

# Greenfunktioner

centralt begrepp i teoretisk fysik



George Green  
1793-1841

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centralt begrepp i teoretisk fysik

BAKGRUND: INHOMOGEN DIFF EKV

$$L_x u(x) = f(x)$$

↑ LINJÄR DIFF OPERATOR

T.ex.  $L_x = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$



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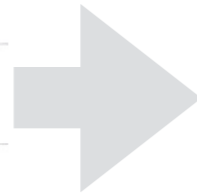
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$$L_x G(x,y) = \delta(x-y)$$

↑ GREEN FUNKTION  
(TILL OPERATORN  $L_x$ )



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$$u(x) = \int G(x,y) f(y) dy$$



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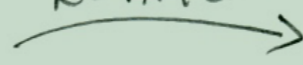
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BAKGRUND: INHOMOGEN DIFF EKV

$$L_x u(x) = f(x)$$

DIRAC  
NOTATION



$$L|u\rangle = |f\rangle$$

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SNABBREPETITION AV DIRAC NOTATION

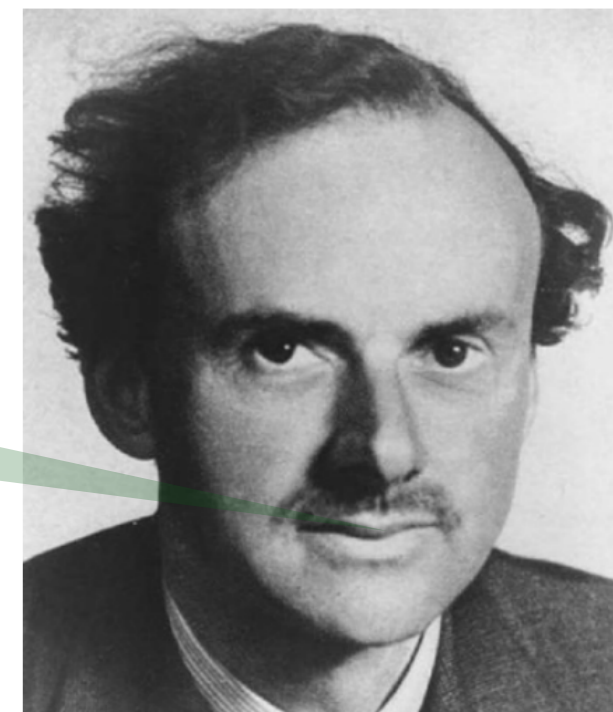
$\mathcal{V}$  KOMPLEX VEKTORUTRYM

$$|\alpha\rangle \in \mathcal{V}, \quad \langle\alpha| \equiv |\alpha\rangle^* \in \mathcal{V}' \quad \left\{ \begin{array}{l} \text{"DUALT TILL"} \\ \mathcal{V} \end{array} \right.$$

$$\text{INRE PRODUKT } \langle\alpha|\beta\rangle = \langle\beta|\alpha\rangle^* \in \mathbb{C}$$

$$\text{ON BAS } \{|\alpha\rangle\}, \quad \langle\alpha|\alpha'\rangle = \delta_{\alpha\alpha'}$$

"God used beautiful mathematics to create the world." Paul Dirac



BAKGRUND: INHOMOGEN DIFF EKV

$$L_x u(x) = f(x)$$

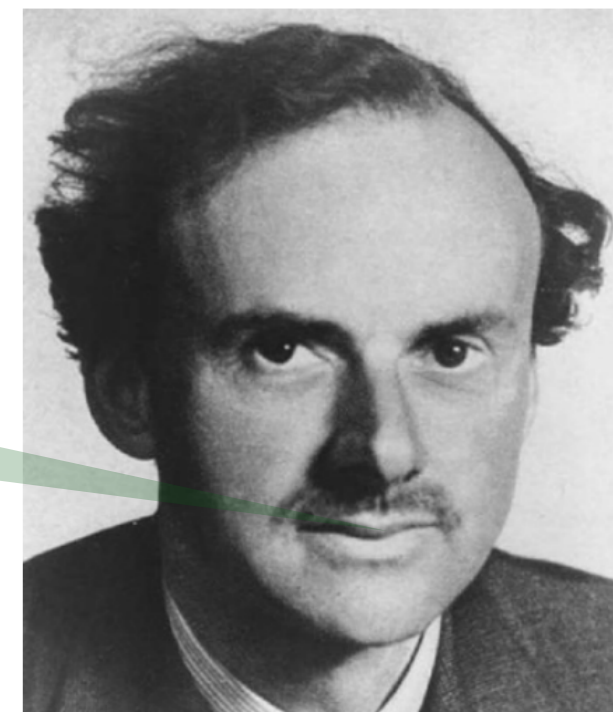
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to create the world." Paul Dirac



$$\mathcal{V} \ni |v\rangle = \sum_{\alpha} \bar{v}_{\alpha} |\alpha\rangle, \quad v_{\alpha} \in \mathbb{C} \quad \text{här antar jag att } \mathcal{V} \text{ har ett uppskattat antal element}$$

$$\langle \beta | v \rangle = \langle \beta | \sum_{\alpha} \bar{v}_{\alpha} |\alpha\rangle = \sum_{\alpha} v_{\alpha} \underbrace{\langle \beta | \alpha \rangle}_{\delta_{\beta\alpha}} = v_{\beta}$$



↑ "IMPROPER BASIS"

BETRÄKTA EN ICKE-UPPRÄKNELIG BAS TILL  $V$

$$\{ |x\rangle \}$$

↑  
 $x \in \mathbb{R}$

$$\sum_x \dots \rightarrow \int dy \dots$$

I EXAKT ANALOGI MED DET UPPRÄKNELIGA FALLET :

$$|f\rangle = \int dy \underbrace{f_y}_{= f(y)} |y\rangle = \int dy f(y) |y\rangle$$

$$\langle x | f \rangle = \langle x | \int dy f(y) |y\rangle = \int dy f(y) \underbrace{\langle x | y \rangle}_{\delta(x-y)} = f(x)$$

## Dirac-konstruktion av operatorer

$$L = |v\rangle\langle v'|$$

EX  $|v\rangle = v_1|1\rangle + v_2|2\rangle$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|v\rangle = v_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + v_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \langle 1|v\rangle \\ \langle 2|v\rangle \end{pmatrix}$$

$$|v\rangle^* \stackrel{\leftarrow |v\rangle^+}{=} \langle v| = (v_1^* \ v_2^*) = (\langle v|1\rangle \ \langle v|2\rangle)$$

$$|v'\rangle = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix}, \quad \langle v'| = (v_1'^* \ v_2'^*)$$

$$|v\rangle\langle v'| = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \otimes (v_1'^* \ v_2'^*)$$

$$= \begin{pmatrix} v_1 (v_1'^* \ v_2'^*) \\ v_2 (v_1'^* \ v_2'^*) \end{pmatrix} = \begin{pmatrix} v_1 v_1'^* & v_1 v_2'^* \\ v_2 v_1'^* & v_2 v_2'^* \end{pmatrix}$$

↑  
MATRIXREPRESENTATION  
AV OPERATOREN L

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↑  
MATRIXREPRESENTATION  
AV OPERATOREN L

Upplösning av identiteten...



# Upplösning av identiteten...

"resolution of identity"



$$|v\rangle = \sum_{\alpha} \underbrace{v_{\alpha}}_{\langle \alpha | v \rangle} |\alpha\rangle = \sum_{\alpha} \langle \alpha | v \rangle |\alpha\rangle = \sum_{\alpha} |\alpha\rangle \langle \alpha | v \rangle$$

$= \mathbb{1}$

$$\Rightarrow \sum_{\alpha} |\alpha\rangle \langle \alpha| = \mathbb{1} \quad \text{om } \{|\alpha\rangle\} \text{ komplett mängd av vektorer, t. ex. ON BAS}$$

ANALOGT FÖR  $100\%$ -UPPRÄKNINGSA FALLET

$$|f\rangle = \int dx f(x) |x\rangle = \int dx \langle x | f \rangle |x\rangle$$

$$= \underbrace{\int dx |x\rangle \langle x|}_{=\mathbb{1}} f \Rightarrow \int dx |x\rangle \langle x| = \mathbb{1}$$

om  $\{|x\rangle\}$  komplett mängd

## Tillbaks till Greenfunktioner!

$$L_x u(x) = f(x) \longrightarrow L|u\rangle = |f\rangle$$

## Tillbaks till Greenfunktioner!

$$L_x u(x) = f(x) \quad \xrightarrow{\quad} \quad L|u\rangle = |f\rangle$$

←  
?

## Tillbaks till Greenfunktioner!

$$L_x u(x) = f(x) \longrightarrow L|u\rangle = |f\rangle$$

KOORDINATREPRESENTATION AV  $L|u\rangle = |f\rangle$

MULTIPLICERA  $L|u\rangle = |f\rangle$  MED  $\langle x|$  FRÅN VÄNSTER

$$\langle x|L|u\rangle = \langle x|f\rangle = f(x)$$

$$\langle x|L|u\rangle = \langle x|L\mathbb{1}|u\rangle = \langle x|L\left(\int dy |y\rangle\langle y|\right)|u\rangle$$

resolution of identity

=  $\mathbb{1}$

$$= \int dy \langle x|L|y\rangle \langle y|u\rangle$$

$L_{xy} = L_x \delta(x-y)$  OM  $L$  ÄR EN "LOKAL OPERATOR"

$$= \int dy L_x \delta(x-y) \langle y|u\rangle = L_x \langle x|u\rangle = L_x u(x)$$

$$\Rightarrow L_x u(x) = f(x)$$



$$L_x u(x) = f(x) \xrightarrow{\text{Fourier}} Lu = |f\rangle$$

KOORDINATREPRESENTATION AV  $Lu = |f\rangle$

MULTIPLICERA  $Lu = |f\rangle$  MED  $\langle x|$  FRÅN VÄNSTER

$$\langle x|Lu\rangle = \langle x|f\rangle = f(x)$$

$$\langle x|Lu\rangle = \langle x|L\mathbb{1}|u\rangle = \langle x|L \left( \int dy |y\rangle\langle y| \right) |u\rangle$$

resolution  
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$L_{xy} = L_x \delta(x-y)$  om  $L$  ÄR EN "LOKAL OPERATOR"

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ANTAG ATT  $L$  ÄR INVERTERBAR! DVS.

$$\text{ANTAG } \exists G : G = L^{-1}$$

$$L_x u(x) = f(x) \iff L|u\rangle = |f\rangle$$

COORDINATREPRESENTATION AV  $L|u\rangle = |f\rangle$

MULTIPLICERA  $L|u\rangle = |f\rangle$  MED  $\langle x|$  FRÅN VÄNSTER

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resolution of identity

=  $\mathbb{1}$

$f(x) = \dots$

operator

$$= \int dy L_x \delta(x-y) \langle y|u\rangle = L_x \langle x|u\rangle = L_x u(x)$$

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$$G L |u\rangle = G |f\rangle$$

$\mathbb{1}$

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resolution of identity

=  $\mathbb{1}$

föreläsning 4/11 (II)

operator

$$= \int dy L_x \delta(x-y) \langle y|u\rangle = L_x \langle x|u\rangle = L_x u(x)$$

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$\mathbb{1}$

$$|u\rangle = G |f\rangle$$

SÄTT IN

$$\mathbb{1} = \int dy |y\rangle\langle y|$$

↓  
-2- är det område  
för  $G$  och  $f$

$$u(x) = Lx|u\rangle = \langle x|G|f\rangle = \langle x|G\mathbb{1}|f\rangle$$

$$= \int dy \langle x|G|y\rangle \langle y|f\rangle = \int dy G(x,y) f(y)$$

$G_{xy} = G(x,y)$

Hur hitta ekvationen för  $G(x,y)$ ? Dirac-notation!

undersök  $\langle x | \overbrace{L G}^{=1} | y \rangle = \langle x | L \underbrace{1}_{=1} G | y \rangle = \langle x | 1 | y \rangle = \langle x | y \rangle = \delta(x-y)$

$\Rightarrow \langle x | L \left( \int dx' |x'\rangle \langle x'| \right) G | y \rangle = \delta(x-y)$

$\Rightarrow \int dx' \underbrace{\langle x | L | x' \rangle}_{L(x,x') = L_x \delta(x-x')} \langle x' | G | y \rangle = \delta(x-y)$

$\Rightarrow L_x G(x,y) = \delta(x-y)$

↑ GREENFUNKTION FÖR  $L_x$

DVS  $G(x,y)$  ÄRE LÖSNINGEN TILL  $L_x G(x,y) = \delta(x-y)$   
 OCH LÖSNINGEN TILL DIFF. KV.  $L_x u(x) = f(x)$  GES AV

$u(x) = \int G(x,y) f(y) dy$

## Låt oss titta på ett exempel: Poissons ekvation!

$$\nabla^2 \phi(\vec{r}) = -\rho(\vec{r})$$

$$\nabla^2 G(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}') \quad \leftarrow \text{punktvärde i } \vec{r} = \vec{r}'$$

TRANSLATIONS  
INVARIANS

$$\nabla^2 G(\vec{r} - \vec{r}') = \delta(\vec{r} - \vec{r}') \quad (1)$$

HUR BESTÄMMA  $G$ ? VANLIGASTE METODEN I FYSIKEN:  
FOURIERTRANSFORMERA!

$$\begin{cases} G(\vec{r} - \vec{r}') = \frac{1}{(2\pi)^3} \int d\vec{k} e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} G(\vec{k}) \\ \delta(\vec{r} - \vec{r}') = \frac{1}{(2\pi)^3} \int d\vec{k} e^{i\vec{k} \cdot (\vec{r} - \vec{r}')} \cdot \text{"FOURIERS INTEGRAL"} \end{cases} \quad (2)$$

$$\text{ut } \& \text{(2)} \Rightarrow -k^2 G(\vec{k}) = 1 \Rightarrow G(\vec{k}) = -\frac{1}{k^2}$$

$$\Rightarrow G(\vec{r} - \vec{r}') = \frac{1}{(2\pi)^3} \int d\vec{k} \left( -\frac{e^{i\vec{k} \cdot (\vec{r} - \vec{r}')}}{k^2} \right) = -\frac{1}{4\pi |\vec{r} - \vec{r}'|}$$

lösningen  
till diff. eq.

$$\Rightarrow \phi(\vec{r}) = \int d\vec{r}' G(\vec{r} - \vec{r}') (-\rho(\vec{r}')) = \int d\vec{r}' \frac{1}{4\pi |\vec{r} - \vec{r}'|} \rho(\vec{r}')$$

RESPONS  
("OUTPUT")RESPONS  
FUNKTION  
(GREEN FUNKTION)STÖRNING  
("INPUT")

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TRANSLATIONS  
INVARIANS

$$\nabla^2 G(\vec{r} - \vec{r}') = \delta(\vec{r} - \vec{r}') \quad (1)$$

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RESPONS  
("OUTPUT")RESPONS  
FUNKTION  
(GREEN FUNKTION)STÖRNING  
("INPUT")

"Linjär respons"



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AN ESSAY  
ON THE  
APPLICATION  
OF  
MATHEMATICAL ANALYSIS TO THE THEORIES OF  
ELECTRICITY AND MAGNETISM.

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BY  
GEORGE GREEN.

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