Greenfunktioner

centralt begrepp i teoretisk fysik

George Green
1793-1841

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BAKGRUND: INHOMOGEN DIFF EKV

$$L_x u(x) = f(x)$$

LINJÄR DIFF OPERATOR

T.EX. $L_x = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$

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$$L_x G(x,y) = \delta(x-y)$$

GREEN FUNKTION (TILL OPERATORN L_x)

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GREEN FUNKTION
(TILL OPERATORN L_x)

$$u(x) = \int G(x,y)\, f(y)\, dy$$

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LINJÄR DIFF OPERATOR

T.EX. $L_x = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$

BAKGRUND: INHOMOGEN DIFF EKV

$$L_x u(x) = f(x) \xrightarrow{\text{DIRAC NOTATION}} L|u\rangle = |f\rangle$$

LINJÄR DIFF OPERATOR

T.EX. $L_x = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$

BAKGRUND: INHOMOGEN DIFF EKV

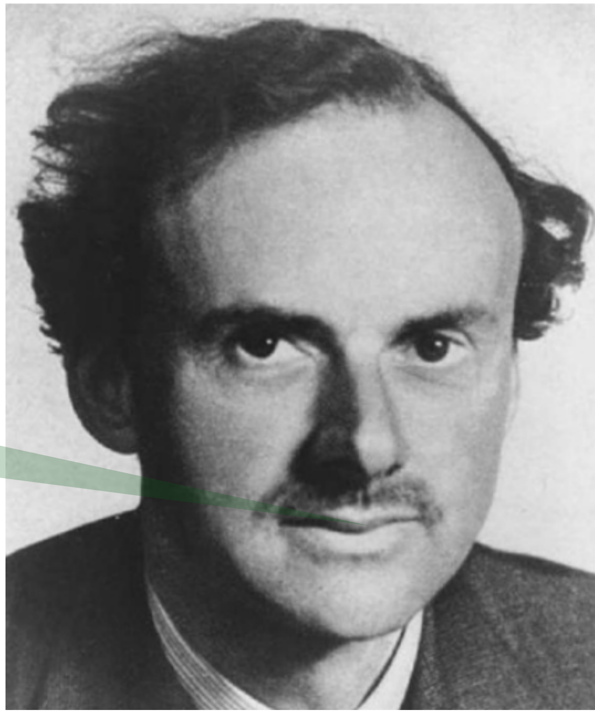
$$L_x\, u(x) = f(x) \xrightarrow[\text{NOTATION}]{\text{DIRAC}} L|u\rangle = |f\rangle$$

↑ LINJÄR DIFF OPERATOR

T.EX. $L_x = \dfrac{d^2}{dx^2} + p(x)\dfrac{d}{dx} + q(x)$

SNABBREPETITION AV DIRAC NOTATION

"God used beautiful mathematics to create the world." Paul Dirac



V KOMPLEX VEKTORRUM

$|\alpha\rangle \in V$, $\langle\alpha| \equiv |\alpha\rangle^* \in V'$ "DUALT TILL V"

INRE PRODUKT $\langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^* \in \mathbb{C}$

ON BAS $\{|\alpha\rangle\}$, $\langle\alpha|\alpha'\rangle = \delta_{\alpha\alpha'}$

BAKGRUND: INHOMOGEN DIFF EKV

$$L_x u(x) = f(x) \xrightarrow{\text{DIRAC NOTATION}} L|u\rangle = |f\rangle$$

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T.ex. $L_x = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$

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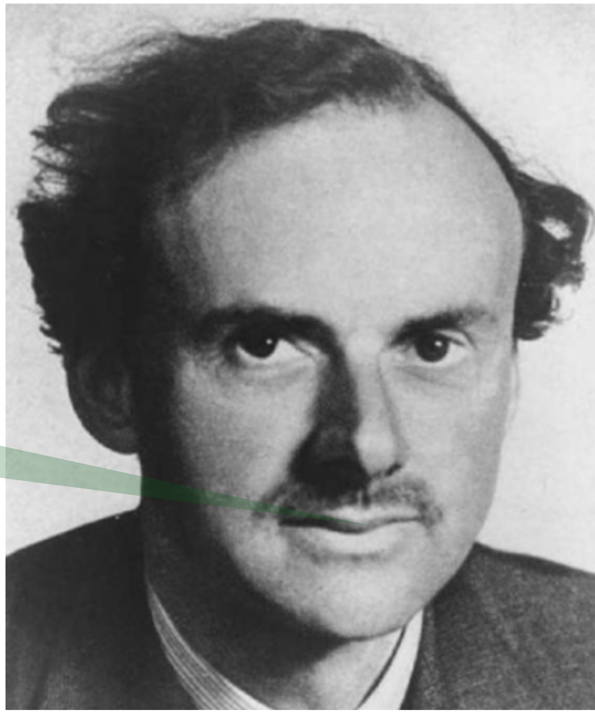

$$V \ni |v\rangle = \sum_\alpha v_\alpha |\alpha\rangle, \quad v_\alpha \in \mathbb{C}$$

här antar jag att V har ett uppräkneligt antal element

$$\langle\beta|v\rangle = \langle\beta|\sum_\alpha v_\alpha|\alpha\rangle = \sum_\alpha v_\alpha \underbrace{\langle\beta|\alpha\rangle}_{\delta_{\beta\alpha}} = v_\beta$$

"IMPROPER BASIS"

BETRAKTA EN ICKE-UPPRÄKNELIG BAS TILL V

$$\{ |x\rangle \}_{x \in \mathbb{R}}$$

$$\sum_\alpha \ldots \rightarrow \int dy \ldots$$

I EXAKT ANALOGI MED DET UPPRÄKNELIGA FALLET:

$$|f\rangle = \int dy \, \underbrace{f_y}_{\equiv f(y)} |y\rangle = \int dy \, f(y) |y\rangle$$

$$\langle x | f \rangle = \langle x | \int dy \, f(y) |y\rangle = \int dy \, f(y) \underbrace{\langle x | y \rangle}_{\delta(x-y)} = f(x)$$

Dirac-konstruktion av operatorer

$$L = |v\rangle\langle v'|$$

EX

$$|v\rangle = v_1|1\rangle + v_2|2\rangle$$

$$|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$|v\rangle = v_1\begin{pmatrix}1\\0\end{pmatrix} + v_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}\langle 1|v\rangle\\ \langle 2|v\rangle\end{pmatrix}$$

$$|v\rangle^* \xleftarrow{} |v\rangle^+ = \langle v| = (v_1^* \;\; v_2^*) = (\langle v|1\rangle \;\; \langle v|2\rangle)$$

$$|v'\rangle = \begin{pmatrix}v_1'\\v_2'\end{pmatrix}, \quad \langle v'| = (v_1'^* \;\; v_2'^*)$$

$$|v\rangle\langle v'| = \begin{pmatrix}v_1\\v_2\end{pmatrix} \otimes (v_1'^* \;\; v_2'^*)$$

$$= \begin{pmatrix} v_1(v_1'^* \;\; v_2'^*) \\ v_2(v_1'^* \;\; v_2'^*) \end{pmatrix} = \begin{pmatrix} v_1 v_1'^* & v_1 v_2'^* \\ v_2 v_1'^* & v_2 v_2'^* \end{pmatrix}$$

MATRISREPRESENTATION
AV OPERATOREN L

Dirac-konstruktion av operatorer

$L = |v\rangle\langle v'|$

Ex $|v\rangle = v_1|1\rangle + v_2|2\rangle$

$|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

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$$= \begin{pmatrix}v_1(v_1'^* \; v_2'^*)\\v_2(v_1'^* \; v_2'^*)\end{pmatrix} = \begin{pmatrix}v_1v_1'^* & v_1v_2'^*\\v_2v_1'^* & v_2v_2'^*\end{pmatrix}$$

MATRISREPRESENT. AV OPERATOREN L

Upplösning av identiteten…

Upplösning av identiteten...

"resolution of identity"

$$|v\rangle = \sum_\alpha \underbrace{v_\alpha}_{\langle\alpha|v\rangle} |\alpha\rangle = \sum_\alpha \langle\alpha|v\rangle|\alpha\rangle = \underbrace{\sum_\alpha |\alpha\rangle\langle\alpha|}_{=\mathbb{1}}\, |v\rangle$$

$$\Rightarrow \sum_\alpha |\alpha\rangle\langle\alpha| = \underline{\mathbb{1}}$$ OM $\{|\alpha\rangle\}$ KOMPLETT MÄNGD AV VEKTORER, T.EX. ON BAS

ANALOGT FÖR ICKE-UPPRÄKNELIGA FALLET

$$|f\rangle = \int dx\, f(x)|x\rangle = \int dx\, \langle x|f\rangle|x\rangle$$

$$= \underbrace{\int dx\, |x\rangle\langle x|}_{=\mathbb{1}}\,|f\rangle \Rightarrow \int dx\, |x\rangle\langle x| = \underline{\mathbb{1}}$$

OM $\{|x\rangle\}$ KOMPLETT MÄNGD

Tillbaks till Greenfunktioner!

$$L_x u(x) = f(x) \longrightarrow \boxed{L|u\rangle = |f\rangle}$$

Tillbaks till Greenfunktioner!

$$L_x u(x) = f(x) \longrightarrow L|u\rangle = |f\rangle$$

?

Tillbaks till Greenfunktioner!

$$L_x u(x) = f(x) \longrightarrow \boxed{L|u\rangle = |f\rangle}$$

KOORDINATREPRESENTATION AV $L|u\rangle = |f\rangle$

MULTIPLICERA $L|u\rangle = |f\rangle$ MED $\langle x|$ FRÅN VÄNSTER

$$\langle x|L|u\rangle = \langle x|f\rangle = f(x)$$

resolution of identity

$$\langle x|L|u\rangle = \langle x|L\mathbb{1}|u\rangle = \langle x|L\Big(\underbrace{\int dy\,|y\rangle\langle y|}_{=\mathbb{1}}\Big)|u\rangle$$

$$= \int dy\,\underbrace{\langle x|L|y\rangle}\langle y|u\rangle$$

$L_{xy} = L_x\,\delta(x-y)$ OM L ÄR EN "LOKAL OPERATOR"

$$= \int dy\,L_x\,\delta(x-y)\langle y|u\rangle = L_x\langle x|u\rangle = L_x u(x)$$

$$\Rightarrow L_x u(x) = f(x)$$

föreläsning 3/11 (II)

$L_x u(x) = f(x) \longrightarrow$ $\boxed{L|u\rangle = |f\rangle}$

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ANTAG ATT L ÄR INVERTERBAR! DVS.

ANTAG $\exists G : G = L^{-1}$

föreläsning 3/11 (II)

$L_x u(x) = f(x) \longrightarrow \boxed{L|u\rangle = |f\rangle}$

KOORDINATREPRESENTATION AV $L|u\rangle = |f\rangle$

MULTIPLICERA $L|u\rangle = |f\rangle$ MED $\langle x|$ FRÅN VÄNSTER

$$\langle x|L|u\rangle = \langle x|f\rangle = f(x)$$

$$\langle x|L|u\rangle = \langle x|L\overset{\text{resolution of identity}}{\mathbb{1}}|u\rangle = \langle x|L\Big(\underbrace{\int dy\, |y\rangle\langle y|}_{=\mathbb{1}}\Big)|u\rangle$$

ERATOR"

$$= \int dy\, L_x\, \delta(x-y)\langle y|u\rangle = L_x\langle x|u\rangle = L_x u(x)$$

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$$\underbrace{GL}_{=\mathbb{1}}|u\rangle = G|f\rangle$$

SÄTT IN

$\mathbb{1} = \int_\Omega dy\, |y\rangle\langle y|$

Ω DEF OMRÅDE FÖR G OCH f

$$|u\rangle = G|f\rangle$$

$$u(x) = \langle x|u\rangle = \langle x|G|f\rangle = \langle x|G\mathbb{1}|f\rangle$$

$$= \int_\Omega dy\, \underbrace{\langle x|G|y\rangle}_{G_{xy} = G(x,y)}\underbrace{\langle y|f\rangle}_{f(y)} = \int dy\, G(x,y) f(y)$$

Hur hitta ekvationen för *G(x,y)*? Dirac-notation!

UNDERSÖK $\langle x | \overbrace{LG}^{=1\!\!1} | y \rangle = \langle x | L 1\!\!1 G | y \rangle = \langle x | 1\!\!1 | y \rangle = \langle x | y \rangle = \delta(x-y)$

EN $\langle x | L \overbrace{\left(\int dx' |x'\rangle\langle x'| \right)}^{1\!\!1} G | y \rangle = \delta(x-y)$

$\Rightarrow \int dx' \underbrace{\langle x | L | x' \rangle}_{L(x,x') = L_x \delta(x-x')} \langle x' | G | y \rangle = \delta(x-y)$

$\Rightarrow L_x G(x,y) = \delta(x-y)$

GREENFUNKTION FÖR L_x

DVS $G(x,y)$ ÄR LÖSNINGEN TILL $L_x G(x,y) = \delta(x-y)$

OCH LÖSNINGEN TILL DIFFEKV. $L_x u(x) = f(x)$ GES AV

$$u(x) = \int G(x,y) f(y) dy$$

Låt oss titta på ett exempel: Poissons ekvation!

$$\nabla^2 \Phi(\vec{r}) = -\rho(\vec{r})$$

PUNKTKÄLLA I $\vec{r} = \vec{r}'$

$$\nabla^2 G(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}')$$

TRANSLATIONS INVARIANS

$$\nabla^2 G(\vec{r} - \vec{r}') = \delta(\vec{r} - \vec{r}') \qquad (1)$$

HUR BESTÄMMA G? VANLIGASTE METODEN I FYSIKEN: FOURIERTRANSFORMERA!

$$\begin{cases} G(\vec{r} - \vec{r}') = \frac{1}{(2\pi)^3} \int d\vec{k}\, e^{i\vec{k}\cdot(\vec{r} - \vec{r}')} G(\vec{k}) \\ \delta(\vec{r} - \vec{r}') = \frac{1}{(2\pi)^3} \int d\vec{k}\, e^{i\vec{k}\cdot(\vec{r} - \vec{r}')} \cdot 1 \quad \text{"FOURIERS INTEGRAL"} \end{cases} \qquad (2)$$

$$(1) \,\&\, (2) \Rightarrow -k^2 G(\vec{k}) = 1 \Rightarrow G(\vec{k}) = -\frac{1}{k^2}$$

$$\Rightarrow G(\vec{r} - \vec{r}') = \frac{1}{(2\pi)^3} \int d\vec{k} \left(-\frac{e^{i\vec{k}\cdot(\vec{r} - \vec{r}')}}{k^2} \right) = -\frac{1}{4\pi |\vec{r} - \vec{r}'|}$$

lösningen till diff. ekv.

$$\Rightarrow \Phi(\vec{r}) = \int d\vec{r}'\, G(\vec{r} - \vec{r}') \left(-\rho(\vec{r}') \right) = \int d\vec{r}'\, \frac{1}{4\pi |\vec{r} - \vec{r}'|} \rho(\vec{r}')$$

RESPONS ("OUTPUT") — RESPONS FUNKTION (GREEN FUNKTION) — STÖRNING ("INPUT")

Låt oss titta på ett exempel: Poissons ekvation!

$$\nabla^2 \Phi(\vec{r}) = -\rho(\vec{r})$$

PUNKTKÄLLA I $\vec{r} = \vec{r}\,'$

$$\nabla^2 G(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,')$$

TRANSLATIONS INVARIANS

$$\nabla^2 G(\vec{r} - \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \qquad (1)$$

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"FOURIERS INTEGRAL"

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lösningen till diff. ekv.

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RESPONS ("OUTPUT")

RESPONS FUNKTION (GREEN FUNKTION)

STÖRNING ("INPUT")

"Linjär respons"

George Green
1793-1841

George Green
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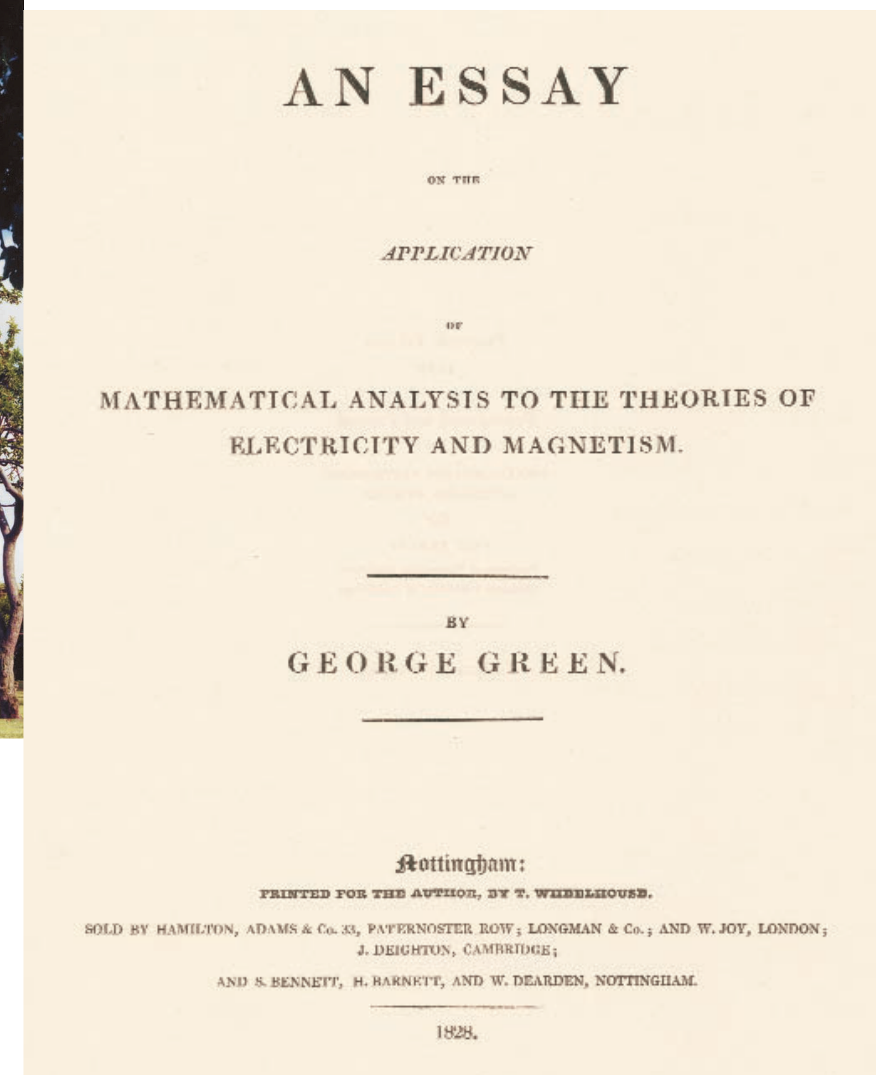

AN ESSAY

ON THE

APPLICATION

OF

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BY

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1828.

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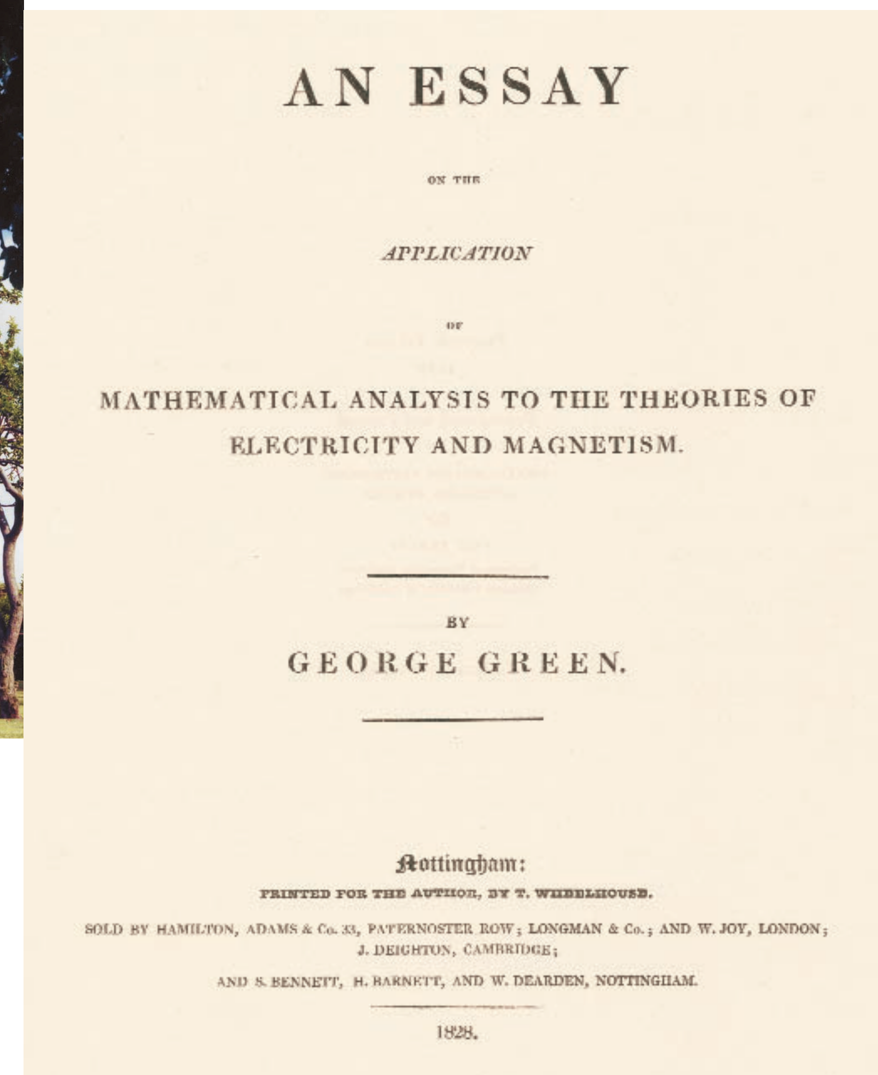

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