Problems in Mathematical Physics (autumn 2006)

1. Solve the differential equation $\cot(y) + x^2 - \frac{x}{\sin^2(y)} y' = 0$

2. Solve the differential equation $2x^3 y' = 1 + \sqrt{1 + 4x^2 y}$

3. Classify all singular points (finite and infinite) of the following differential equations:

   (a) $y'' = xy$ (Airy equation)

   (b) $x^2 y'' + xy' + (x^2 - \nu^2) y = 0$ (Bessel equation)

   (c) $xy'' + (b - x) y' - ay = 0$ (Kummer's confluent hypergeometric equation)

   (d) $y'' + (h - 2\theta \cos(2x)) y = 0$ (Mathieu equation)

4. The Atomic Energy Commission in the US put nuclear waste in sealed containers and dumped them in the ocean. The containers do not break (on impact) if they hit the sea floor with a speed that is less than 12 ms^{-1}. Newton's second law and Archimedes' principle yield the equation of motion

$$m \frac{dv}{dt} = W - B - kv, \qquad v(0) = 0$$

   where W is the weight of the container, B its buoyancy, and $-kv$ is water resistance. Determine the maximum safe ocean depth for $W = 2.3kN, B = 2.1kN, k = 0.6kgs^{-1}$.

5. Evaluate $\int_0^\infty dy\, \frac{e^{-ay} - e^{-by}}{y}$.

6. Evaluate $\int_0^\infty dx\, \frac{\cos \alpha x}{1 + x^2}$.

7. Evaluate $\int d^3x \mathbf{x}\, e^{i\mathbf{a} \cdot \mathbf{x}} e^{-bx^2}$

8. Evaluate $\int d\Omega \frac{\hat{\mathbf{r}}}{1 + \mathbf{k} \cdot \hat{\mathbf{r}}}$ for $k < 1$. The integral runs over the unit sphere in the $\mathbf{r}$ space.

9. Evaluate

$$\mathbf{P}(\mathbf{E}) = \int d^3x\, \mathbf{x}\, e^{-\beta\left(\frac{1}{2}kx^2 - q\mathbf{x} \cdot \mathbf{E}\right)}$$

10. When a realistic electrostatic problem is approximated by one that involves point charges on a (one dimensional) lattice, one often runs into trouble with divergent integrals, *e.g.* the interaction energy of charges at positions x_i and x_j, $V_{ij} = C/|x_i - x_j|$, diverges for $x_j = x_i$. To overcome such problems the integrals must be regularized in some way. One way to regularize them is to smear out the point charges to smooth charge distributions around the lattice points.

    (a) Show that smearing in one dimension is not sufficient, that is, define

    $$V_{ij} = \int dx \int dx' \rho_i(x)\rho_j(x') \frac{1}{|x - x'|}$$

    with $\rho_i(x) = q_i\sqrt{\frac{\alpha}{\pi}}e^{-\alpha(x-x_i)^2}$, and show that V_{ii} still diverges.

    (b) Now consider smearing in two dimension: set $\rho_i(x) = q_i\frac{\alpha}{\pi}e^{-\alpha[(x-x_i)^2+y^2]}$ and evaluate V_{ii}.

    Hint: write $\frac{1}{\sqrt{(x-x_i)^2+y^2}}$ as an auxiliary Gaussian integral over a dummy variable s, and perform the x- and y-directional integrals using relative and center-of-mass coordinates.

11. Evaluate $\int_{-\infty}^{\infty} dx\, \delta'(x) f(x)$ where $f(x)$ is a sufficiently well-behaved function. What does 'sufficiently well-behaved' mean in this case?

12. Try to evaluate $\int_{-\infty}^{\infty} dx\, \delta(x^2) f(x)$.

    Hint: Consider a sequence of functions $\delta_n(x)$ such that $\delta_n(x) \to \delta(x)$ as $n \to \infty$, and evaluate the integral as the limit $\lim_{n\to\infty} \int_{-\infty}^{\infty} dx\, \delta_n(x^2) f(x)$. Check your result for consistency using $\delta_n^{(1)}(x) = \frac{n}{\pi}\frac{1}{n^2x^2+1}$ and $\delta_n^{(2)}(x) = \sqrt{\frac{n}{\pi}}e^{-nx^2}$.

13. Show that $\delta(f(x)) = \frac{1}{|f'(x_0)|}\delta(x - x_0)$ where x_0 is a root of f and x is confined to values close to x_0. Hint: Make change of variables to $y = f(x)$.

14. (*Augustin-Louis Cauchy's pathological function*)

    Consider the function $f(x) = \begin{cases} e^{-\frac{1}{x^2}}, x \neq 0 \\ 0, x = 0 \end{cases}$ where $x \in \mathbb{R}$.

    (a) Show that $f(x)$ is continuous and differentiable at $x = 0$.

(b) Expand $f(x)$ as a Taylor series around $x = 0$, $\sum_{n=0}^{\infty} a_n x^n$.

(c) For which values of x is the sum of the series equal to $f(x)$? Explain!

15. MW 3-12

16. MW 3-16

17. Evaluate $\sum_{n=-\infty}^{\infty} \frac{1}{n^3+a^3}$ for $a \in \mathbb{R}$.

18. Evaluate $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n^4+a^4}$ for $a \in \mathbb{R}$.

19. In the analysis of 2-dimensional classical systems or 1-dimensional quantum mechanical systems one often encounters so-called elliptic functions, which are in some sense generalizations of trigonometric functions (they are analytic except for finite number of poles in a period parallelogram) and possess two periods τ_1 and τ_2 such that $f(z + n\tau_1 + m\tau_2) = f(z)$ for $n, m \in \mathbb{Z}$, and $\tau_1/\tau_2 \notin \mathbb{R}$). One elliptic function (which is not really elliptic according to the above definition but can be used to generate doubly periodic functions) is called the Jacobi theta function of the third kind, and is defined through

$$\vartheta_3(u, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{i2nu}$$

where $0 < q < 1$ and $u \in \mathbb{C}$.

(a) Find an approximate expression for $\vartheta_3(u, q)$ valid for small q.

(b) Find an approximate expression for $\vartheta_3(u, q)$ valid for large q ($i.e.$, $q \lesssim 1$).

20. Sum the series $1 - 2 + 4 - 8 + \ldots$ using Euler and Borel summation and show that the results agree.

21. Compute the (generalized) Borel sum $S(x) = \sum_{n=0}^{\infty} (-1)^n (2n)!\, x^n$.

22. MW 1-35

23. Consider the modified Bessel equation

$$x^2 y'' + xy' - (x^2 + n^2)y = 0$$

and

a) determine all finite and infinite singular points (both regular and irregular)

b) determine the leading behaviors of the solutions $y(x)$ for small x

c) determine the asymptotic behavior of the solutions for large x

d) comment on the analytic properties of the solutions near ordinary points, regular singular points, and irregular singular points

24. Find the asymptotic behavior as $x \to 0^+$ of the solutions of $x^4 y'' - 3x^2 y' + 2y = 0$.

25. Find the asymptotic behavior as $x \to \infty$ of the solution of $y'' = (\ln x)^2 y$, *i.e.* find the terms in $S(x)$ ($y(x) = e^{S(x)}$) that do not vanish as $x \to \infty$.

26. Using the method of dominant balance, find the leading behavior as $x \to \infty$ of a solution of $y'' + x^2 y = \sin x$.

27. Consider the differential equation

$$x^3 f''(x) - x f'(x) + (3 - 2x) f(x) - 3 = 0$$

with boundary condition $f'''(0) = 36$.

   (a) Find a power series solution $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

   (b) What is the radius of convergence of your solution?

   (c) Transform your sum to an integral using Borel summation technique — when does the integral converge?

   (d) Verify that the integral satisfies the differential equation in its domain of convergence (OK to use Mathematica or some other symbolic manipulation software).

28. Find a function for which the series $\sum_{n=0}^{\infty} (-1)^n x^n \Gamma\left(\frac{n+1}{p}\right)$ is asymptotic as $x \to 0^+$.

29. Evaluate $\int_0^{\infty} dt\, e^{xt - e^t}$ for $x \gg 1$.

30. Energy in a star is produced by nuclear reactions. The number of nuclear collision with center of mass kinetic energy in the range from E to $E + dE$ is $Ne^{-E/k_BT} E\, dE$ per unit time. The probability of a collision resulting in a nuclear reaction is $Me^{-\alpha/\sqrt{E}}$ where M and α are constants. Find an approximate expression for the total number of nuclear reactions per unit time in the low-temperature regime $k_BT/\alpha^2 \ll 1$.

31. Use the method of stationary phase to find the leading behavior of $\int_0^1 dt\, \cos(xt^4) \tan(t)$ as $x \to +\infty$.

32. Evaluate the leading order behavior of the integral

$$\int_{-1}^{1} dt\, \sinh(t) \sin[x(t - \sin(t)]$$

for large x.

33. Consider the integral $I(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, e^{-\frac{1}{2}x^2 - \frac{g}{4}x^4}$.

    (a) What is the saddle point approximation for $I(g)$?

    (b) Expand the integrand as a power series of g and integrate the series termwise to get an expression for $I(g)$ as a power series, $I(g) = \sum_{n=0}^{\infty} a_n g^n$.

    (c) What is the radius of convergence of the series? How do you see this immediately from the definition of $I(g)$?

    (d) For a fixed g, which term in the power series is smallest? Call the corresponding exponent $N(g)$. Define $I_1(g) = \sum_{n=0}^{N(g)} a_n g^n$. Using a computer, plot the relative error $(I_1(g) - I(g))/I(g)$ (using some numerical integration routine in Matlab or Mathematica for $I(g)$). Plot first for $g = 0.01 \ldots 1$, and then $g = 0.001 \ldots 0.01$.

The series you obtained is an example of an asymptotic series which diverges but is still useful. The most familiar asymptotic series is the Stirling formula $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \ldots\right)$ which diverges for all n but is nevertheless frequently used (you just need to know where to stop).

34. Find the asymptotic expression for

$$\int_0^\infty dt \, \frac{\sin t}{t+z}$$

valid for large z, *i.e.*, express the integral as a power series in $1/z$.

35. The total energy flux Φ emitted by a black body is given by

$$\Phi = 2\pi c^2 \hbar \left(\frac{kT}{\hbar c}\right)^4 \int_0^\infty du \, \frac{u^3}{e^u - 1}$$

In this problem we analyze the definite integral that appears in this expression.

(a) Show that

$$\int_0^\infty du \, \frac{u^3}{e^u - 1} = 6 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

Hint: write the integrand as a series.

(b) Evaluate the above sum using residue calculus.

36. MW 6-1

37. MW 6-11

38. The Pauli spin matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(a) Show that all 2×2 matrices can be written as linear combinations of the Pauli matrices and the unit matrix.

(b) Evaluate $\exp(\alpha\sigma_j)$, $j = 1, 2, 3$.

(c) Evaluate $\exp(\sigma_1 + \sigma_2)$, and compare the result with $\exp(\sigma_1)\exp(\sigma_2)$.

39. Try to solve $X^2 + AX + B = 0$ where X is an unknown 2×2 matrix, and

(a) $A = \begin{pmatrix} 3 & -\sqrt{3} \\ \sqrt{3} & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$.

(b) $A = \begin{pmatrix} 3 & -\sqrt{3} \\ \sqrt{3} & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & \sqrt{3} \\ -\sqrt{3} & 0 \end{pmatrix}$.

40. The definition of a determinant involves a sum over permutations with each permutation weighted by the factor $(-1)^P$ where P is the parity of the permutation (*i.e.*, P tells if the permutation can be constrcucted by an even or odd number of successive pairwise interchanges [transpositions]). A consequence of this is that the sign of the determinant changes if two rows (or columns) of the matrix are interchanged.

   (a) Using this result, and power counting, evaluate (up to an overall sign) the so-called Vandermonde determinant $\det(A)$ where $A_{ij} = x_i^j$.

   (b) In quantum mechanics, the many particle wave function of a system of N fermions is given by a Slater determinant, which is the determinant of a matrix whose entries are $\psi_j(\mathbf{r}_i)$. Here $\{\mathbf{r}_i\}_{i=1}^N$ are the coordinates of the fermions, and $\psi_j(\mathbf{r})$ are the wavefunctions of the occupied one-particle states ($j = 1, \ldots, N$). In the quantum Hall effect (Nobel prizes in 1985 and 1998), electrons are confined to move in two dimensions, and the single particle states are given by $\psi_j(x, y) = A_j(x - iy)^{j-1} \exp[-(x^2 + y^2)/(4\ell_B^2)]$ where $(j-1)\hbar$ is the z-directional angular momentum ($j = 1, \ldots$), A_j is a normalization constant, and ℓ_B is the magnetic length ($\ell_B = \sqrt{\frac{\hbar}{eB}}$). Evaluate the many-particle wave function $\Psi(z_1, \ldots, z_N)$ that corresponds to the lowest z-directional angular momentum (here $z_i = x_i - iy_i$ is a complex position coordinate).

41. The solution of $y'' + \omega^2 y = g(x)$ for $0 \le x \le 2\pi$, subject to periodic boundary conditions, can be written as $y(x) = \int_0^{2\pi} dx'\, G(x, x') g(x')$. Find the Green's function G in a closed form.

42. The static deflection $u(x)$ of a thin x-directional beam follows the equation

$$EI \frac{\partial^4 u}{\partial x^4} = -q(x)$$

where E is the Young's modulus of the beam material and I is the areal momentum of inertia for the beam cross section. Here $q(x)$ is the load, *i.e.* force per unit length. Assume that the beam cross section remains constant for the length of the beam and is unaffected by the load. Write the beam deflection $u(x)$ as an integral of the load function $q(x)$ for a doubly clamped beam with boundary conditions $u(0) = u(L) = 0$ and $u'(0) = u'(L) = 0$.

43. Find a solution to the inhomogeneous Euler equation

$$t^2 y''(t) + t y'(t) + a^2 y(t) = F(t),$$

valid for $t \geq 1$ and satisfying the boundary conditions $y(1) = y_0$ and $y'(1) = v_0$, where a is a real parameter and $F(t)$ vanishes for $t < 1$ but is otherwise arbitrary. Hint: it may be useful to write $t = e^s$.

44. MW 9-8

45. MW 9-9

46. The Schrödinger equation for a particle in potential $V(\mathbf{r})$ can be written as

$$[H_0 + V(\mathbf{r}) - E_\alpha]\psi_\alpha(\mathbf{r}) = 0$$

where H_0 is the Hamiltonian for the free particle (or a Hamiltonian with a simpler potential than V). Show that this differential equation can be alternatively written as an integral equation

$$\psi_\alpha(\mathbf{r}) = \psi_\alpha^{(0)}(\mathbf{r}) + \int d^d r' \, G_0(E_\alpha; \mathbf{r}, \mathbf{r}') V(\mathbf{r}') \psi_\alpha(\mathbf{r}')$$

where $G_0(E_\alpha; \mathbf{r}, \mathbf{r}')$ is the Green's function of the simpler equation $[H_0 - E_\alpha]\psi_\alpha^{(0)}(\mathbf{r}) = 0$ and $\psi_\alpha^{(0)}(\mathbf{r})$ is a solution of the simpler equation. This integral equation, known as the Lippmann-Schwinger equation, is often useful in describing various scattering problems. *N.B.*: the sign of the term on the right hand side depends on whether how you exactly define the Green's function.

47. Consider the integral equation

$$f(x) = x^2 + \lambda \int_{-1}^{1} dy \, (1 + xy) f(y)$$

and solve it using

a) Neumann series — for which λ do you obtain a solution?

b) the theory for degenerate kernels — for which λ do you now obtain a solution?

c) the Schmidt-Hilbert theory

48. Find the eigenvalues λ_n and eigenfunctions $y_n(x)$ of $y''(x)+\epsilon\sin(\pi x/L)y(x) = \lambda y(x)$, $0 \le x \le L$, $y(0) = y(L) = 0$, to lowest order in ϵ.

49. Solve the integral equation

$$f(x) = x^3 + \lambda \int_{-\pi}^{\pi} dy\, \cos^2\left(\frac{x-y}{4}\right) f(y).$$

For which values of λ does a solution exist?

50. MW 10-6

51. MW 10-7

52. MW 10-8

53. MW 10-9

54. Consider the Schrödinger equation for a particle in a box, $-\psi''(x) + V_0(x)\psi(x) = E\psi(x)$ where $V_0(x) = 0$ for $|x| < L/2$ and $V(x) = +\infty$ otherwise.

   (a) Determine the energy eigenvalues E_n and the corresponding eigenstates $\psi_n(x)$.

   (b) Consider now a perturbation $V_1(x) = V_1\cosh(\alpha x)$, and determine the energy eigenvalues and eigenstates for the potential $V(x) = V_0(x)+V_1(x)$ to linear order in V_1.

   (c) When do you expect the linear approximation to be accurate?

55. High frequency electromagnetic signals are often transmitted in so-called wave guides, hollow metallic tubes. The signals that can be transmitted through a wave guide can be divided into transverse magnetic (TM) and transverse electric (TE) waves, the former satisfying $B_z = 0$ everywhere and $E_z = 0$ on the surface of the conductor, and the latter satisfying $E_z = 0$ everywhere and $\nabla B_z \cdot \hat{\mathbf{n}} = 0$ on the surface of the conductor (the wave guide runs in the z-direction). In this problem we only consider transverse magnetic waves of the form $\mathbf{E} = E_z\hat{\mathbf{z}} + \mathbf{E}_t$ where the transverse component can be written as

$\mathbf{E}_t = \pm\frac{ik}{\gamma^2}\nabla_t\psi(x,y)$ ($\nabla_t = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y}$). The longitudinal component is given by $E_z = e^{\pm ikz}\psi$, and the function ψ satisfies the eigenvalue equation

$$\nabla_t^2\psi + \gamma^2\psi = 0, \qquad \psi(x,y) = 0 \text{ for } x, y \text{ on the surface.}$$

The frequency ω of the mode is related to the wave vector k and the eigenvalue γ by $\gamma^2 = \mu\epsilon\frac{\omega^2}{c^2} - k^2$ where μ and ϵ are material parameters and c is the speed of light.

(a) Find the frequencies $\omega(k)$ for TM waves in a wave guide with rectangular cross section $-a/2 < x < a/2$, $-b/2 < y < b/2$. What is the lowest frequency that can be transmitted through the wave guide (so-called cutoff frequency)? Assume that a/b is not a rational number so that there are no degeneracies to worry about.

(b) If the walls of the wave guide are not perfect conductors, the boundary condition for TM becomes $E_z = f\nabla E_z \cdot \hat{\mathbf{n}}$ where f is proportional to the surface impedance of the walls. Find the perturbed frequencies ω to first order in f.

56. Damped harmonic oscillator

(a) Find the Green's function for the damped harmonic oscillator $(m\partial_t^2 + 2mQ^{-1}\Omega\partial_t + m\Omega^2)G(t,t') = \delta(t-t')$ in the overdamped ($Q < 1$) and underdamped ($Q > 1$) cases.

(b) Using the Green's function, obtain the solution of the driven oscillator with driving force $F(t)$, and show that the solution at time t only depends on the driving force at earlier times.

(c) Consider now a modified oscillator with an additional term $m\tau\partial_t^3 x(t)$, and show that such a term is not compatible with causality for $Q \to \infty$. A model of this type was, *e.g.*, put forth by Abraham and Lorenz in 1903 to describe the energy loss of an accelerating charge due to radiation; the model predicts that the accelaration of a charge at time t depends on the electric field not only at times before t but also up to time $\tau \approx 10^{-23}s$

after t, *i.e.* the particle anticipates that soon a field will be turned on! The resolution of the problem was provided by quantum mechanics.

57. Find leading order uniform asymptotic approximation to the solution of

$$\epsilon y'' + (\cosh x)y' - y = 0, \quad y(0) = y(1) = 1, 0 \le x \le 1$$

58. *Particles in a gravitational potential.*

    (a) Show that the WKB approximation for the eigenvalues of $-\epsilon^2 y'' + [V(x) - E]y = 0$, $y(0) = y(+\infty) = 0$, where $V(x)$ increases monotonically and $V(0) = 0$, is given by $(1/\epsilon)\int_0^{x_0} \sqrt{E - V(x)}\, dx = (n - \frac{1}{4})\pi + O(\epsilon)$, $\epsilon \to 0^+$, where $V(x_0) = E$.

    (b) Find an approximation of eigenvalues of a Schrödinger equation for a particle in a vertical container subject to the (gravitational) potential $V(x) = mgx$ for $x > 0$ and $V(x) = +\infty$ for $x < 0$ that is valid for large energies.

59. Consider the eigenvalue problem $y'' + E\cos(x)y = 0, y(\pm\pi) = 0$. Find an approximation to E valid for large E using the WKB method.

60. Consider the nonlinear van der Pol oscillator $y''(t) + \omega_0^2 y(t) - \epsilon(1 - y^2(t))y'(t) = 0$. For arbitrary initial conditions the solution of this equation approaches a limit cycle. Find the approach to this limit cycle using multiple scale perturbation theory.

61. Solve the integral equation

$$f(x) = x + \lambda \int_{-1}^{1} dy\, (y - x) f(y)$$

    (a) Using the method of separable kernels

    (b) Using the Neumann series. When does the Neumann series converge?

62. Min(x, y) is the smaller of x and y.

(a) Find the eigenvalues and eigenfunctions that satisfy

$$\int_0^1 dy\, \mathrm{Min}(x,y) f(y) = k f(x)$$

Hint: use the equation and its derivative to obtain boundary conditions for $f(x)$, and use the second derivative to obtain a differential equation for $f(x)$.

(b) Add a term $g(x)$ to the left hand side of the above equation, and examine if a simple iteration will converge (note that $|\langle u|K|u\rangle| \leq |\kappa| \langle u|u\rangle$ where κ is the largest eigenvalue of K).

(c) What is the Hilbert series for the solution of the inhomogeneous equation? Note that the series could be summed using the methods we discussed earlier in the course, however, the procedure is rather laborious and you need not do it.

63. (a) For which values of λ does the equation

$$f(x) = \phi(x) + \lambda \int_0^1 dy\, (1 - 3xy) f(y)$$

have a solution for a general $\phi(x)$? Find the solution!

(b) For the remaining values of λ (*i.e.*, those that do not have a solution for a general $\phi(x)$), what conditions must $\phi(x)$ satisfy in order for a solution $f(x)$ to exist? Find the solutions!

64. Solve the integral equation

$$f(x) = x + \lambda \int_0^\pi dy\, (x \cos(y) + x^2 \cos(2y)) f(y)$$

65. Euler-Lagrange equations:

(a) Find the function $u(x)$ that minimizes the functional

$$I_\pi[u] = \int_0^\pi dx\, \left[(u'(x))^2 - (u(x))^2\right]$$

and satisfies the boundary conditions $u(0) = u(\pi) = 0$.

(b) Derive and solve the Euler-Lagrange equation for the minimization problem

$$I_{2\pi}[u] = \int_0^{2\pi} dx\ \left[(u'(x))^2 - (u(x))^2\right]$$

where $u(0) = u(2\pi) = 0$.

(c) Evaluate $I_{2\pi}[\sin(x/2)]$. Comments?

66. Consider a uniform beam with Young's modulus E and moment of inertia I. The beam is attached at points $x = 0$ and $x = L$ so that $y(0) = y(L) = 0$. Use Hamilton's principle to derive the equation of motion for small transverse oscillations in the $x - y$ plane. Hint: the potential energy of the beam due to bending is $V = \frac{1}{2}\int_0^L dx\, EI[y''(x)]^2$.

67. *Buckling of a rod.* The potential energy of a rod of length L is given by $U = \int_0^L dx\,[(u'')^2 - P(u')^2]$ where $u(x)$ is the transverse displacement, $u(0) = u(L) = 0$ and P is the force applied to the ends of the rod. For small P the potential energy is minimized for a straight rod $u(x) = 0$ but for a large force the rod buckles, that is, bends. Regard this as a variational problem, derive the appropriate Euler-Lagrange equation, and determine the critical force P. Hint: Note that for a straight rod $U = 0$ whereas for a buckled rod $U < 0$, and therefore the largest value of P such that the minimum of U is zero can be related to minimizing $\int_0^L dx\,(u'')^2$ while keeping $\int_0^L dx\,(u')^2$ fixed.

68. Consider a cylinder with radius R with straight vertical walls and a horizontal flat bottom. The cylinder is filled with a fluid with volume V and density ρ. Determine the shape of the fluid surface when the fluid rotates with angular velocity ω. Hint: minimize the potential energy in the rotating coordinate system where the particles experience a centrifugal force $\mathbf{F} = m\omega^2 r^2\hat{\mathbf{r}}$.

69. Consider the following problem:

(a) Derive the condition that the function $\phi(x, y, z)$ must satisfy in order for the integral

$$I[\phi] = \int\int\int_V L(\phi, \nabla\phi, x, y, z)dxdydz$$

to have an extremum. Here V is a simply connected three-dimensional region and $L(\phi, \nabla\phi, x, y, z)$ and arbitrary continuous function of its arguments.

(b) In electrostatics the energy of the electric field $\mathbf{E}(x, y, z)$ is given by

$$W = \int\int\int_V E^2(x, y, z)dxdydz.$$

Write $\mathbf{E} = -\nabla\phi$ and show that ϕ must satisfy the Laplace equation $\nabla^2\phi = 0$.

70. Consider the following problem:

Use a rope of length 100m to enclose a piece of land in such a way that the value of the enclosed land is maximized. Both ends of the rope must attach to an (immovable, point-like) apple tree. The land that is no more than twenty meters north of the tree costs one thousand krona per square meter and land that is more than twenty meters north of the tree costs two thousand krona per square meter. What is the optimal shape of the perimeter of the enclosed land, and what is the maximal value of the land enclosed by the rope?

In other words, if the apple tree is in the origin, and the x-coordinate points to the north, the land value is 1000 SEK$/m^2$ for $x < 20m$ and 2000 SEK$/m^2$ for $x \geq 20m$. You can neglect the curvature of the Earth.

Some of the equations determining the shape of the optimal perimeter may need to be solved numerically. The same applies to evaluating the integral that yields the total value of the land.

71. *Global metro.* Consider an underground transportation system consisting of motorless trains running in thin tunnels connecting pairs of cities to each other. The train cars move under the influence of gravity through the frictionless tunnels. Determine the shape of the tunnel that allows fastest transportation between cities A and B located such that the angle AOB, where O is the center of the Earth, is Θ. Regard the Earth as a solid sphere with a constant density. What would be the transport time between Göteborg and Stockholm? Göteborg and Los Angeles?

Hints: (i) find speed as a function of depth, write $dt = ds/v$ and write ds in terms of $d\theta$; (ii) derive Euler-Lagrange equation for $r(\theta)$ and note that it can be integrated once with relative ease; (iii) note that at the deepest point of the trajectory $r'(\theta) = 0$, which allows you to determine the optimal path $r(\theta)$ or $\theta(r)$ in terms of the deepest trajectory depth r_0; (iv) substitute the trajectory to the integral that yields the travelling time T and obtain T as a function of r_0; (v) by solving $r(\pm\Theta/2) = R$, or $\theta(R) = \pm\Theta/2$, obtain r_0 in terms of Θ; (vi) look up Θ for the two trips and obtain the travelling times. *Hint:* If you solve this proble as most people, you will encounter an integral that contains a square root of an expression involving r^2, R^2 and r_0^2, which cannot be found in Beta. If you make a variable substitution and introduce a new variable ζ that equals the expression inside the square root, you can simplify the integrand to $\sqrt{\zeta}P(\zeta)$ where $P(\zeta)$ is a rational function. Dividing $P(\zeta)$ into two partial fractions yields integrals that can be evaluated by elementary means.

72. *Superconductor below T_c.* In the lecture we discussed a superconductor above the transition temperature when $a > 0$. Below the transition temperature $a < 0$ and the Landau-Ginzburg free energy must contain an additional term for stability, $\beta F_{LG}[\psi] = \int dx \left[\frac{1}{2}K|\nabla\psi|^2 + \frac{1}{2}a|\psi|^2 + \frac{1}{4}b|\psi|^4\right]$ where $b > 0$.

   - Determine the mean field value of the order parameter $\psi_0(x)$ that minimizes βF_{LG}. Note that having $\nabla\psi \neq 0$ always increases free energy, and thererefore ψ_0 is a constant independent of x. The phase of ψ_0 is not determined by this procedure, so for concreteness choose ψ_0 to be real and positive.
   - Write $\psi(x) = \psi_0 + \delta\psi(x)$ and expand $\beta F_{LG}[\psi]$ to second order in $\delta\psi$.
   - Express ψ in terms of its Fourier transform ψ_k and obtain βF_{LG} in terms of ψ_k
   - Calculate now $\langle \mathrm{Re}(\delta\psi(x))\mathrm{Re}(\delta\psi(x'))\rangle$ and $\langle \mathrm{Im}(\delta\psi(x))\mathrm{Im}(\delta\psi(x'))\rangle$ using path integrals. Which fluctuates more, the real or imaginary part of ψ? In this case when ψ_0 is real, fluctuations of the imaginary part correspond transverse fluctuations while the fluctuations of the real part are

longitudinal fluctuations.

- Show that the long wavelength transverse fluctuations diverge if $d \leq 2$, which implies that two-dimensional and one-dimensional systems with continuous order parameters cannot have ordered phases, a result known as the (Coleman-) Mermin-Wagner theorem.

Problem selection:

Problem set 1	5, 8
Problem set 2	18, 24, 27
Problem set 3	25, 30, 32
Problem set 4	42, 56
Problem set 5	47, 57, 58
Problem set 6	68, 71

Maximum points for home problems:

2 points: 5, 8

3 points: 27, 47

4 points: 18, 24, 25, 30, 42, 57, 68

5 points: 32, 56

6 points: 58, 71

Total 60 points.