{dy} = \frac{1}{((y/c)^2-1)^{1/2}} \Rightarrow \tag{5}$$

$$\Rightarrow \int dx = \int \frac{1}{((y/c)^2-1)^{1/2}}\, dy \tag{6}$$

$$\Rightarrow x = c \cdot \operatorname{arccosh}(y/c) + c' \tag{7}$$

$$\Rightarrow y = c\cosh\left(\frac{x-c'}{c}\right) \tag{8}$$

Bestäm c och c' från randvillkoren $y(x_1) = y_1,\ y(x_2) = y_2$

Euler ger bara ett nödvändigt villkor för en lösning… vår fysikaliska intuition säger att det **måste finnas en lösning där energin (dvs. arean) är minimal**

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Men vi har ett annat problem: lösningen gäller inte säkert för alla val av parametrar, t.ex. $x_0$ = avståndet mellan det två ringarna

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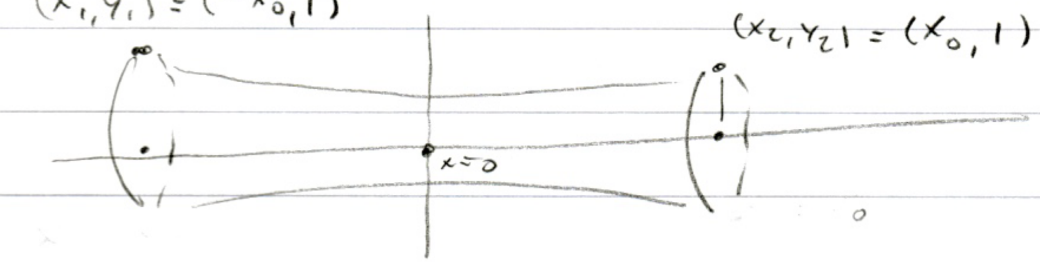
Men vi har ett annat problem: lösningen gäller inte säkert för alla val av parametrar, t.ex. för $x_0$ = avståndet mellan det två ringarna

Analys!

VÄLJ TVÅ LIKA STORA KONCENTRISKA RINGAR!




$$y = c\cosh\left(\frac{x-c'}{c}\right) \Rightarrow c' = 0 \text{ TY SYMMETRI KRING } x=0$$

$$y = c\cosh\left(\frac{x}{c}\right)$$

RANDVILLKOR: $1 = c\cosh\left(\frac{x_0}{c}\right)$ TVÅ LÖSNINGAR!

ANTAG: $x_0 = 1/2$

$$\Rightarrow 1 = c\cosh\left(\frac{1}{2c}\right)$$

$c \approx 0.24 \Rightarrow A \approx 6.85$

$c \approx 0.85 \Rightarrow A \approx 6.00$

min lösning

TESTA!

$$A = 2\pi\int_{-x_0}^{x_0} y\underbrace{(1+y'^2)^{1/2}}_{y/c}dx = \frac{4\pi}{c}\int_0^{x_0} y^2(x)\,dx = \frac{4\pi}{c}\int_0^{x_0}\left(c\cosh\left(\frac{x}{c}\right)\right)^2 dx$$

från rad 3 på föregående sida

ANTAG ISTÄLLET ATT $x_0 = 1$

$\Rightarrow \quad 1 = c\cosh\left(\frac{1}{c}\right) \quad \Rightarrow$ INGEN REELLVÄRD LÖSNING !

Tolkning: vi kan inte dra isär ringarna godtyckligt mycket för då brister såpbubblan!

Kritiskt värde $x_0 = x_{0c}$ då såpbubblan brister? ("Goldschmidtpunkten")

$$1 = c\cosh\left(\frac{x_0}{c}\right) \iff x_0 = c\,\mathrm{arccosh}\left(\frac{1}{c}\right)$$

HAR REELLVÄRD LÖSNING OMM

$c < 1$ MED MAXIMALT $x_0$

DÅ $\frac{dx_0}{dc} = 0$

$\Rightarrow \quad x_{0\,max} \approx x_{0c} \approx 0.66$

FÖR $c \approx 0.55$

Generalisering I av Euler:

# Flera beroende variabler

$$I = \int_{x_1}^{x_2} F\left[ y_1, y_2, \ldots, y_n ; y_1', y_2', \ldots, y_n' ; x \right] dx$$

$$\Downarrow \quad \delta I = 0$$

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\frac{\partial F}{\partial y_i'} = 0 \quad , \quad i = 1,2,\ldots,n$$

AWH, SID 1096f

VIKTIG TILLÄMPNING: EULER – LAGRANGE FÖR n PARTIKLAR

$$y_i \to x_i \, , \; y_i' \to \dot{x}_i \, , \; x \to t, \; F \to L$$

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = 0 \quad , \quad i = 1,2,\ldots,n$$

$$y_i \to x_i \to q_i \; , \; y_i' \to \dot{x}_i = \dot{q}_i \; , \; x \to t \, , \; F \to L \, , \; I \to S$$

$$L = K - V$$

$$S = \int_{t_1}^{t_2} L\left[ q_1(t), q_2(t), \ldots, q_n(t) ; \dot{q}_1(t), \dot{q}_2(t), \ldots, \dot{q}_n(t) ; t \right] dt$$

$$\Downarrow \quad \delta S = 0$$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0$$

EULER-LAGRANGE FÖR n PARTIKLAR (*)

$i = 1,2,\ldots,n$

Generalisering II av Euler:

# Flera oberoende variabler

EX VÄLJ $x_1 = x$, $x_2 = y$, $x_3 = z$

GENERALISERING TILL FLERA OBEROENDE VARIABLER $x_1, x_2, \ldots, x_n$

u=y i tidigare notation

$$I = \int F[u; u_x, u_y, u_z; x, y, z]\, dx\, dy\, dz$$

$$u_x = \frac{\partial u}{\partial x}$$

$$\Downarrow \quad \delta I = 0$$

$$\frac{\partial F}{\partial u} - \underbrace{\frac{d}{dx}\frac{\partial F}{\partial u_x}}_{} - \frac{d}{dy}\frac{\partial F}{\partial u_y} - \frac{d}{dz}\frac{\partial F}{\partial u_z} = 0$$

$$\frac{d}{dx}\frac{\partial}{\partial u_x} F[u(x,y,z); u_x(x,y,z), u_y(x,y,z), u_z(x,y,z); x, y, z]$$

AWH, SID 1100f