Magnetic impurity in a Luttinger liquid: A conformal field theory approach

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We study the low-temperature properties of a spin-$\frac{1}{2}$ magnetic impurity coupled to a one-dimensional interacting electron system. Using a formalism by Affleck and Ludwig, with a scale invariant boundary condition replacing the impurity, we exploit boundary conformal field theory to deduce the impurity thermal and magnetic response. In the case of only forward electron scattering off the impurity, we predict the same critical scaling as for the two-channel Kondo effect for noninteracting electrons, but with a different Wilson ratio. Backward electron scattering off the impurity destabilizes this behavior and drives the system to a new fixed point. In the case of equal amplitudes for forward and backward scattering *(Kondo interaction)*, we show that there are only two types of scaling behaviors consistent with the symmetries of the problem: *either* a local Fermi liquid *or* a critical theory with an anomalous specific heat. The latter case agrees with a recent poor-man's scaling result proposed by Furusaki and Nagaosa.

I. INTRODUCTION

Quantum many-particle systems sometimes exhibit a growth of an effective coupling at low energies, resulting in a nonperturbative ground state. The maybe simplest example of this phenomenon is the Kondo effect,[1] arising from the exchange interaction between a spin-1/2 magnetic impurity and a gas of free quasiparticles (''dressed electrons'') in an s-wave band. As the temperature is decreased, the system crosses over from weak electron-impurity coupling to strong coupling, with a complete screening of the impurity spin at $T=0$. The resulting ground state is of Fermi liquid type, with the single quasiparticle wave functions acquiring a phase shift (''one-channel Kondo effect'').[2] The picture changes when electrons in degenerate orbital bands are allowed to interact with the impurity, and the ground state is now described by a non-Fermi liquid fixed point (''multichannel Kondo effect'').[3]

What is the corresponding scenario for an *interacting one-dimensional electron system* coupled to a magnetic impurity? The question may soon become of experimental relevance, considering the rapid progress in the fabrication and study of very narrow conduction channels (''quantum wires''), obtained, for example, by gating two-dimensional (2D) electron gases in GaAs inversion layers.[4] A possible laboratory realization would be a single conduction channel with a trapped atom containing two (or several) spin levels. (The related problem of tunneling through potential barriers in 1D correlated systems is already being addressed by experimentalists,[5] following the pioneering work of Kane and Fisher.[6]) The question is also interesting considering recent work on exotic superconductivity, where the analogue to the multichannel Kondo effect has been exploited.[7] Other realizations of Kondo physics that have been proposed include two-level tunneling in metallic glasses,[8] and certain heavy-fermion materials.[9] Treating the effect of a magnetic impurity in the presence of interacting electrons may offer a new perspective on these intriguing connections.

More importantly, one is faced here with an archetype problem of describing the interplay between direct fermion correlations (from interaction and statistics) and correlations induced via a coupling to a local quantum mechanical degree of freedom. As is well known, the notion of free quasiparticles *(low-temperature Fermi liquid)* breaks down in one dimension: any arbitrarily small electron-electron interaction wipes out the single-particle poles of the electron propagator, leaving behind only collective charge- and spin-density excitations. In the limit of weak electron-electron interaction, these excitations are well described by a system of noninteracting spin-charge separated bosonic modes *(Luttinger liquid)*.[10,11] By adding a localized magnetic impurity, one confronts the problem of how to incorporate its coupling to *single electrons* in the description of the bosonic collective degrees of freedom.

A first attack on the problem was launched by Lee and Toner,[12] employing Abelian bosonization followed by a perturbative scaling analysis of a resulting ''kink-gas'' action. For the case of a spin-1/2 impurity, and with the electron gas away from half-filling, it was found that the Kondo temperature T_K (setting the scale for the weak-to-strong coupling crossover) depends on the bare Kondo coupling λ_K in a power-law fashion, $T_K \sim (\lambda_K \tau_0)^{2/\eta}$. Here τ_0 is a short-time cutoff, and η is the exponent characterizing the equal-time spin-spin correlation in a Luttinger liquid. For temperatures $T<T_K$, the physics is controlled by some strong-coupling fixed point — as in the ordinary Kondo problem — not directly accessible via this kind of analysis. In a recent work, Furusaki and Nagaosa[13] derived a set of improved scaling equations in the weak-coupling regime, preserving the spin SU(2) symmetry of the problem. By tentatively extending these equations to the strong-coupling regime, Furusaki and Nagaosa conjectured that the fixed-point Hamiltonian consists of two semi-infinite Luttinger liquids and a completely screened impurity (decoupled spin singlet). The low-

temperature impurity contributions to the specific heat and magnetic susceptibility were calculated to $C_{\rm imp}\sim T^{(1/K_\rho)-1}$ and $\chi_{\rm imp}\sim T^0$, respectively, with K_ρ the usual Luttinger liquid charge parameter.[10,11] Support for this scenario can be found in earlier work[14] on impurity spins in antiferromagnetic spin-1/2 chains (a ''stripped-down'' version of the Kondo effect in a Luttinger liquid). In the case of an external $S=1/2$ impurity coupled to a single site on the chain, the impurity was found to be completely screened, severing the chain at the impurity site.

In this paper we explore the problem using *exact methods*, expanding upon results announced in Ref. 15. We shall begin by studying a simplified model. Specifically, we consider a spin-1/2 impurity coupled with equal strength to two next-nearest-neighbor sites on a Hubbard chain. In the continuum limit, and with quarter-filling of the band, this becomes a Tomonaga-Luttinger (TL) model[16,17] with *forward electron-impurity scattering* only:

$$\mathscr{H}=\mathscr{H}_{\rm TL}+\mathscr{H}_F\,, \tag{1.1}$$

where

$$\begin{aligned}\mathscr{H}_{\rm TL}=\frac{1}{2\pi}\int dx\Big\{ v_F\Big[&:\psi^\dagger_{L,\sigma}(x)\,i\frac{d}{dx}\,\psi_{L,\sigma}(x):\\ &-:\psi^\dagger_{R,\sigma}(x)\,i\frac{d}{dx}\,\psi_{R,\sigma}(x):\Big]\\ &+\frac{g}{2}\sum_{r,s=L,R}:\psi^\dagger_{r,\sigma}(x)\,\psi_{r,\sigma}(x)::\psi^\dagger_{s,-\sigma}(x)\,\psi_{s,-\sigma}(x):\\ &+g:\psi^\dagger_{R,\sigma}(x)\,\psi_{L,\sigma}(x)\,\psi^\dagger_{L,-\sigma}(x)\,\psi_{R,-\sigma}(x):\Big\}\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}\mathscr{H}_F=\lambda[&:\psi^\dagger_{L,\sigma}(0)\;\tfrac{1}{2}\boldsymbol{\sigma}_{\sigma\mu}\psi_{L,\mu}(0):\cdot\boldsymbol{S}\\ &+:\psi^\dagger_{R,\sigma}(0)\;\tfrac{1}{2}\boldsymbol{\sigma}_{\sigma\mu}\psi_{R,\mu}(0):\cdot\boldsymbol{S}].\end{aligned} \tag{1.3}$$

Here $\psi_{L,\sigma}(x)$ and $\psi_{R,\sigma}(x)$ are the left- and right-moving components, respectively, of the electron field $\Psi_\sigma(x)$, with spin projection $\sigma=\uparrow,\downarrow$, expanded about the Fermi momenta $\pm k_F$ in the long-wavelength limit:

$$\Psi_\sigma(x)=e^{-ik_Fx}\psi_{L,\sigma}(x)+e^{ik_Fx}\psi_{R,\sigma}(x). \tag{1.4}$$

The fields are normalized such that

$$\{\psi_{r,\sigma}(x),\psi^\dagger_{s,\mu}(y)\}=2\pi\delta_{rs}\delta_{\sigma\mu}\delta(x-y) \tag{1.5}$$

and summation over repeated (Greek) indices is implied. The first term in (1.2) describes free left- and right-moving electrons, whereas the second and the third terms describe forward and backward electron-electron scattering, respectively. The couplings g (>0) and λ (>0) depend on the microscopic parameters of the lattice model and v_F is the Fermi velocity. Normal ordering :: is carried out with respect to the filled Dirac sea.

We shall treat the model in (1.1) using the newly developed conformal field theory approach to quantum impurity problems by Affleck and Ludwig.[18–20] The basic idea of this method is to replace the impurity by a *boundary condition*, in the spirit of Nozières' local Fermi liquid theory of the ordinary Kondo effect.[2] At the low-temperature fixed point, the long-wavelength properties are described by a conformally invariant boundary condition and *boundary conformal field theory*[21–23] essentially determines all universal properties. Specifically, the theory predicts all boundary scaling operators that govern the asymptotic autocorrelation functions in the neighborhood of the impurity. As was realized by Nozières many years ago,[2] the impurity response to an external bulk field is governed by the leading irrelevant boundary operators. Thus, knowing these, one can directly deduce the impurity critical behavior. The difficulty, though, is to identify the right boundary condition, although frequently the symmetries of the problem cut down the list of candidates to a small number. In short, each boundary condition is associated with a selection rule for combining the various degrees of freedom (such as charge and spin) *at the boundary*. The problem thus reduces to identifying the right selection rule. This can be done, according to the fusion-rule hypothesis of Affleck and Ludwig[19,24] by applying conformal field theory *fusion rules*.[25]

With only forward electron-impurity scattering present in (1.3), the model can easily be cast on a form where boundary conformal field theory applies. Writing the Hamiltonian in terms of charge and spin currents (Sugawara construction[25]), the effect of the impurity is traded for a new boundary condition in the spin sector. At this point, however, two new elements enter the problem (as compared to the treatment of the single-impurity Kondo effect for free electrons). First, charge and spin excitations — although dynamically decoupled — are still connected via a selection rule of the type mentioned above. This must be carefully analyzed in the basis of states which diagonalize the interacting Hamiltonian. Second, by having both left- and right-moving electrons coupled to the impurity, left- and right-moving spin excitations are no longer separately conserved, only the total spin remains conserved. In technical terms (to be made precise), the chiral spin $SU(2)\times SU(2)$ symmetry of the critical *bulk* theory is not recovered at the fixed-point value of the electron-impurity coupling λ. This symmetry breaking introduces boundary operators with noninteger scaling dimensions by a mechanism similar to that operating in the two-impurity Kondo problem.[24]

The picture that emerges is consistent with that recently suggested by Furusaki and Nagaosa[13] in their ''poor-man's scaling'' analysis of the problem: the system renormalizes onto a low-temperature fixed point with the same critical exponents for impurity specific heat and susceptibility as in the two-channel Kondo problem.[19] Any asymmetry in the left-right electron-impurity coupling destabilizes this critical point, driving the system to a one-channel (Fermi-liquid-like) fixed point. An interesting feature of our solution is that the leading correction-to-scaling boundary operator at the symmetric fixed point is unique, in contrast to previous treatments of two-channel Kondo physics.[19] Specifically, for vanishing electron-electron interaction, this implies a unique Wilson ratio also at low temperatures.

To make contact with possible future experiments, one necessarily has to add electron back scattering off the impurity. This is so, since a ''real'' (Kondo type) spin exchange

$$\mathscr{H}_K=\lambda:\Psi_\sigma^\dagger(0)\ \tfrac{1}{2}\boldsymbol{\sigma}_{\sigma\mu}\Psi_\mu(0):\cdot\boldsymbol{S}, \tag{1.6}$$

with $\Psi_\sigma(x)$ the electron field in (1.4), decomposes into

$$\mathscr{H}_K=\mathscr{H}_F+\mathscr{H}_B, \tag{1.7}$$

$\mathscr{H}_F$ being the forward scattering term in (1.3), and

$$\mathscr{H}_B=\lambda[\psi_{L,\sigma}^\dagger(0)\ \tfrac{1}{2}\boldsymbol{\sigma}_{\sigma\mu}\psi_{R,\mu}(0)\cdot\boldsymbol{S} + \psi_{R,\sigma}^\dagger(0)\ \tfrac{1}{2}\boldsymbol{\sigma}_{\sigma\mu}\psi_{L,\mu}(0)\cdot\boldsymbol{S}]. \tag{1.8}$$

The back scattering term $\mathscr{H}_B$ mixes left and right electron fields, and thus breaks both chiral spin $[\mathrm{SU}(2)\times\mathrm{SU}(2)]$ *and* charge $[\mathrm{U}(1)\times\mathrm{U}(1)]$ symmetry of the bulk critical theory. This results in the appearance of a relevant boundary operator which takes the system to a new fixed point, describing Kondo scattering in a Luttinger liquid. Turning off the electron-electron interaction, the system may still be represented by a two-channel Hamiltonian, but now coupled to a magnetic impurity in only one of the channels. This is known to give a fixed-point theory with Fermi liquid exponents (as for the ordinary one-channel Kondo problem).[3,26] To include the effect of the electron-electron interaction is a more delicate problem: the mixing of left- and right-moving electrons in (1.8) obstructs a Sugawara construction in a basis where the interaction remains local.

Not being able to attack the problem directly at a Hamiltonian level, we shall make the natural assumption that the full Kondo interaction may nonetheless be described by a renormalized boundary condition on the critical bulk theory. This is in accord with the expected behavior of *any* quantum impurity interaction, as discussed in Ref. 20. Note that the relevant operator due to the Kondo interaction only couples to the boundary, i.e., the new fixed point in this scheme is a new boundary fixed point with the critical bulk theory unchanged, and that all conformally invariant boundary conditions are scale invariant and correspond to such boundary fixed points. By demanding that the noninteracting limit is correctly reproduced, with analytic scaling in temperature for the impurity specific heat C_{imp} and susceptibility χ_{imp}, it turns out that conformal invariance together with the symmetry of the problem restrict the possible types of critical behavior to only two: *Either* the theory remains a local Fermi liquid in the presence of electron-electron interaction, *or* electron correlations drive the system to a new fixed point where (to leading order in temperature)

$$\begin{aligned} C_{\mathrm{imp}}&=c_1[(1/K_\rho)-1]^2T^{(1/K_\rho)-1}+c_2T,\\ \chi_{\mathrm{imp}}&=c_3T^0, \end{aligned} \tag{1.9}$$

$K_\rho=(1+2g/v_F)^{-1/2}$ being the Luttinger liquid charge parameter, and $c_{1,2,3}$ amplitudes depending on the scaling fields. The second case (1.9) agrees with the finding by Furusaki and Nagaosa referred to above.[13] In the conformal field theory scheme this scaling is implied by a certain selection rule for recombining the degrees of freedom at the impurity site, and we will discuss its properties *in extenso*.

It should be noted that a faithful modeling of a magnetic impurity must allow for the possibility that the impurity carries a net charge giving rise to a screened local potential. In the case of a Fermi liquid its effect can be absorbed by passing to a new electronic basis with renormalized single-particle energies (which is the reason why potential scattering is often neglected in the ordinary Kondo problem). However, for a Luttinger liquid the effect of a local potential is more dramatic, as shown in Ref. 6. For this reason, potential scattering must here be treated on equal footing with the spin exchange interaction.[27] Some attempts in this direction have recently been discussed in Ref. 28, but we will not address the problem here.

The rest of the paper is organized as follows. In the next section we introduce the Hubbard chain coupled to an impurity spin, and perform a continuum limit retaining only forward electron scattering off the impurity. In Sec. III we derive the finite-size energy spectrum and the corresponding spectrum of boundary operator dimensions, employing a particular variant of Affleck and Ludwig's fusion-rule hypothesis. This section also contains a matching of the Luttinger liquid selection rule for combining charge and spin excitations against that from a *Bethe-ansatz* analysis of the Hubbard model.[29–31] Employing the results for the boundary operator spectrum, the impurity critical behavior is identified in Sec. IV as that of the two-channel Kondo problem. In Sec. V we then consider the effect of adding electron back scattering off the impurity, thus treating the full Kondo interaction in a Luttinger liquid. Section VI, finally, summarizes our results. Throughout the paper we try to provide sufficient information to make it essentially self-contained for a reader with some acquaintance with conformal theory.

II. THE MODEL

A. The Hubbard chain

To make the physical picture clear we start by considering an explicit model for interacting electrons on a one-dimensional lattice, coupled to a single $S=1/2$ impurity. The Hamiltonian

$$\mathscr{H}=\mathscr{H}_H+\mathscr{H}_I \tag{2.1}$$

consists of a periodic Hubbard chain with nearest-neighbor hopping (t) and repulsive on-site interactions (U)

$$\mathscr{H}_H=-t\sum_n\ (c_{n,\sigma}^\dagger c_{n+1,\sigma}+c_{n+1,\sigma}^\dagger c_{n,\sigma})+U\sum_n\ n_{n,\uparrow}n_{n,\downarrow}, \qquad U>0, \tag{2.2}$$

and couplings (J_n) of the electron spins to the impurity

$$\mathscr{H}_I=\sum_n\ J_n c_{n,\sigma}^\dagger\ \tfrac{1}{2}\boldsymbol{\sigma}_{\sigma\mu}c_{n,\mu}\cdot\boldsymbol{S}, \tag{2.3}$$

where we implicitly sum over repeated Greek indices as before. The electron creation and annihilation operators satisfy canonical anticommutation relations

$$\{c_{m,\sigma},c_{n,\mu}^\dagger\}=\delta_{nm}\delta_{\sigma\mu}, \tag{2.4}$$

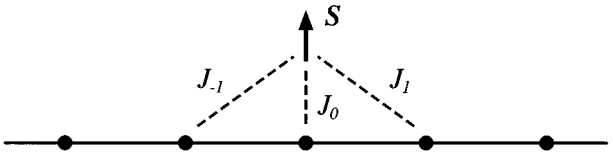

FIG. 1. The Hubbard chain interacting with an impurity spin $\boldsymbol{S}$ at a few sites. Solid lines represent electron hopping and dashed lines spin interactions.

and the number of electrons with spin σ at site n is given by $n_{n,\sigma}=c^{\dagger}_{n,\sigma}c_{n,\sigma}$ (without summation). The electron density n_e is the expectation value of $n_{n,\uparrow}+n_{n,\downarrow}$.

Without interactions, (2.2) yields a free electron dispersion $\epsilon(k)=-2t\cos ak$, with a the lattice constant. The Fermi surface consists of the two points $k=\pm k_F$, $k_F=n_e\pi/2a$. For small excitations (weak interaction) one may linearize the spectrum around the Fermi points, which gives rise to left- and right-moving particles with velocities $\pm v_F$, $v_F=2at\sin(\pi n_e/2)$. In terms of electron operators one has

$$c_{n,\sigma}=e^{-ik_F na}c_{L,n,\sigma}+e^{ik_F na}c_{R,n,\sigma}, \tag{2.5}$$

with $c_{L,n,\sigma}$ and $c_{R,n,\sigma}$ referring to the excitations around $k=-k_F$ and $k=k_F$, respectively. The sign of k_F in (2.5) is a matter of convention; we use that $c_{n,\sigma}$ is expanded in $\{e^{ikna}\}$. These electron operators, defined on the lattice, are replaced in the continuum limit by the Dirac fields $\psi_{r,\sigma}(x)$:

$$c_{r,n,\sigma}\sim\sqrt{\frac{a}{2\pi}}\psi_{r,\sigma}(na), \tag{2.6}$$

with normalization given by (1.5). It is then straightforward to verify that the free part of the Hamiltonian (2.2) equals $\mathscr{H}_{\mathrm{TL}}$ in (1.2) with $g=0$.

As for the electron-electron interaction, it follows from substituting (2.5) in (2.2) that

$$\begin{aligned}\mathscr{H}_H\big|_{t=0}=U\sum_n\Big\{&\frac{1}{2}\sum_{r,s=L,R}c^{\dagger}_{r,n,\sigma}c_{r,n,\sigma}c^{\dagger}_{s,n,-\sigma}c_{s,n,-\sigma}\\&+c^{\dagger}_{R,n,\sigma}c_{L,n,\sigma}c^{\dagger}_{L,n,-\sigma}c_{R,n,-\sigma}\\&+[e^{-i2\pi nn_e}c^{\dagger}_{R,n,\uparrow}c_{L,n,\uparrow}c^{\dagger}_{R,n,\downarrow}c_{L,n,\downarrow}+\mathrm{H.c.}]\Big\},\end{aligned} \tag{2.7}$$

describing forward, backward, and Umklapp electron-electron scattering, respectively. Due to phase oscillations, however, the Umklapp term cancels away from half-filling ($n_e\neq 1$), and in the continuum limit we recover the interaction part of (1.2) with $g=Ua/2\pi$. In this paper we will consider gapless excitations only, which restricts us to $U>0$ and $n_e\neq 1$.

Finally, we rewrite the interaction with the impurity spin in (2.3). It interacts locally with a few sites and we may concentrate on $n=0$ and $n=\pm 1$, see Fig. 1. With $J_{-1}=J_1$ we recover in the continuum limit the previous electron-impurity interactions (1.3) and (1.8) with the following couplings for forward and backward scattering against the impurity spin:

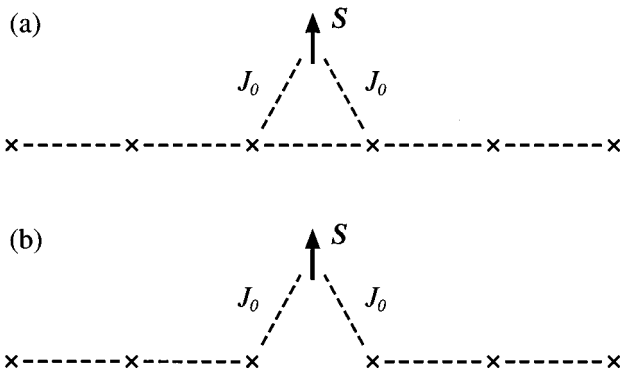

FIG. 2. The Heisenberg antiferromagnet in 1D coupled via two neighboring sites to an impurity $\boldsymbol{S}$. In (a) the chain is closed, in (b) open, although linked via the impurity.

$$\lambda_F=\frac{a}{2\pi}(J_0+2J_1), \tag{2.8a}$$

$$\lambda_B=\frac{a}{2\pi}(J_0+2J_1\cos\pi n_e). \tag{2.8b}$$

Note that backward, but not forward, electron-impurity scattering depends on the filling factor when the impurity is coupled to more than one site.[32] The first case of impurity interaction addressed in this paper concerns forward scattering only, and from this construction it is clear that one can cancel backward scattering ($\lambda_B=0$) by coupling the impurity to the two nearest-neighboring sites at quarter-filling, i.e., $J_0=0$ and $n_e=1/2$. For other filling fractions we may also fulfill $\lambda_B=0$ by allowing $J_0\neq 0$. The next case of interest is the Kondo interaction, $\lambda_F=\lambda_B\neq 0$. It corresponds to coupling the impurity to one site only at arbitrary filling, i.e., $J_1=0$ in (2.8).

Let us mention that another way of canceling the backscattering term is to couple two neighboring sites to $\boldsymbol{S}$ at half-filling and choose $J_0=J_1$ (and all other $J_n=0$). With $n_e=1$, Umklapp processes now come into play in (2.7), causing a mass gap in the charge sector.[25] The spin sector remains massless and describes a spin-1/2 antiferromagnetic Heisenberg chain with two neighboring sites coupled antiferromagnetically to an impurity spin [Fig. 2(a)]:

$$\mathscr{H}_{\mathrm{spin}}=J\sum_n \boldsymbol{S}_n\cdot\boldsymbol{S}_{n+1}+J_0(\boldsymbol{S}_0+\boldsymbol{S}_1)\cdot\boldsymbol{S},\quad \boldsymbol{S}_N=\boldsymbol{S}_0, \tag{2.9}$$

with $J=4t^2/U$. This situation is similar to that considered by Eggert and Affleck,[14] who studied a spin chain with two open ends coupled symmetrically to a single impurity [Fig. 2(b)]. On the basis of bosonization and numerical renormalization it was concluded that this system is in the same universality class as the two-channel Kondo model. The case above [Fig. 2(a)] where the spins coupled to the impurity interact mutually has been discussed by Clarke *et al.*[33] Also using bosonization, these authors proposed that two-channel

Kondo behavior is manifest for this case as well. Their argument is based on a formal similarity between the bosonized versions of the Hamiltonians for the impurity-spin chain system and the two-channel Kondo effect, and the situation is somewhat less clear than for the open chain. It is therefore of interest to reconsider the problem, and we shall return to it below.

B. Sugawara form

In what follows we focus on the case away from half-filling ($n_e \neq 1$), described by the Tomonaga-Luttinger model (1.2), together with the forward electron-impurity interaction (1.3). It is convenient to rewrite the Hamiltonian in terms of charge and spin currents

$$J_r(x) = :\psi^\dagger_{r,\sigma}(x)\psi_{r,\sigma}(x):, \tag{2.10a}$$

$$\boldsymbol{J}_r(x) = :\psi^\dagger_{r,\sigma}(x)\ \tfrac{1}{2}\boldsymbol{\sigma}_{\sigma\mu}\psi_{r,\mu}(x):, \tag{2.10b}$$

with $r = L, R$. These obey the (level-two) U(1) and level-one SU(2) affine Kac-Moody algebras[34]

$$[J^L_R(x), J^L_R(y)] = \pm 4\pi i\delta'(x-y), \tag{2.11a}$$

$$[J^a_L(x), J^b_L(y)] = i\epsilon^{abc}J^c_L 2\pi\delta(x-y) \pm \frac{\pi i}{2}\delta^{ab}\delta'(x-y), \tag{2.11b}$$

respectively, with J^a_r the components of $\boldsymbol{J}_r = (J^x_r, J^y_r, J^z_r)$. The normal-ordered products of the fields in (2.10) are defined by the usual point-splitting procedure:

$$:\psi^\dagger_{r,\sigma}(x)\psi_{r,\sigma}(x): \equiv \lim_{\delta\to 0}[\psi^\dagger_{r,\sigma}(x+\delta)\psi_{r,\sigma}(x) - \langle\psi^\dagger_{r,\sigma}(x+\delta)\psi_{r,\sigma}(x)\rangle]. \tag{2.12}$$

The terms of the Hamiltonian (1.1) can now be identified with combinations of the quadratic forms $:J_r(x)J_s(x):$ and $:\boldsymbol{J}_r(x)\cdot\boldsymbol{J}_s(x):$. As an example, let us consider the case $r = s = L$. The normal ordering is again defined via point splitting,

$$:J_L(x)J_L(x): \equiv \lim_{\delta\to 0}[J_L(x+\delta)J_L(x) - \langle J_L(x+\delta)J_L(x)\rangle], \tag{2.13}$$

and we need to evaluate

$$J_L(x+\delta)J_L(x) = :\psi^\dagger_{L,\sigma}(x+\delta)\psi_{L,\sigma}(x+\delta): \times :\psi^\dagger_{L,\mu}(x)\psi_{L,\mu}(x):. \tag{2.14}$$

This can be done by using Wick's theorem and the Green's functions

$$\langle\psi_{L,\sigma}(x+\delta)\psi^\dagger_{L,\mu}(x)\rangle = \langle\psi^\dagger_{L,\sigma}(x+\delta)\psi_{L,\mu}(x)\rangle = \frac{\delta_{\sigma\mu}}{i\delta}, \tag{2.15}$$

with the result that

$$J_L(x+\delta)J_L(x) = :\psi^\dagger_{L,\sigma}(x+\delta)\psi_{L,\sigma}(x+\delta)\psi^\dagger_{L,\mu}(x)\psi_{L,\mu}(x): + \frac{1}{i\delta}:\psi_{L,\sigma}(x+\delta)\psi^\dagger_{L,\sigma}(x): + \frac{1}{i\delta}:\psi^\dagger_{L,\sigma}(x+\delta)\psi_{L,\sigma}(x): - \frac{2}{\delta^2}. \tag{2.16}$$

Hence,

$$:J_L(x)J_L(x): = :\psi^\dagger_{L,\sigma}(x)\psi_{L,\sigma}(x)\psi^\dagger_{L,-\sigma}(x)\psi_{L,-\sigma}(x): + 2:\psi^\dagger_{L,\sigma}(x)i\frac{d}{dx}\psi_{L,\sigma}(x): \tag{2.17}$$

up to a total derivative from a partial integration. The analogous procedure for the spin currents yields (using that $\boldsymbol{\sigma}_{\sigma\mu}\cdot\boldsymbol{\sigma}_{\nu\eta} = 2\delta_{\sigma\eta}\delta_{\mu\nu} - \delta_{\sigma\mu}\delta_{\nu\eta}$)

$$:\boldsymbol{J}_L(x)\cdot\boldsymbol{J}_L(x): = -\frac{3}{4}:\psi^\dagger_{L,\sigma}(x)\psi_{L,\sigma}(x)\psi^\dagger_{L,-\sigma}(x)\psi_{L,-\sigma}(x): + \frac{3}{2}:\psi^\dagger_{L,\sigma}(x)i\frac{d}{dx}\psi_{L,\sigma}(x):, \tag{2.18}$$

and it is clear from (2.17) and (2.18) that the first term of (1.2) can be written entirely in terms of currents

$$:\psi^\dagger_{L,\sigma}(x)i\frac{d}{dx}\psi_{L,\sigma}(x): = \frac{1}{4}:J_L(x)J_L(x): + \frac{1}{3}:\boldsymbol{J}_L(x)\cdot\boldsymbol{J}_L(x): \tag{2.19}$$

(up to a total derivative). In the same manner, one straightforwardly replaces all other terms and arrives at the Sugawara form[25] of the Hamiltonian in (1.1):

$$\mathscr{H}_{\rm TL} = \frac{1}{2\pi}\int dx\Big\{\frac{v_F+g}{4}[:J_L(x)J_L(x): + :J_R(x)J_R(x):] + \frac{v_F-g}{3}[:\boldsymbol{J}_L(x)\cdot\boldsymbol{J}_L(x): + :\boldsymbol{J}_R(x)\cdot\boldsymbol{J}_R(x):] + \frac{g}{2}[J_L(x)J_R(x) - 4\boldsymbol{J}_L(x)\cdot\boldsymbol{J}_R(x)]\Big\}, \tag{2.20}$$

and

$$\mathscr{H}_F = \lambda[\boldsymbol{J}_L(0) + \boldsymbol{J}_R(0)]\cdot\boldsymbol{S}, \tag{2.21}$$

with $g = aU/2\pi$ for the Hubbard model. Rewriting the spin-current part of the Hamiltonian on matrix form $:\boldsymbol{J}_r(x)\cdot\boldsymbol{J}_r(x): = 1/2:\mathscr{J}_{r,\sigma\mu}(x)\mathscr{J}_{r,\mu\sigma}(x):$ where $\mathscr{J}_{r,\sigma\mu}(x) \equiv :\psi^\dagger_{r,\sigma}(x)\psi_{r,\mu}(x): - (1/2)\delta_{\sigma\mu}:\psi^\dagger_{r,\nu}(x)\psi_{r,\nu}(x):$ such that $\boldsymbol{J}_r(x) = (1/2)\boldsymbol{\sigma}_{\sigma\mu}\mathscr{J}_{r,\sigma\mu}(x)$, one notes that the spin currents are traceless [$\mathscr{J}_{r,\sigma\sigma}(x) = 0$], and hence have no charge components. Thus, spin-charge separation is manifest in (2.20).

The spin-interaction term $\boldsymbol{J}_L(x)\cdot\boldsymbol{J}_R(x)$ can be shown to be marginally irrelevant for $g > 0$,[25] and will be dropped henceforth.[35] [The spin interaction is the only term in (2.20) that renormalizes, the remaining terms in $\mathscr{H}_{\rm TL}$ being exactly marginal.] The piece of the Hamiltonian containing charge currents is diagonalized via the Bogoliubov transformation

$$J^L_R(x) = \cosh\theta\, j^L_R(x) - \sinh\theta\, j^R_L(x). \tag{2.22}$$

The transformation is canonical, with the new currents j^L_R also satisfying the U(1) Kac-Moody algebra

$$[j^L_R(x), j^L_R(y)] = \pm 4\pi i \delta'(x-y). \tag{2.23}$$

Inserting (2.22) into (2.20), the transformation is found to diagonalize the charge Hamiltonian when

$$\tanh 2\theta = \frac{g}{v_F + g}. \tag{2.24}$$

Collecting the results,

$$\mathscr{H} = \mathscr{H}^*_{\mathrm{TL}} + \mathscr{H}_F \tag{2.25}$$

with

$$\mathscr{H}^*_{\mathrm{TL}} = \frac{1}{2\pi}\int dx \left\{ \frac{v_c}{4}[:j_L(x)j_L(x):+:j_R(x)j_R(x):] + \frac{v_s}{3}[:\boldsymbol{J}_L(x)\cdot\boldsymbol{J}_L(x):+:\boldsymbol{J}_R(x)\cdot\boldsymbol{J}_R(x):] \right\} \tag{2.26}$$

the resulting critical bulk Hamiltonian, and with $\mathscr{H}_F$ defined in (2.21). Here

$$v_c = v_F\sqrt{1+\frac{2g}{v_F}}, \tag{2.27a}$$

$$v_s = v_F - g \tag{2.27b}$$

with v_F and g defined above. It is easy to see that $\mathscr{H}^*_{\mathrm{TL}}$ is invariant under independent global U(1) and SU(2) transformations on the left- and right-moving (chiral) fields:

$$\psi_{r,\sigma} \to e^{i\phi_r}\psi_{r,\sigma}, \quad \psi_{r,\sigma} \to U_{r,\sigma\mu}\psi_{r,\mu}, \quad r = L,R \tag{2.28}$$

with ϕ_r a constant, and $U_{r,\sigma\mu}$ an element of SU(2). In fact, the Sugawara form of $\mathscr{H}^*_{\mathrm{TL}}$ implies invariance under the larger chiral

$$\mathscr{G} = \mathrm{U}(1)_L \times \mathrm{U}(1)_R \times \mathrm{SU}(2)_{1,L} \times \mathrm{SU}(2)_{1,R} \tag{2.29}$$

Kac-Moody algebra[34] [with the subscript ''1'' denoting the level of the SU(2) algebras]. However, the chiral spin symmetry gets broken by the impurity-electron interaction $\mathscr{H}_F$ for *any value of the coupling* λ . This is in contrast to the Kondo effect for free electrons. In that case too there is only forward electron scattering off the impurity, but only with one type of chiral electrons (say, left movers). The critical bulk Kac-Moody symmetry is $\mathrm{U}(1)\times\mathrm{SU}(2)_1$ [or $\mathrm{U}(1)\times\mathrm{SU}(2)_2\times\mathrm{SU}(2)_2$ in the case of two channels], and this symmetry is restored precisely at the strong coupling fixed point.[19] We return to this below.

It should be emphasized that although the theory contains two distinct velocities, v_c and v_s, Lorentz invariance is manifest separately in the charge and spin sectors of $\mathscr{H}^*_{\mathrm{TL}}$. This can be seen via bosonization.[25] The terms containing charge currents represent a free boson theory, while the spin-current terms represent an SU(2) $k=1$ Wess-Zumino-Witten (WZW) model. Both theories are conformally invariant, implying also Lorentz invariance. This observation will be important for the applications to come.

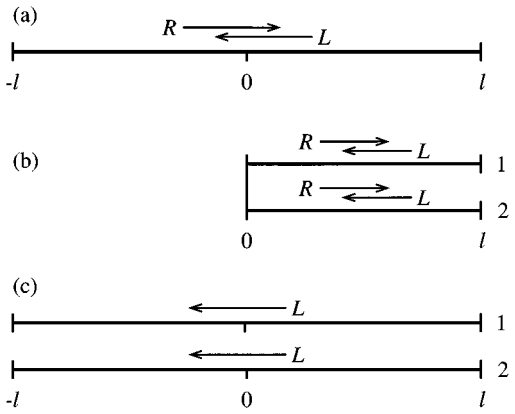

FIG. 3. Equivalent representations of the Luttinger liquid. The original $c=2$ system (a) with left- and right-moving excitations on $[-\ell,\ell]$ is folded into a two-channel $c=4$ theory (b) on $[0,\ell]$. By analytic continuation, this may be written as a chiral $c=4$ system on $[-\ell,\ell]$ with only left-moving excitations.

C. Boundary formulation

At this point we reformulate the problem so that Cardy's boundary conformal field theory[21–23] applies. The boosted currents are defined in two-dimensional space-time (τ,x), with the impurity sitting on the time axis. We can think of the time axis as a boundary with periodic boundary conditions imposed on the currents

$$j_r(\tau,0_+) = j_r(\tau,0_-), \quad \boldsymbol{J}_r(\tau,0_+) = \boldsymbol{J}_r(\tau,0_-). \tag{2.30}$$

In Cardy's formalism no excitations (or two-momentum) may flow through the boundary, and hence periodic boundary conditions are excluded. To circumvent this restriction, we confine the system to the interval $-\ell \leqslant x \leqslant \ell$ (taking $\ell\to\infty$ at the end), fold it in half, double the currents, and identify the two points $x=-\ell$ and $x=\ell$,[36] see Figs. 3(a) and 3(b). The new currents, defined for $x\geqslant 0$ only, are related to the old ones by

$$\begin{aligned} j^1_L(x) &\equiv j_L(x), \quad \boldsymbol{J}^1_L(x) \equiv \boldsymbol{J}_L(x), \\ j^2_L(x) &\equiv j_R(-x), \quad \boldsymbol{J}^2_L(x) \equiv \boldsymbol{J}_R(-x), \\ j^1_R(x) &\equiv j_R(x), \quad \boldsymbol{J}^1_R(x) \equiv \boldsymbol{J}_R(x), \\ j^2_R(x) &\equiv j_L(-x), \quad \boldsymbol{J}^2_R(x) \equiv \boldsymbol{J}_L(-x), \end{aligned} \tag{2.31}$$

and the periodic boundary conditions in (2.30) become

$$\begin{aligned} j^1_L(0) &= j^2_R(0), \quad \boldsymbol{J}^1_L(0) = \boldsymbol{J}^2_R(0), \\ j^2_L(0) &= j^1_R(0), \quad \boldsymbol{J}^2_L(0) = \boldsymbol{J}^1_R(0). \end{aligned} \tag{2.32}$$

Hence, a flow of excitations across $x=0$ in the original system corresponds to having them come in through one channel and then reflected back through the other.

With this procedure, the critical bulk Hamiltonian is now defined on the positive x axis only:

$$\mathscr{H}^{*}_{\mathrm{TL}}=\frac{1}{2\pi}\sum_{i=1,2}\int_{0}^{\ell}dx\Big\{\frac{v_c}{4}[:j^i_L(x)j^i_L(x):+:j^i_R(x)j^i_R(x):] + \frac{v_s}{3}[:\boldsymbol{J}^i_L(x)\cdot\boldsymbol{J}^i_L(x):+:\boldsymbol{J}^i_R(x)\cdot\boldsymbol{J}^i_R(x):]\Big\}, \tag{2.33}$$

with the boundary condition (2.32) imposed at $x=0$. We also see that the electron-impurity interaction can be written as

$$\mathscr{H}_F=\lambda[\boldsymbol{J}^1_L(0)+\boldsymbol{J}^2_L(0)]\cdot\boldsymbol{S}. \tag{2.34}$$

The new Hamiltonian (2.33) is invariant under the full $\mathscr{G}=\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_1$ Kac-Moody algebra and has conformal charge $c=4$, i.e., we *represent* our $c=2$ theory defined on $-\ell\leq x\leq\ell$ by a $c=4$ theory on $0\leq x\leq\ell$. However, we may analytically continue the left-moving currents in (2.33) to the negative x axis[37] [Fig. 3(c)]. From (2.32) it then follows that these can be identified with the right-moving currents on the positive axis:

$$\begin{aligned} j^1_L(-x)&=j^2_R(x), \quad \boldsymbol{J}^1_L(-x)=\boldsymbol{J}^2_R(x),\\ j^2_L(-x)&=j^1_R(x), \quad \boldsymbol{J}^2_L(-x)=\boldsymbol{J}^1_R(x). \end{aligned} \tag{2.35}$$

Hence we can formulate the theory in terms of left-moving currents only. This leads to the form of the Hamiltonian that we shall mostly use:

$$\mathscr{H}=\mathscr{H}^{*}_{\mathrm{TL}}+\mathscr{H}_F, \tag{2.36}$$

where

$$\mathscr{H}^{*}_{\mathrm{TL}}=\frac{1}{2\pi}\sum_{i=1,2}\int_{-\ell}^{\ell}dx\Big\{\frac{v_c}{4}:j^i_L(x)j^i_L(x): + \frac{v_s}{3}:\boldsymbol{J}^i_L(x)\cdot\boldsymbol{J}^i_L(x):\Big\}, \tag{2.37}$$

and with $\mathscr{H}_F$ given in (2.34). The analytically continued currents satisfy periodic boundary conditions:

$$j^i_L(-\ell)=j^i_L(\ell), \quad \boldsymbol{J}^i_L(-\ell)=\boldsymbol{J}^i_L(\ell), \quad i=1,2, \tag{2.38}$$

yielding a theory defined on a ring with circumference 2ℓ.

Before proceeding, it is instructive to look at the special case of *noninteracting* $(g=0)$ electrons, i.e., with $v_c=v_s$ in (2.37). For this case, an alternative construction is possible: Introducing two channels (''flavors'') of left-going fields,

$$\psi^1_{L,\alpha}(x)\equiv\psi_{L,\alpha}(x), \quad \psi^2_{L,\alpha}(x)\equiv-\psi_{R,\alpha}(-x) \tag{2.39}$$

the free part of the bulk Hamiltonian in (1.2) attains the form

$$\mathscr{H}_0=\frac{v_F}{2\pi}\int dx:\psi^{j\dagger}_{L,\sigma}(x)i\frac{d}{dx}\psi^{j}_{L,\sigma}(x): \tag{2.40}$$

with the electron-impurity interaction written as

$$\mathscr{H}_F=\lambda:\psi^{j\dagger}_{L,\sigma}(0)\ \tfrac{1}{2}\boldsymbol{\sigma}_{\sigma\mu}\psi^{j}_{L,\sigma}(0):\cdot\boldsymbol{S}. \tag{2.41}$$

With the simple transformation in (2.39), one thus arrives at a chiral (left-going) representation of the *two-channel Kondo model*.[3] This has the Sugawara form[18]

$$\mathscr{H}_0+\mathscr{H}_F=\frac{v_F}{2\pi}\int_{-\ell}^{\ell}dx\Big\{\frac{1}{8}:J_L(x)J_L(x):+\frac{1}{4}:\boldsymbol{J}_L(x)\cdot\boldsymbol{J}_L(x): + \frac{1}{4}:\boldsymbol{J}^F_L(x)\cdot\boldsymbol{J}^F_L(x):\Big\}+\lambda\boldsymbol{J}_L(0)\cdot\boldsymbol{S}, \tag{2.42}$$

where the currents J_L (charge), $\boldsymbol{J}_L$ (spin), and $\boldsymbol{J}^F_L$ (flavor) generate the affine U(1), $\mathrm{SU}(2)_2$, and $\mathrm{SU}(2)_2$ algebras, respectively.

The structure in (2.42) is not easy to obtain in the presence of electron-electron interaction. Technically, the construction requires all excitations to have the same velocity, which is the case only when $g=0$. The result nonetheless suggests that the interacting problem exhibits two-channel Kondo behavior.[13] Since the forward electron scattering off the impurity only affects the spin sector, the fact that the electron-electron interaction pushes the spin and charge excitations apart by endowing them with different velocities seems irrelevant. However, to put the conclusion on firm ground, one must carefully check the role of the selection rule for combining charge and spin excitations when interactions are present. One should here recall that although the electron-impurity interaction is entirely in the spin sector, the charge sector may nonetheless contribute correction-to-scaling operators, as in the (one-channel) Kondo effect for noninteracting electrons. A second reason for dealing with the electron-electron interaction ''head on'' is that it gives us an inroad to attack the problem of Kondo interaction in a Luttinger liquid [by including backward electron scattering off the impurity (see Sec. V)].

Returning to (2.37), we see that the bulk Hamiltonian $\mathscr{H}^{*}_{\mathrm{TL}}$ separately conserves the U(1) and SU(2) excitations in the two channels. This simply reflects the fact that the Kac-Moody symmetry is given by $\mathscr{G}=\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_1$, as it must. The electron-impurity interaction, however, breaks the $\mathrm{SU}(2)_1\times\mathrm{SU}(2)_1$ symmetry. This appears similar to the effect of the Kondo interaction on free electrons. In a left- (or right-) moving description, the Kondo term breaks the single-spin $\mathrm{SU}(2)_1$ symmetry [or $\mathrm{SU}(2)_2$ in the two-channel case] of the chiral electron Hamiltonian. However, at a special value of the Kondo coupling, $\lambda_{\mathrm{Kondo}}=\lambda^{*}_{\mathrm{Kondo}}$, the impurity spin can be ''absorbed'' in the electron spin current via a canonical transformation. The impurity-electron interaction disappears from the Hamiltonian, and the $\mathrm{SU}(2)_1$ [or two-channel $\mathrm{SU}(2)_2$] symmetry is restored (now generated by the spin current of the *combined* electron-impurity system). In this scheme, $\lambda^{*}_{\mathrm{Kondo}}$ *defines* the local strong coupling fixed point. It is tempting to proceed in an analogous way for the present problem, and try to absorb the impurity via the transformations $\boldsymbol{J}^i_L(x)\rightarrow\boldsymbol{J}^i_L(x)+\boldsymbol{S}\delta(x)$, $i=1,2$, judiciously choosing a special value of the impurity-electron coupling λ^{*}. However, these transformations are not canonical and couple the two spin currents at the impurity site: the $\mathrm{SU}(2)_1$ Kac-Moody algebras for channels 1 and 2 are no longer independent. Thus, the full $\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_1$ symmetry of $\mathscr{H}^{*}_{\mathrm{TL}}$ is *not* recovered at λ^{*}. Now, suppose we could rewrite the spin part of $\mathscr{H}^{*}_{\mathrm{TL}}$ in

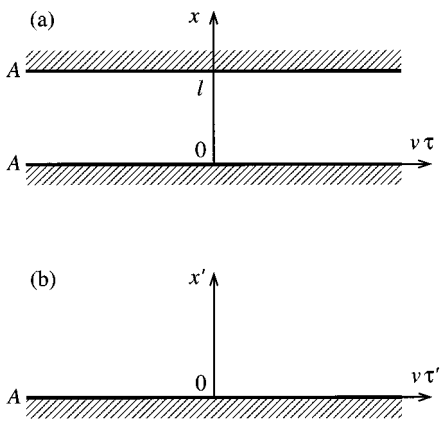

FIG. 4. In Euclidean space-time, the finite-size theory (a) on the strip $[0,\ell]$ with boundary condition A at both ends is conformally mapped to a semi-infinite plane (b) with the same boundary condition A applied at the boundary.

terms of the *total* electron spin current $\boldsymbol{J}(x)\equiv\boldsymbol{J}_L^1+\boldsymbol{J}_L^2$ (in addition to some auxiliary degrees of freedom). The total current $\boldsymbol{J}(x)$ generates the diagonal subgroup of $\mathrm{SU}(2)_1\times\mathrm{SU}(2)_1$, and, as may be easily verified, satisfies a level-two SU(2) Kac-Moody algebra

$$[J^a(x),J^b(y)]=i\epsilon^{abc}J^c(x)2\pi\delta(x-y)+\pi i\delta^{ab}\delta'(x-y). \tag{2.43}$$

The impurity may now be absorbed without problem, using the *single* canonical transformation $\boldsymbol{J}(x)\rightarrow\boldsymbol{J}(x)+\boldsymbol{S}\delta(x)\equiv\boldsymbol{J}'(x)$. The combined electron-impurity current $\boldsymbol{J}'(x)$ is conserved, and the $\mathrm{SU}(2)_2$ Kac-Moody symmetry is hence restored.

To carry out this program, one needs a dictionary to translate from the $\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_1$ formulation of the problem to another representation in terms of $\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{SU}(2)_2\times\mathscr{G}$, where $\mathscr{G}$ is the symmetry group of some auxiliary degrees of freedom. Fortunately, such a dictionary already exists in conformal field theory [the *coset construction* of Goddard, Kent, and Olive[38] (GKO)], and provides an elegant solution to the problem. Before exploiting it, however, we set up the formalism for studying the finite-size spectrum of the theory.

III. FINITE-SIZE SPECTRUM AND BOUNDARY OPERATORS

The finite-size spectrum of a 1+1D scale-invariant theory provides important information about its critical behavior. This follows from a well-known result in conformal field theory:[39] The energy levels in a finite geometry are directly connected to the (boundary) scaling dimensions of operators in the (semi-) infinite plane. More precisely, consider a conformally invariant theory defined on the strip $\{w=v\tau+ix|-\infty<\tau<\infty,\ 0\leqslant x\leqslant\ell\}$ in Fig. 4(a), with τ ''imaginary time'' and x the space coordinate. (The velocity of the excitations of the Hamiltonian is denoted by v.) Then impose a conformally invariant boundary condition, call it A, at the edges $x=0$ and $x=\ell$, and map the strip onto the semi-infinite plane $\{z=v\tau'+ix'|x'\geqslant 0\}$ in Fig. 4(b), using the conformal transformation $z=\exp(\pi w/\ell)$ (implying boundary condition A at $x'=0$). With E^0 the ground-state energy, one has

$$E=E^0+\frac{\pi v\Delta}{\ell}, \tag{3.1}$$

where $\{E\}$ is the spectrum of excited energy levels in $0\leqslant x\leqslant\ell$, and $\{\Delta\}$ is the spectrum of *boundary scaling dimensions* in the semi-infinite plane. In a 1+1D quantum mechanical realization the boundary $x'=0$ coincides with the time axis (τ' also being an imaginary time), and it follows that the boundary dimensions determine the asymptotic autocorrelation functions. In other words, for $|\tau'|\gg x'$,

$$\langle\mathscr{O}(\tau',x')\mathscr{O}(0,x')\rangle-\langle\mathscr{O}(\tau',x')\rangle\langle\mathscr{O}(0,x')\rangle\sim\frac{1}{|\tau'|^{2\Delta}}, \tag{3.2}$$

with $\mathscr{O}$ an operator with boundary dimension Δ.

For the present problem two additional features appear, not present in the standard scenario discussed above. First, the chiral (here, ''left-moving'') Hamiltonian in (2.37) represents a full 1+1D theory on the cylinder [via the folding procedure in (2.31)]: the second channel of left-moving currents simulates the presence of right-moving currents. Therefore, *bulk dimensions* appear in the finite-size scaling formula, disguised as sums of dimensions of left-moving operators labeled by the channel index. Second, $\mathscr{H}_{\mathrm{TL}}^*$ supports *two* kinds of excitations, charge and spin, with distinct velocities v_c and v_s (when $g\neq 0$). However, as we already noted, the charge and spin excitations are dynamically decoupled, and conformal invariance (including Lorentz invariance) holds separately in the two sectors. Summing up, one expects that (3.1) is replaced by

$$E-E^0=E_c+E_s-(E_c^0+E_s^0), \tag{3.3}$$

where

$$\begin{aligned}E_c-E_c^0&=\frac{\pi v_c}{\ell}(\Delta_c^1+\Delta_c^2),\\ E_s-E_s^0&=\frac{\pi v_s}{\ell}(\Delta_s^1+\Delta_s^2),\end{aligned} \tag{3.4}$$

$\{\Delta_a^j\}$ being the boundary dimensions in channel $j=1,2$ and sector $a=c$ (charge), s (spin). (This structure of the spectrum has been exhibited in a Bethe-ansatz analysis of the Hubbard chain;[29] cf. the following section.)

As we have seen in Sec. II C, it is convenient to represent a theory defined on a strip by a *chiral* theory on a cylinder. Formally, this follows from the vanishing of the energy-momentum tensor at the boundary,[21] implying that left- and right-moving operators coincide at the boundary. This is precisely what we used in (2.35) when we continued the charge and spin currents to the negative x axis. We know that the energy spectrum of this theory is in one-to-one correspondence to the boundary scaling dimensions due to the boundary conditions applied at the edges of the strip. However, these scaling dimensions are only a subset of all possible chiral dimensions of the bulk conformal field theory. One is thus faced with the task to pick out those chiral dimensions that represent the wanted boundary condition. Formally, this may be done by connecting the boundary condition to a *se-*

lection rule that prescribes how charge and spin excitations are combined at that boundary. Knowing the selection rule, and allowing only boundary operators that preserve the symmetry of the Hamiltonian, the formalism unambiguously predicts the set of possible boundary dimensions. For a quantum impurity problem that we are dealing with here, the more intricate task is to identify the correct boundary condition (or selection rule) that represents the presence of the impurity: According to a conjecture by Affleck and Ludwig,[19,20] any quantum impurity renormalizes, at the fixed point, into a particular conformally invariant boundary condition on the critical theory that carries the extended degrees of freedom (in our case, $\mathscr{H}_{\rm TL}$). *One* way of identifying this boundary condition is to start with some known, trivial boundary condition on the critical theory, with no coupling to the impurity. The associated selection rule simply describes the allowed combinations of charge and spin excitations when there is no impurity present. Now, place a spin impurity at the boundary, and couple it to the electrons. By redefining the spin current as that of electrons *and* impurity, $\boldsymbol{J}(x)\to\boldsymbol{J}(x)+\boldsymbol{S}\delta(x)\equiv\boldsymbol{J}'(x)$, the electron-impurity interaction is removed at the fixed point. The spin quantum numbers $\{j\}$ will be shifted accordingly: $\{j\}\to\{j'\}$. The new selection rule, describing the renormalized boundary condition, is then obtained from the old by substituting $\{j'\}$ for $\{j\}$. The fusion-rule hypothesis by Affleck and Ludwig[19] suggests that the shift of quantum numbers is precisely governed by the conformal field theory fusion rules, in our case those of the $SU(2)_2$ Kac-Moody algebra: The set of states *(conformal tower)* labeled by a quantum number j is mapped onto new sets labeled by j', where

$$j'=|j-\tfrac{1}{2}|,|j-\tfrac{1}{2}|+1,\ldots,\min\{j+\tfrac{1}{2},2-j-\tfrac{1}{2}\}. \quad (3.5)$$

This is the essence of the conformal field theory approach, to be exploited below.

In this section we study the finite-size spectrum of $\mathscr{H}^*_{\rm TL}$, and derive expressions for the scaling dimensions $\{\Delta^j_a\}$ on a form adapted to the impurity problem. We verify our result by matching it to that of the exact Bethe-ansatz analysis of the Hubbard model. Bringing the electron-impurity interaction into play, we then use the *coset construction*[38] to make a conformal embedding of the original $SU(2)_1\times SU(2)_1$ spin currents into $SU(2)_2\times\mathbb{Z}_2$. [This corresponds to writing the spin part of $\mathscr{H}^*_{\rm TL}$ as a single $SU(2)_2$ Sugawara Hamiltonian together with an Ising model.] With this proviso, we suggest a particular application of the fusion-rule hypothesis,[19,24] and absorb the impurity spin in the total electron spin current $\boldsymbol{J}(x)$, using the conformal field theory fusion rules for the $SU(2)_2$ Kac-Moody algebra. From this, the spectrum of boundary scaling dimensions in the presence of the impurity spin is read off.

A. Finite-size spectrum of $\mathscr{H}^*_{\rm TL}$

Before applying these techniques to our Hamiltonian (2.37), we need to rewrite it in Fourier space. Introducing the Fourier-transformed currents

$$\begin{pmatrix} j^i_m \\ \boldsymbol{J}^i_m \end{pmatrix}=\frac{1}{2\pi}\int_{-\ell}^{\ell}dx\; e^{im\frac{\pi}{\ell}x}\begin{pmatrix} j^i_L(x) \\ \boldsymbol{J}^i_L(x) \end{pmatrix} \quad (3.6)$$

for $i=1,2$ and $m\in\mathbb{Z}$, the mode-expanded Hamiltonian takes the form

$$\mathscr{H}^*_{\rm TL}=\sum_{i=1,2}\mathscr{H}^{\,i}_{\,c}+\mathscr{H}^{\,i}_{\,s}, \quad (3.7)$$

where

$$\mathscr{H}^{\,i}_{\,c}=\frac{\pi v_c}{\ell}\left(\frac{1}{4}j^i_0j^i_0+\frac{1}{2}\sum_{m=1}^{\infty}j^i_{-m}j^i_m-\frac{1}{24}\right), \quad (3.8a)$$

$$\mathscr{H}^{\,i}_{\,s}=\frac{\pi v_s}{\ell}\left(\frac{1}{3}\boldsymbol{J}^i_0\cdot\boldsymbol{J}^i_0+\frac{2}{3}\sum_{m=1}^{\infty}\boldsymbol{J}^i_{-m}\cdot\boldsymbol{J}^i_m-\frac{1}{24}\right). \quad (3.8b)$$

This result is easy to verify. Consider first the U(1) currents, with

$$:j^i_L(x)j^i_L(x):\equiv\lim_{\delta\to0}[j^i_L(x+\delta)j^i_L(x)-\langle j^i_L(x+\delta)j^i_L(x)\rangle]. \quad (3.9)$$

Noting that j^i_L is a dimension-one analytic operator with argument $z=v_c\tau+ix$, it follows from the operator product expansion[40] of j^i_L with itself that[41]

$$\langle j^i_L(x+\delta)j^i_L(x)\rangle=\frac{2}{(i\delta)^2}+\frac{1}{\ell^2}O\left[\left(\frac{\delta}{\ell}\right)^0\right], \quad (3.10)$$

where the last term is the correction due to the finite ℓ. The inverse of (3.6),

$$\begin{pmatrix} j^i_L(x) \\ \boldsymbol{J}^i_L(x) \end{pmatrix}=\frac{\pi}{\ell}\sum_{m=-\infty}^{\infty}e^{-im\frac{\pi}{\ell}x}\begin{pmatrix} j^i_m \\ \boldsymbol{J}^i_m \end{pmatrix}, \quad (3.11)$$

together with (3.9) and (3.10), implies that

$$\frac{1}{2\pi}\int_{-\ell}^{\ell}dx:j^i_L(x)j^i_L(x):=\lim_{\delta\to0}\left[\frac{\pi}{\ell}\sum_n e^{-in\frac{\pi}{\ell}\delta}j^i_nj^i_{-n}+\frac{2\ell}{\pi\delta^2}\right]. \quad (3.12)$$

Using the U(1) Kac-Moody algebra (2.23) in Fourier space,

$$[j^i_n,j^k_m]=2n\delta_{n+m,0}\delta^{ik}, \quad (3.13)$$

we may write

$$\sum_n e^{-in\frac{\pi}{\ell}\delta}j^i_nj^i_{-n}=j^i_0j^i_0+2\sum_{n>0}\cos\left(n\frac{\pi}{\ell}\delta\right)j^i_{-n}j^i_n+2\sum_{n>0}ne^{-in\frac{\pi}{\ell}\delta}, \quad (3.14)$$

so that the last term cancels the singular part of (3.12):

$$\frac{2\pi}{\ell}\sum_{n>0}ne^{-in\frac{\pi}{\ell}\delta}=-\frac{2\ell}{\pi\delta^2}-\frac{\pi}{6\ell}+\frac{1}{\ell}O\left[\left(\frac{\delta}{\ell}\right)^2\right]. \quad (3.15)$$

Hence, $\mathscr{H}^{\,i}_{\,c}$ (3.8a) follows from comparing (2.37) with (3.12). The analogous treatment of the Fourier transformed spin currents, satisfying the $SU(2)_1$ Kac-Moody algebra

$$[J^{ia}_n,J^{kb}_m]=i\epsilon^{abc}J^{ic}_{n+m}\delta^{ik}+\frac{n}{2}\delta_{n+m,0}\delta^{ab}\delta^{ik}, \quad (3.16)$$

leads to (3.8b). As can be seen from the derivation, the Schwinger-type terms in (3.8a) and (3.8b), including the constant 1/24, are due to the regularization of the spectrum (normal ordering of the quadratic currents). These terms encode the conformal anomaly[40] of the theory, and we shall discuss their role below.

Given $\mathscr{H}^{\,i}_{\,c}$ (3.8a) and $\mathscr{H}^{\,i}_{\,s}$ (3.8b), it is now easy to extract the finite-size spectrum. Let us start with the charge Hamiltonian, $\mathscr{H}_c=\mathscr{H}_c^1+\mathscr{H}_c^2$, and make the connection to the original electron fields in the Tomonaga-Luttinger Hamiltonian (1.2). By construction [cf. Eqs. (2.31) and (2.35)] we can identify

$$j_m^1=j_{L,m}\,,\quad j_m^2=j_{R,m}\,, \tag{3.17}$$

where

$$j^{L}_{\,R},_m=\frac{1}{2\pi}\int_{-\ell}^{\ell}dx e^{\pm im(\pi/\ell)x}j^L_R(x) \tag{3.18}$$

are the Fourier transforms of the left–right-moving U(1) currents introduced in (2.22). Thus

$$\begin{aligned} j_m^1&=\cosh\theta J_{L,m}+\sinh\theta J_{R,m}\,,\\ j_m^2&=\cosh\theta J_{R,m}+\sinh\theta J_{L,m}\,, \end{aligned} \tag{3.19}$$

with

$$J^{L}_{\,R},_m=\frac{1}{2\pi}\int_{-\ell}^{\ell}dx\; e^{\pm im(\pi/\ell)x}{:}\psi^{\dagger}_{L\atop R},_\sigma(x)\psi_{L\atop R},_\sigma(x){:} \tag{3.20}$$

the Fourier components of the original U(1) currents in (2.10a). For later convenience we impose antiperiodic boundary conditions[42] on $\psi_{r,\sigma}(x)$, which then can be expanded as

$$\psi^{L}_{\,R},_\sigma(x)=\frac{\pi}{\ell}\sum_n\; e^{\mp i(n+1/2)(\pi/\ell)x}\psi^{L}_{\,R},_{\sigma,n} \tag{3.21}$$

This definition implies that the momenta for $\psi^{L}_{\,R},_{\sigma,n}$ are given by $k=\mp(\pi/\ell)(n+1/2)$, so that the "single-particle" energy levels $\mp v_c k$, with respect to the Fermi level $k=0$, both satisfy

$$\epsilon_n=\frac{v_c\pi}{\ell}(n+\tfrac{1}{2}). \tag{3.22}$$

As follows from the definition of the integrand of (3.20) in terms of point splitting (2.12), we need the (finite-size) Green's function

$$\langle\psi^{\dagger}_{L\atop R},_\sigma(x+\delta)\psi^{L}_{\,R},_\sigma(x)\rangle=\frac{2}{(\pm i\delta)}+\frac{1}{\ell}O\left(\frac{\delta}{\ell}\right) \tag{3.23}$$

to obtain

$$J^{L}_{\,R},_m=\lim_{\delta\to0}\left[\frac{\pi}{\ell}\sum_n\; e^{\pm i(n+1/2)(\pi/\ell)\delta}\psi^{\dagger}_{L\atop R},_{\sigma,n}\psi_{L\atop R},_{\sigma,n+m}\mp\frac{2\ell\delta_{m0}}{i\pi\delta}\right]. \tag{3.24}$$

It is now convenient to introduce normal ordering in Fourier space. The filled Fermi sea occupies all levels for $n<0$, and hence

$$:\psi^{\dagger}_{r,\sigma,n}\psi_{r,\sigma,m}:=\begin{cases}\psi^{\dagger}_{r,\sigma,n}\psi_{r,\sigma,m}\,, & n\neq m \quad\text{or}\quad n=m\geqslant0,\\ -\psi_{r,\sigma,m}\psi^{\dagger}_{r,\sigma,n}\,, & n=m<0,\end{cases} \tag{3.25}$$

for $r=L$ or R. Using $\{\psi_{r,\sigma,m}\,,\psi^{\dagger}_{s,\mu,n}\}=(\ell/\pi)\delta_{rs}\delta_{\sigma\mu}\delta_{mn}$ and

$$\sum_{n>0}\; e^{\pm i(n+\frac{1}{2})\frac{\pi}{\ell}\delta}=\pm\frac{\ell}{i\pi\delta}+O\left(\frac{\delta}{\ell}\right), \tag{3.26}$$

we may finally write

$$J^{L}_{\,R},_m=\frac{\pi}{\ell}\sum_n\;{:}\psi^{\dagger}_{L\atop R},_{\sigma,n}\psi_{L\atop R},_{\sigma,n+m}{:}. \tag{3.27}$$

The interpretation of $J^{L}_{\,R},_m$ is now straightforward: For $m=0$ it counts the net number of left- (right-) moving particles with respect to the Fermi sea, whereas for $m\neq0$ it excites particles m steps.

Next we introduce a set of *Kac-Moody primary states* $|\Lambda_P\rangle$ with respect to the charge and spin currents,[40] defined by

$$J^{L}_{\,R},_m|\Lambda_P\rangle=0,\quad \boldsymbol{J}^{L}_{\,R},_m|\Lambda_P\rangle=0,\quad m>0, \tag{3.28}$$

from which all other states can be generated. In an occupation number representation,[43]

$$|\Lambda_P\rangle\equiv|Q_{L\uparrow}\,,Q_{L\downarrow}\,,Q_{R\uparrow}\,,Q_{R\downarrow}\rangle \tag{3.29}$$

with the constraint

$$|Q^{L}_{\,R}{}_\uparrow-Q^{L}_{\,R}{}_\downarrow|\leqslant1. \tag{3.30}$$

Here $|Q_{r\sigma}\rangle$ denotes a nonexcited state with $Q_{r\sigma}$ the number of $r=L$, R electrons, carrying spin σ, added to the filled Fermi sea. Hence, the total charge in channel r, $Q_r=Q_{r\uparrow}+Q_{r\downarrow}$, is the eigenvalue of $J_{r,0}$ in (3.27). Combining (3.8a), (3.19), and (3.27), we thus have

$$\mathscr{H}^{\,i}_{\,c}|\Lambda_P\rangle=\frac{\pi v_c}{\ell}\left\{\frac{(q^i)^2}{4}-\frac{1}{24}\right\}|\Lambda_P\rangle,\quad i=1,2, \tag{3.31}$$

where we have introduced the eigenvalues of j_0^i,

$$q^{\,1}_{\,2}=Q\frac{e^{\theta}}{2}\pm\Delta Q\frac{e^{-\theta}}{2} \tag{3.32}$$

labeled by the quantum numbers

$$\begin{aligned} Q&\equiv Q_L+Q_R=Q_{L\uparrow}+Q_{L\downarrow}+Q_{R\uparrow}+Q_{R\downarrow}\,,\\ \Delta Q&\equiv Q_L-Q_R=Q_{L\uparrow}+Q_{L\downarrow}-Q_{R\uparrow}-Q_{R\downarrow}\,. \end{aligned} \tag{3.33}$$

As $Q_L\,,Q_R\in\mathbb{Z}$, it follows that $Q,\Delta Q\in\mathbb{Z}$ with the restriction that $Q\pm\Delta Q$ is even.

Turning to the spin Hamiltonian, $\mathscr{H}_s$, we proceed analogously. Writing $\boldsymbol{J}_0^i\cdot\boldsymbol{J}_0^i=3J_0^{iz}J_0^{iz}$ in (3.8b), and identifying Fourier modes,

$$J_m^{1z}=J_{L,m}^z, \quad J_m^{2z}=J_{R,m}^z, \tag{3.34}$$

the spin Hamiltonian (3.8b) and the Fourier transform of (2.10b) imply that

$$\mathscr{H}_s^i|\Lambda_P\rangle=\frac{\pi v_s}{\ell}\left(\frac{1}{4}(Q_\uparrow^i-Q_\downarrow^i)^2-\frac{1}{24}\right)|\Lambda_P\rangle. \tag{3.35}$$

(In obvious notation, $Q^1=Q_L$ and $Q^2=Q_R$.) Introducing spin quantum numbers $j^i=0,1/2$, $i=1,2$, connected to the particle numbers by

$$\frac{4}{3}j^i(j^i+1)=(Q_\uparrow^i-Q_\downarrow^i)^2\leqslant 1, \tag{3.36}$$

the eigenvalues in (3.35) are expressed as

$$E_s^i(j^i)=\frac{\pi v_s}{\ell}\left\{\frac{1}{3}j^i(j^i+1)-\frac{1}{24}\right\}, \quad j^i=0,\tfrac{1}{2}. \tag{3.37}$$

[This result can also be obtained directly by noting that $\boldsymbol{J}_0^i\cdot\boldsymbol{J}_0^i$ is the Casimir invariant of $SU(2)_1$, with eigenvalues $j^i(j^i+1)$ when acting on the primary states.[34]]

As a consequence of $[\mathscr{H}_c,\mathscr{H}_s]=0$, the primary states factorize in charge and spin,

$$|\Lambda_P\rangle=|\Lambda_P\rangle_c\otimes|\Lambda_P\rangle_s, \tag{3.38}$$

and as the Hamiltonians are diagonal in channels 1 and 2,

$$|\Lambda_P\rangle_s=|j^1\rangle^{(1)}\otimes|j^2\rangle^{(2)}, \tag{3.39a}$$

$$|\Lambda_P\rangle_c=|q^1\rangle^{(1)}\otimes|q^2\rangle^{(2)}, \tag{3.39b}$$

the superscripts denoting the two channels. Note that the q^i's mix the original L and R channels and are defined by (3.32) in terms of the quantum numbers Q and ΔQ.

To obtain a state of arbitrary particle number $(Q_{L\uparrow},Q_{L\downarrow},Q_{R\uparrow},Q_{R\downarrow})$ we undo the constraints (3.30) by applying the operators $J_{r,-m}^{\pm}=J_{r,-m}^x\pm J_{r,-m}^y$, with $m>0$, on $|\Lambda_P\rangle$. These operators flip spin within the left- and right-moving branches. The resulting states, call them $|\Lambda\rangle$, are still labeled by $Q_{\substack{L\\R}\alpha}$, which now may be any integer. Furthermore, by acting with the operators $J_{r,-m}$ and $J_{r,-m}^z$ $(m>0)$ on $|\Lambda\rangle$, we also remove the constraint that $|Q_{r\sigma}\rangle$ is nonexcited, i.e., we allow nonfilled levels below the highest occupied level: As is readily verified, the Fourier modes $(1/2)J_{r,-m}\pm J_{r,-m}^z$ $(m>0)$ create "particle-hole excitations" from the states $|\Lambda\rangle$ within each $Q_{r\sigma}$ branch. *Any* state of particle number $(Q_{L\uparrow},Q_{L\downarrow},Q_{R\uparrow},Q_{R\downarrow})$ can thus be obtained from the set of primary states. (In the following, we label our states in the diagonal basis of $\mathscr{H}$, though.) As follows from (3.19) and (3.28), the primary states $|\Lambda_P\rangle$ are also primary with respect to the diagonalized currents j_{-m}^i and $\boldsymbol{J}_{-m}^i$ and *any* other excited state is obtained by applying these operators for $m>0$.

To extract the spectrum including the energy levels of the *descendant states* just exposed, it is sufficient to note that the operators $\{j_{-m}^k;m>0,\ k=1,2\}$ and $\{\boldsymbol{J}_{-m}^k;m>0,\ k=1,2\}$ act as "raising operators" with respect to the primary states in (3.39a) and (3.39b). Explicitly, from (3.8a) and (3.13),

$$[\mathscr{H}_c^{\ i},j_{-m}^k]=\frac{\pi v_c}{\ell}mj_{-m}^k\delta^{ik}. \tag{3.40}$$

Thus, comparing with (3.31), the descendant levels in (the diagonal basis of) the charge sector are obtained by adding energy to the primary levels in steps of $\pi v_c/\ell$. The resulting finite-size spectrum organizes into *conformal towers*[44] of equally spaced energy levels, each tower having a primary level q^i as base:

$$E_c^i(q^i,N_c^i)=\frac{\pi v_c}{\ell}\left\{\frac{(q^i)^2}{4}-\frac{1}{24}+N_c^i\right\}, \quad N_c^i\in\mathbb{N},\ \ i=1,2, \tag{3.41}$$

with

$$q^{\frac{1}{2}}=Q\frac{e^\theta}{2}\pm\Delta Q\frac{e^{-\theta}}{2}, \quad Q,\ \Delta Q\in\mathbb{Z}. \tag{3.42}$$

Similarly, (3.8b) and (3.16) imply

$$[\mathscr{H}_s,\boldsymbol{J}_{-m}^k]=\frac{\pi v_s}{\ell}m\boldsymbol{J}_{-m}^k. \tag{3.43}$$

A comparison with (3.35) yields the finite-size spectrum in the spin sector, with two conformal towers ($j^i=0,1/2$) per channel:

$$E_s^i(j^i,N_s^i)=\frac{\pi v_s}{\ell}\left\{\frac{1}{3}j^i(j^i+1)-\frac{1}{24}+N_s^i\right\}, \quad N_s^i\in\mathbb{N},\ \ i=1,2. \tag{3.44}$$

Putting $Q=\Delta Q=j^1=j^2=0$ and summing over the channels in (3.41) and (3.44), we obtain the ground-state energy $E^0=E_c^0+E_s^0$ with

$$E_j^0=-\frac{\pi v_j}{12\ell}, \quad j=c,s. \tag{3.45}$$

As is well known, the finite-size correction to the ground-state energy $E^0=E_L^0+E_R^0$ of a conformally invariant Hamiltonian (defined on a ring of circumference L_x) scales as $E_{\substack{L\\R}}^0=-\pi v c/12L_x$,[45] v being the velocity of the massless excitations. In the present case, we use a chiral formulation and must accordingly compare (3.45) with E_L^0 only. Putting $L_x=2\ell$, we obtain $c=2$ in the charge and spin sectors, respectively, yielding a total $c=4$. This is what we expect, since we are using a $c=4$ representation of our original $c=2$ theory.

Given the finite-size spectrum, we now identify the corresponding scaling dimensions. Using (3.4), together with (3.41) and (3.45), we have

$$\Delta_c^i=\frac{1}{4}\left(e^\theta\frac{Q}{2}-(-1)^ie^{-\theta}\frac{\Delta Q}{2}\right)^2+N_c^i, \quad i=1,2,\ \ N_c^i\in\mathbb{N}. \tag{3.46}$$

Similarly, (3.4), (3.44), and (3.45) imply for the scaling dimensions in the spin sector:

$$\Delta_s^i=\frac{1}{3}j^i(j^i+1)+N_s^i, \quad j^i=0,\tfrac{1}{2},\ \ i=1,2,\ \ N_s^i\in\mathbb{N}. \tag{3.47}$$

The dimension of a composite operator is thus given by

$$\Delta=\Delta_c^1+\Delta_c^2+\Delta_s^1+\Delta_s^2. \tag{3.48}$$

We recognize the scaling dimensions in (3.46) as those of a U(1) Gaussian theory represented by free bosons with periodicity $\phi=\phi+2\pi\mathscr{R}$, with $\mathscr{R}=e^{-\theta}/\sqrt{2}$.[46] The same structure is hidden in (3.47). Substituting $Q^{\pm}=Q_{\uparrow}^1-Q_{\downarrow}^1 \pm(Q_{\uparrow}^2-Q_{\downarrow}^2)$ for Q and ΔQ, respectively, choosing $e^{-\theta}=1$ and using (3.36), (3.46) gives (3.47). For this special value of the periodicity ($\mathscr{R}=1/\sqrt{2}$), it is known that the symmetry is enhanced to SU(2),[47] and the scaling dimensions become those of an SU(2) $k=1$ WZW model.

It may here be worthwhile to recall the key elements of the analysis: Eq. (3.48) gives the spectrum of *chiral dimensions* of the analytically continued theory in the full complex plane. As we pointed out above, *boundary scaling dimensions* in the half-plane correspond to subsets of this spectrum, with one subset for each particular boundary condition. By the ''folding procedure'' in (2.31), the subset corresponding to the boundary condition (2.32) exactly coincides with the set of *bulk dimensions* of the original theory in the full plane. To pick out these dimensions from (3.48) we must first identify the rule for assigning values to the quantum numbers $(Q,\Delta Q,j^1,j^2)$ in the presence of the boundary condition (2.32). By the equivalence with the bulk problem, this is the same as identifying the selection rule governing the bulk energy spectrum of $\mathscr{H}_{\mathrm{TL}}^*$.

Assuming that the electron-electron interaction does not obstruct the choice of particle number $(Q_{L\uparrow},Q_{L\downarrow},Q_{R\uparrow},Q_{R\downarrow})$, (3.33) and (3.36) imply the *Luttinger liquid selection rule:*

$$\begin{aligned} j^1&=\frac{1}{4}(Q+\Delta Q)\ \mathrm{mod}1,\\ j^2&=\frac{1}{4}(Q-\Delta Q)\ \mathrm{mod}1. \end{aligned} \tag{3.49}$$

Remember that $Q\pm\Delta Q$ is even, which is consistent with the allowed values of j^i. At this point we would also like to point out that this selection rule includes an implicit relation between the two charge channels 1 and 2: The definition of q^i in terms of Q and ΔQ only permits certain combinations of the q^1 and q^2 conformal towers. In contrast, the conformal towers in the spin sector are not constrained by such a relation.

The scaling dimensions of the possible boundary operators are now obtained from (3.48) by feeding into (3.46) and (3.47) the allowed values of $(Q,\Delta Q,j^1,j^2)$ according to (3.49). Not all of these operators appear, though, in the effective theory describing the scaling region. In general, this can be written as an expansion

$$\mathscr{H}=\mathscr{H}^*+\sum_k g_k\mathscr{O}_k(0), \tag{3.50}$$

where $\mathscr{H}^*$ is the fixed-point Hamiltonian, and g_k and $\mathscr{O}_k$ are the associated scaling fields and boundary operators. Corrections to scaling are produced by the irrelevant operators, and these must be invariant under the continuous symmetries of $\mathscr{H}$ (as must any relevant operators in the absence of external perturbations). Applied to our case, $\mathscr{H}=\mathscr{H}_{\mathrm{TL}}^*+\Sigma_k g_k\mathscr{O}_k(0)$, invariant under chiral $\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{SU}(2)_1\times\mathrm{SU}(2)_1$. In other words, charge and spin are conserved separately in the two channels, implying $Q=\Delta Q=0$ and $j^1=j^2=0$ for $\mathscr{O}_k$. Hence, the only boundary operators appearing at the fixed point are descendants of the identity. This trivial content of correction-to-scaling operators could of course have been predicted directly from symmetry arguments, without invoking the finite-size spectrum. However, having the spectrum in hand, *including* the selection rule in (3.49), we can attack the more challenging problem of electron-impurity scattering.

Before doing so, however, we shall compare our spectrum and selection rule to those from an exact Bethe-ansatz analysis of the Hubbard chain (of which $\mathscr{H}_{\mathrm{TL}}^*$ is the fixed-point theory). As was first shown by Woynarovich[29] (see also Refs. 30 and 31), the Bethe-ansatz spectrum of a Hubbard model on a finite ring also organizes into conformal towers. Away from half-filling:

$$E-E_0\sim\frac{2\pi v_c}{N}(\Delta_c^+ +\Delta_c^-)+\frac{2\pi v_s}{N}(\Delta_s^+ +\Delta_s^-)+O\left(\frac{1}{N^2}\right). \tag{3.51}$$

Here N is the number of sites on the ring, v_c and v_s are model dependent charge and spin velocities, and $\Delta_c^{\pm}$ and $\Delta_s^{\pm}$ are given by

$$\Delta_c^{\pm}=\frac{1}{2}\left(\frac{I_c}{2\xi_c}\pm\xi_c\left(D_c+\frac{D_s}{2}\right)\right)^2+N_c^{\pm}, \tag{3.52a}$$

$$\Delta_s^{\pm}=\frac{1}{4}\left(I_s-\frac{I_c}{2}\pm D_s\right)^2+N_s^{\pm}, \tag{3.52b}$$

with $N_c^{\pm},N_s^{\pm}\in\mathbb{N}$. As in the conformal approach, the positive integers $N^{\pm}$ label ''particle-hole excitations'' (although the notion of a ''particle'' or ''hole'' in a Bethe-ansatz basis is different from that used here). The parameter ξ_c is a nonuniversal function of the microscopic parameters, while (I_c,I_s,D_c,D_s) are quantum numbers subject to the Bethe-ansatz selection rule

$$\begin{aligned} D_c&=\frac{1}{2}(I_c+I_s)\quad(\ \mathrm{mod}1),\\ D_s&=\frac{1}{2}I_c\quad(\ \mathrm{mod}1), \end{aligned} \tag{3.53}$$

where $2D_c,2D_s,I_c,I_s\in\mathbb{Z}$.

To match our result in (3.46) and (3.47) to that in (3.52), we note that the dressed charge ξ_c and our parameter e^{θ} both measure the strength of the Hubbard interaction. Explicitly, using the parametrization in Sec. II, we have that $e^{\theta}=\{1+U/[2\pi t\sin(ak_F)]\}^{1/4}$, while $\xi_c=\sqrt{2}\{1-U/[8\pi t\sin(ak_0)]\}$,[30] k_0 playing the role of a ''Fermi momentum'' in the Bethe-ansatz formalism. Putting $k_0=k_F$, it follows that $\xi_c=\sqrt{2}e^{-\theta}+O[(U/t)^2]$. To lowest order in U/t, $\Delta_c^1+\Delta_c^2$ and $\Delta_s^1+\Delta_s^2$ in (3.46) and (3.47) indeed exhaust the same *combined* spectrum of scaling dimensions as $\Delta_c^+ +\Delta_c^-$ and $\Delta_s^+ +\Delta_s^-$ in (3.52). This can be seen from the following analysis.

Let us first generate all combinations of scaling dimensions according to the Luttinger liquid selection rule (3.49) and show that they equal allowed combinations of scaling dimensions according to the Bethe-ansatz selection rule (3.53). We may concentrate on $N_c^i=N_s^i=0$, as higher levels can be trivially mapped once the relation for $N_c^i=N_s^i=0$ has been established. Moreover, if we formally set $Q=I_c$ and use $\xi_c=\sqrt{2}e^{-\theta}$, we see that it is sufficient to check that ΔQ and $4D_c+2D_s$ span the same range of integers when $\Delta_s^1+\Delta_s^2=\Delta_s^++\Delta_s^-$. Whether this correspondence is possible or not depends on the selection rule.

Consider $Q=2n$, $n\in\mathbb{Z}$. By construction, ΔQ is even, and from (3.47) and (3.49) it follows that for $\Delta_s^1+\Delta_s^2=0$ we have $\Delta Q=4m$ for n even and $\Delta Q=4m+2$ for n odd ($m\in\mathbb{Z}$). This is exactly reproduced in the Bethe-ansatz spectrum by choosing $D_s=0$ (which is allowed as I_c is even) and $I_s=n$. According to the selection rule, the allowed values of D_c are p for n even and $p+\frac{1}{2}$ for n odd ($p\in\mathbb{Z}$), i.e., $4D_c+2D_s$ gives the same range of integers as ΔQ. The other possibility, $\Delta_s^1+\Delta_s^2=1/2$, is analogously combined with the above values of ΔQ shifted by 2, which is reproduced by choosing $D_s=0$, $I_s=n+1$, and $D_c=p+1/2$ for n even and $D_c=p$ for n odd ($p\in\mathbb{Z}$).

Next, consider $Q=2n+1$, $n\in\mathbb{Z}$. Then $\Delta_s^1+\Delta_s^2=\frac{1}{4}$ and ΔQ spans all odd integers. Choosing $I_s=n$ and $D_s=\pm 1/2$ is consistent with (3.53) and gives $\Delta_s^++\Delta_s^-=\frac{1}{4}$. For I_s even, the allowed values of D_c are $p+\frac{1}{2}$ and for I_s odd, they are p ($p\in\mathbb{Z}$). In both cases, $4D_c+2D_s$ spans all odd integers as required. This completes the first part of our comparison.

In order to prove that the Luttinger liquid and Bethe-ansatz spectra are identical, we must also check that any other allowed combination of quantum numbers of the latter only reproduce energy levels already obtained. (Note that we only compare energy levels and not their degeneracies.) So far we have exhausted all combinations of I_c and D_c. By also allowing I_s and D_s to take any permissible value, one can show that all energy levels in the Bethe-ansatz solution for $N_c^\pm=N_s^\pm=0$ correspond to allowed combinations of $\Delta_c^1+\Delta_c^2$ and $\Delta_s^1+\Delta_s^2$ with $N_c^i=0$ and $N_s^i\in\mathbb{N}$, respectively. The calculation is straightforward, so we leave it out for brevity. As before, it is trivial to extend the mapping to include $N_c^\pm, N_s^\pm\in\mathbb{Z}^+$, and thus the desired result follows.

B. Coset construction: Boundary operators in presence of $\mathscr{H}_F$

Having established the spectrum of $\mathscr{H}_{\rm TL}^*$, we now include the electron-impurity term

$$\mathscr{H}_F=\lambda[\boldsymbol{J}_L^1(0)+\boldsymbol{J}_L^2(0)]\cdot\boldsymbol{S} \tag{3.54}$$

and explore how this interaction affects the boundary scaling dimensions. For this purpose, it is useful to rewrite the Hamiltonian $\mathscr{H}_{\rm TL}^*+\mathscr{H}_F$ in terms of the total spin current $\boldsymbol{J}=\boldsymbol{J}_L^1+\boldsymbol{J}_L^2$: The chiral $\mathrm{SU}(2)_1\times\mathrm{SU}(2)_1$ symmetry of $\mathscr{H}_{\rm TL}^*$ gets broken by $\mathscr{H}_F$, whereas the diagonal subgroup $\mathrm{SU}(2)_2$, generated by $\boldsymbol{J}$, remains as a symmetry (cf. our discussion in Sec. II).

In the conformal field theory formalism it is sufficient to work at the level of representations of the $\mathrm{SU}(2)_k$ algebras. Given a direct product of two irreducible representations (conformal towers) of $\mathrm{SU}(2)_1$, we thus ask how the states reappear using representations of $\mathrm{SU}(2)_2$. The answer is immediately obtained from the GKO coset construction:[38] Products of two $\mathrm{SU}(2)_1$ conformal towers $(j^1)_1\times(j^2)_1$ decompose into products of $\mathrm{SU}(2)_2$ and Ising model conformal towers $(j)_2\times(\phi)$ according to

$$\begin{aligned}(0)_1\times(0)_1&=(0)_2\times(\mathbb{1})+(1)_2\times(\epsilon),\\ (0)_1\times(\tfrac{1}{2})_1&=(\tfrac{1}{2})_2\times(\sigma),\\ (\tfrac{1}{2})_1\times(\tfrac{1}{2})_1&=(0)_2\times(\epsilon)+(1)_2\times(\mathbb{1}).\end{aligned} \tag{3.55}$$

[Note that $(1/2)_1\times(0)_1$ is degenerate with $(0)_1\times(1/2)_1$.] Here $j^i=0,1/2$ and $j=0,1/2,1$ label the conformal towers of $\mathrm{SU}(2)_1$ and $\mathrm{SU}(2)_2$, respectively, while $\phi=\mathbb{1}$ (identity), σ (order parameter), and ϵ (energy density) label the three conformal towers of the Ising model ($c=1/2$). The scaling dimensions in the $\mathrm{SU}(2)_2$ sector are given by

$$\Delta_S=\frac{1}{4}j(j+1)+N_S,\quad N_S\in\mathbb{N}, \tag{3.56}$$

while

$$\Delta_{\rm Ising}=\begin{cases}0 & + & N & (\mathbb{1}),\\ \dfrac{1}{16} & + & N & (\sigma),\quad N\in\mathbb{N}\\ \dfrac{1}{2} & + & N & (\epsilon),\end{cases} \tag{3.57}$$

are those of the Ising sector.

All primary states in the new representation are now labeled by $(Q,\Delta Q,j,\phi)$, with the new selection rule obtained by combining (3.49) and (3.55):

$$(j,\phi)=\begin{cases}(0,\mathbb{1})\ \text{ or }\ (1,\epsilon), & Q\ \text{ even},\ \dfrac{1}{2}(Q+\Delta Q)\ \text{ even},\\ (0,\epsilon)\ \text{ or }\ (1,\mathbb{1}), & Q\ \text{ even},\ \dfrac{1}{2}(Q+\Delta Q)\ \text{ odd},\\ \left(\dfrac{1}{2},\sigma\right), & Q\ \text{ odd}.\end{cases} \tag{3.58}$$

At the Hamiltonian level, the conformal embedding $\mathrm{SU}(2)_1\times\mathrm{SU}(2)_1\to\mathrm{SU}(2)_2\times\mathrm{Ising}$ implies that the spin part of $\mathscr{H}_{\rm TL}$ decomposes into a sum of an $\mathrm{SU}(2)_2$ Sugawara and a free Majorana Hamiltonian:

$$\tilde{\mathscr{H}}_s=\frac{v_s}{2\pi}\int_{-\ell}^{\ell}dx\left\{\frac{1}{4}{:}\boldsymbol{J}(x)\cdot\boldsymbol{J}(x){:}+\eta_L(x)i\frac{d}{dx}\eta_L(x)\right\}, \tag{3.59}$$

with η_L a left-moving Majorana fermion. This enables us to absorb the electron-impurity term $\mathscr{H}_F=\lambda\boldsymbol{J}(0)\cdot\boldsymbol{S}$ into $\tilde{\mathscr{H}}_s$ by making the canonical transformation

$$\boldsymbol{J}(x)\to\boldsymbol{J}(x)+\boldsymbol{S}\delta(x), \tag{3.60}$$

and choosing $\lambda=\lambda^*\equiv v_s/4\pi$. In this scheme λ^* defines the (nonuniversal) fixed-point coupling: the elimination of $\mathscr{H}_F$

by (3.60) restores translational (and thereby conformal) invariance, implying a fixed-point theory.

The redefinition of the total current in (3.60) changes the rule for coupling conformal towers. Effectively, (3.60) adds an extra spin-1/2 degree of freedom to the $SU(2)_2$ sector, leaving the U(1) and Ising sectors intact. As a result, the $SU(2)_2$ conformal towers $(j)_2$ get replaced, according to

$$(0)_2 \to (\tfrac{1}{2})_2, \quad (\tfrac{1}{2})_2 \to (0)_2 \text{ or } (1)_2, \quad (1)_2 \to (\tfrac{1}{2})_2. \tag{3.61}$$

These are the $SU(2)_2$ *fusion rules*, see (3.5), describing the effect of combining a tower of spin j with a spin-1/2 tower. Together with (3.58) this leads to the new selection rule

$$(j,\phi) = \begin{cases} (\tfrac{1}{2},\mathbb{1}) \text{ or } (\tfrac{1}{2},\epsilon), & Q \text{ even}, \\ (0,\sigma) \text{ or } (1,\sigma), & Q \text{ odd}, \end{cases} \tag{3.62}$$

which governs the finite-size spectrum of $\mathscr{H}^*_{\mathrm{TL}} + \mathscr{H}_F$, or equivalently, the spectrum of $\mathscr{H}^*_{\mathrm{TL}}$ with a modified boundary condition at $x=0$ representing the impurity spin.

However, to obtain the scaling dimensions related to a boundary condition, we recall from the beginning of Sec. III that these are in one-to-one correspondence to the finite-size spectrum of the Hamiltonian with that boundary condition applied at *both* ends. In terms of the Hamiltonian (2.33), defined on $x \in [0,\ell]$, this corresponds to having impurity spins at both ends of the space interval. Passing over to the chiral formulation (2.37), we thus see that the boundary scaling dimensions of $\mathscr{H} = \mathscr{H}^*_{\mathrm{TL}} + \mathscr{H}_F$ are in one-to-one correspondence to the energy levels of the *auxiliary Hamiltonian*

$$\mathscr{H}' = \mathscr{H}^*_{\mathrm{TL}} + \lambda \int_{-\ell}^{\ell} dx \boldsymbol{J}(x)[\boldsymbol{S}_1 \delta(x) + \boldsymbol{S}_2 \delta(x-\ell)]. \tag{3.63}$$

The two impurity terms are removed at the fixed point by the canonical transformation

$$\boldsymbol{J}(x) \to \boldsymbol{J}(x) + \boldsymbol{S}_1 \delta(x) + \boldsymbol{S}_2 \delta(x-\ell), \tag{3.64}$$

by which *two* extra spin-1/2 degrees of freedom are added to the $SU(2)_2$ sector. The fusion hypothesis[19] suggests that the effect is described by *two* repeated fusions of the spin j conformal towers with a $j=1/2$ tower. The new selection rule that emerges is readily extracted, using (3.61) twice to replace the $SU(2)_2$ towers in (3.58) by the resulting set of conformal towers. One finds

$$(j,\phi) = \begin{cases} (0 \text{ or } 1, \mathbb{1} \text{ or } \epsilon), & Q \text{ even}, \\ (\tfrac{1}{2},\sigma), & Q \text{ odd}. \end{cases} \tag{3.65}$$

As before, ΔQ has the same parity as Q, i.e., they are both even or both odd.

Summarizing, the spectrum of boundary dimensions in the presence of $\mathscr{H}_F$ is obtained from

$$\Delta = \Delta_c^1 + \Delta_c^2 + \Delta_S + \Delta_{\mathrm{Ising}}, \tag{3.66}$$

using (3.65) to insert the allowed combinations of $(Q,\Delta Q,j,\phi)$ into the expressions for Δ_c^i $(i=1,2)$, Δ_S, and Δ_{Ising} in (3.46), (3.56), and (3.57), respectively.

IV. CRITICAL BEHAVIOR

We now turn to the question of the *impurity critical behavior*, exploiting the result derived in the previous section. To do so requires two steps. First, the *symmetry preserving* boundary operators must be identified from the spectrum in (3.66). With these in hand, one then selects the operator of lowest dimension to perturbatively calculate the leading finite-size corrections to the fixed-point theory. Treating the (inverse) length as a temperature variable allows for the calculation of the finite-T scaling of physical response functions, such as the impurity contribution to the specific heat and magnetic susceptibility.

It is tempting to make a shortcut and refer to the logic of our procedure to conclude that the impurity critical behavior must be that of the two-channel Kondo problem: The selection rule in (3.65), describing the renormalized boundary condition due to the impurity, descends from the Luttinger liquid selection rule in (3.49) via the coset construction and ''double fusion.'' This selection rule in turn is nothing but the free electron selection rule in disguise, ''rotated'' to a diagonal basis in the charge sector. One thus infers that the interacting problem cannot differ in essence, provided that the charge sector does not contribute a (nontrivial) leading correction-to-scaling boundary operator (which, by chiral charge conservation, is excluded). Although this line of reasoning is essentially correct, it is instructive to carry out the analysis ''by hand,'' and observe how two-channel Kondo physics emerges in the scheme proposed here. In fact, certain novel features appear: the leading correction-to-scaling boundary operator (LCBO) is found to be *unique* at the fixed point, suggesting a universal Wilson ratio in the low-temperature, strong-coupling limit. Also, by working out the solution explicitly in the new scheme we gain a number of important insights, to be exploited when extending the analysis to Kondo scattering in a Luttinger liquid (Sec. V).

A. Finite-temperature scaling

To provide some background, let us briefly review the finite-temperature scaling approach to the problem, including the rather exotic mechanism that brings the *irrelevant* boundary operators to center stage. For more details, we refer to Refs. 19 and 20.

Consider the (folded) system confined to a spatial interval $[0,\ell]$ and in the presence of an external magnetic field h. By treating temperature T as an inverse length $1/\beta$, we may extract the low-temperature thermodynamics via finite-size scaling on a cylinder with circumference $v\beta$. A convenient choice is to map the upper half-plane $\mathbb{C}^+ = \{\mathrm{Im} z > 0\}$ [$T=0$ geometry in Fig. 4(b)] onto a cylinder $\Gamma^+ = \{w = (v\beta/\pi)\arctan z\}$ (finite-T geometry in Fig. 5). At finite temperature the free energy separates into two pieces, describing the bulk and the impurity, respectively,

$$F(\beta,h,\lambda_I) = 2\ell f_{\mathrm{bulk}}(\beta,h) + f_{\mathrm{imp}}(\beta,h,\lambda_I), \tag{4.1}$$

with λ_I the leading irrelevant boundary scaling field.[48] By writing 2ℓ instead of ℓ in (4.1), we let f_{bulk} refer to the free-energy density of the original (unfolded) system; cf. Figs. 3(a) and 3(b) and the remark after (2.34).

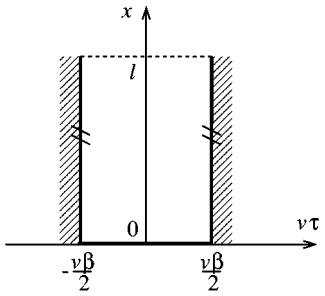

FIG. 5. Finite-temperature geometry with periodic boundary condition in imaginary time τ. The temperature T is identified as $1/\beta$ and the circumference $v\beta$ of the cylinder satisfies $v\beta \ll \ell$. For $\ell \to \infty$, this geometry is related to the zero-temperature geometry in Fig. 4(b) via the conformal map $w=(v\beta/\pi)\arctan(z)$.

From the standard finite-size scaling hypothesis,[49] it follows that the reduced free-energy density of a critical theory, defined on a cylinder with circumference $v\beta$, scales as

$$\beta F/\ell = f_0 v\beta + Y(h\beta)/v\beta + \cdots, \tag{4.2}$$

where the universal amplitude $Y_{\rm bulk}(0) = -\pi c/6$.[45] In our case, the critical theory is a sum of two conformal theories, each with $c=2$, and with effective velocities $v = v_c$ and v_s, respectively. Moreover, as the magnetic field h only couples to the spin sector, we may simply add the contributions from the two sectors:

$$f_{\rm bulk}(\beta,h) = E_{\rm bulk} - \frac{\pi}{6v_c\beta^2} + \frac{1}{2v_s\beta^2} Y_s(h\beta) + \cdots, \tag{4.3}$$

with $E_{\rm bulk}$ a nonuniversal quantity and $Y_s(0) = -\pi/3$. Putting $\beta = 1/T$ one thus obtains for the bulk specific heat:

$$C_{\rm bulk}(T) = -T\frac{\partial^2 f_{\rm bulk}}{\partial T^2} = \frac{\pi}{3}\left(\frac{1}{v_c} + \frac{1}{v_s}\right)T + \cdots, \tag{4.4}$$

with "$\cdots$" denoting subleading terms as $T \to 0$. The bulk magnetic susceptibility can similarly be obtained from $f_{\rm bulk}$ by expanding Y_s to second order in h/T, and one finds

$$\chi_{\rm bulk}(T) = -\left.\frac{\partial^2 f_{\rm bulk}}{\partial h^2}\right|_{h=0} = \frac{1}{2\pi v_s} + \cdots. \tag{4.5}$$

In the same manner as for $f_{\rm bulk}$ one can write down a finite-size scaling ansatz for the impurity part of the free energy:

$$f_{\rm imp}(\beta,h,\lambda_I) = E_{\rm imp} + \frac{1}{\beta} Y_{\rm imp}(h\beta, \lambda_I \beta^y) + \cdots, \tag{4.6}$$

with $E_{\rm imp}$ nonuniversal. The exponent y is a renormalization-group eigenvalue, connected to the dimension Δ of the leading irrelevant boundary operator by $y = 1-\Delta$. As the boundary scaling field λ_I may couple to both the charge and the spin sector ($\Delta = \Delta_c + \Delta_s$), $Y_{\rm imp}$ is not just a sum of a charge and a spin part. This will not be important, though, as we will later calculate $f_{\rm imp}$ without any assumptions about $Y_{\rm imp}$. However, for $\lambda_I = 0$ there is only one scaling field h and it couples only to the spin sector. The h dependence of $Y_{\rm imp}$ is therefore given by the same universal function as in a theory with a single velocity.

From the scaling form of $Y_{\rm imp}$ one expects nonanalytic temperature terms in $f_{\rm imp}$ for noninteger y and $\lambda_I \neq 0$. Remarkably, the impurity specific heat and susceptibility vanish identically when $\lambda_I = 0$, so these nonanalytic terms become dominant. The mechanism behind this unusual behavior is particularly transparent within the conformal field theory approach: Since, *at the fixed point*, the impurity spin has been absorbed in the spin current $\boldsymbol{J}$, the fixed point theory has lost all memory of the impurity. Specifically, the total magnetization

$$\int_{-\infty}^{\infty} dx \langle J^z(x)\rangle = \frac{\partial F(\beta,h,\lambda_I=0)}{\partial h} \tag{4.7}$$

is insensitive to the impurity, and thus, from (4.6),

$$\frac{\partial Y_{\rm imp}(h\beta,\lambda_I=0)}{\partial(h\beta)} = 0. \tag{4.8}$$

This implies that exactly *at* the fixed point

$$\chi_{\rm imp}(\lambda_I=0) = -\left.\frac{\partial^2 f_{\rm imp}}{\partial h^2}\right|_{h=0} = -\beta \left.\frac{\partial^2 Y_{\rm imp}(h\beta,\lambda_I=0)}{\partial(h\beta)^2}\right|_{h=0} = 0, \tag{4.9}$$

and analogously

$$C_{\rm imp}(\lambda_I=0) = -T\frac{\partial^2 f_{\rm imp}}{\partial T^2} = 0. \tag{4.10}$$

To "put back" the effect of the impurity, the leading irrelevant boundary operator must be added, and this operator then produces the *dominant* term in the scaling in temperature.

B. Boundary operators

Following the procedure outlined in the beginning of this section, we first pick out the scaling dimensions in (3.66) corresponding to the symmetry preserving operators.

It is easiest to start in the charge sector. Chiral U(1) invariance implies charge conservation in each channel, hence $Q = \Delta Q = 0$. The selection rule in (3.65) then implies that possible boundary operators can appear only in the products of conformal towers

$$(Q,\Delta Q,j,\phi) = (0,0,0,\bar{1}),(0,0,1,\bar{1}),(0,0,0,\epsilon), \text{ and } (0,0,1,\epsilon). \tag{4.11}$$

The first and fourth structures in (4.11) contain only operators with integer scaling dimensions. As we shall see in the next section, these produce Fermi-liquid-like, analytic scaling in temperature, and will be disregarded at this point. Turning to the second and third structures, we note that both contribute a *relevant* boundary operator, with $\Delta < 1$.[50] The second possibility, $(0,0,1,\bar{1})$, contains the spin-1 primary field, call it $\boldsymbol{\phi}$, with $\Delta = 1/2$. However, the *total* spin is conserved, requiring all operators in the $SU(2)_2$ sector to transform as singlets. This expels $\boldsymbol{\phi}$, leaving only descendant operators in this sector with $\Delta \geq 3/2$. The third structure,

$(0,0,0,\epsilon)$, contains the Ising energy density as a primary field, also with dimension $\Delta=1/2$. Is it also suppressed by symmetry, or do we have to ''fine-tune'' the theory to stay at the fixed point? Consider the latter alternative. The operator ϵ originates from the conformal embedding of $SU(2)_1 \times SU(2)_1$ into $SU(2)_2 \times \mathbb{Z}_2$. The only parameter in $\mathcal{H}=\mathcal{H}^*_{\rm TL}+\mathcal{H}_F$ multiplying the $SU(2)_1$ currents is the spin velocity $v_s=v_F-g$. A ''fine-tuning'' scenario thus implies that we can stay at the fixed point only for some privileged value of g, which, considering the known solution for noninteracting electrons, must be $g=0$. Although not excluded *a priori*, this is not a likely situation. Let us instead explore whether there is any symmetry that suppresses ϵ.

The important symmetry to consider is invariance under channel exchange: $\mathcal{E}:1\leftrightarrow 2$, under which $\boldsymbol{J}^1_L(x)\leftrightarrow\boldsymbol{J}^2_L(x)$. To test for $\mathcal{E}$, we connect ϵ to the spin currents via the following observation: The difference of $SU(2)_1$ currents, $\boldsymbol{J}^1_L(x)-\boldsymbol{J}^2_L(x)$, transforms in the adjoint (spin-1) representation of *global* SU(2) (with the Schwinger term subtracted):

$$[J^{1a}_L(x)+J^{2a}_L(x),J^{1b}_L(y)-J^{2b}_L(y)]=i\epsilon^{abc}[J^{1c}_L(x) - J^{2c}_L(x)]\,\delta(x-y). \tag{4.12}$$

Knowing that the currents carry dimension one, one is led to the identification

$$\boldsymbol{J}^1_L(x)-\boldsymbol{J}^2_L(x)\sim\boldsymbol{\phi}(x)\times\boldsymbol{\epsilon}(x). \tag{4.13}$$

This gives the correct assignment of spin $(=1)$ and scaling dimension $(\Delta_{\boldsymbol{\phi}}+\Delta_{\epsilon}=1/2+1/2=1)$. Now consider a channel exchange $\mathcal{E}$. According to (4.13),

$$\mathcal{E}:\boldsymbol{\phi}\times\boldsymbol{\epsilon}\rightarrow-\boldsymbol{\phi}\times\boldsymbol{\epsilon}. \tag{4.14}$$

A consistent conformal representation hence requires that we assign evenness (oddness) to ϵ ($\boldsymbol{\phi}$) under $\mathcal{E}$, *or* vice versa. Consider the first possibility: ϵ even and $\boldsymbol{\phi}$ odd. Since $\mathcal{H}$ is invariant under $\mathcal{E}$, $\boldsymbol{\phi}$ and all its descendants become suppressed. The operator ϵ, on the other hand, is allowed, and the required fine-tuning leaves us with a critical theory only at $g=0$, according to the argument above. The first descendant of ϵ, $L_{-1}\epsilon$ with scaling dimension 3/2, here becomes the leading irrelevant operator. But this is exactly the boundary operator that drives two-impurity Kondo behavior,[24] in contradiction to the *known* two-channel behavior at $g=0$. Agreement with established results in the noninteracting limit forces us to make the alternative assignment under $\mathcal{E}$: ϵ odd and $\boldsymbol{\phi}$ even. Now the ϵ conformal tower gets suppressed, with no fine-tuning necessary. Our analysis shows that there is no relevant boundary operator present: The only candidates appearing in (4.11), $\boldsymbol{\phi}$ $(\Delta=1/2)$ and ϵ $(\Delta=1/2)$ are expelled by symmetry. This yields the important conclusion that the theory flows onto a *stable* fixed point in the absence of external perturbations.

The leading irrelevant boundary operator, call it $\mathcal{O}_I$, is obtained by contracting $\boldsymbol{\phi}$ with the first $SU(2)_2$ raising operator $\boldsymbol{J}_{-1}$: $\mathcal{O}_I=\boldsymbol{J}_{-1}\cdot\boldsymbol{\phi}$, with dimension 3/2. As expected, $\mathcal{O}_I$ is the same boundary operator that appears in the analysis of the two-channel Kondo effect, using the ''charge-spin-flavor'' scheme,[19] reviewed in Sec. II. In that formulation, however, the flavor sector contributes a second operator with dimension 3/2, call it $\mathcal{O}_{\rm flavor}$. This is different from our approach where the leading correction-to-scaling operator $\mathcal{O}_I$ is unique. Consistency between the two schemes requires that the scaling field conjugate to $\mathcal{O}_{\rm flavor}$ vanishes at the fixed point. Based on the related case of ''underscreening,'' it has been argued that this indeed happens: When the impurity carries unit spin both the charge and flavor sectors contribute correction-to-scaling operators with the same dimension $(=2)$ as the spin sector. Given a proper regularization of the theory, one can show that the conjugate charge and flavor scaling fields $\rightarrow 0$ at the trivial $T>T_K$ fixed point. There is evidence that an analogous mechanism is at play at the ''overscreened'' $T<T_K$ fixed point (with $s=1/2$), leaving only the operator $\mathcal{O}_I$ effective. Our result supports this picture, and allows for the definition of a universal Wilson ratio in the limit of vanishing electron-electron interaction.

C. Low-temperature thermodynamics

The low-temperature thermodynamics driven by $\mathcal{O}_I$ has been studied in Ref. 19. In the next section we shall review this analysis, and extend it to theories with more than one velocity and boundary operators of arbitrary dimensions. Here we only quote the result for the impurity specific heat $C_{\rm imp}$ and magnetic susceptibility $\chi_{\rm imp}$:

$$C_{\rm imp}(T,\lambda_I)=\frac{\lambda_I^2 9\pi^2}{v_s^3}T\,\ln\left(\frac{1}{\tau_0 T}\right)+O(\tau_0 T), \tag{4.15}$$

$$\chi_{\rm imp}(T,\lambda_I)=\frac{\lambda_I^2 18}{v_s^3}\ln\left(\frac{1}{\tau_0 T}\right)+O[(\tau_0 T)^0], \tag{4.16}$$

with τ_0 a short-time cutoff, playing the role of an inverse ''Kondo temperature.'' The scaling field λ_I (conjugate to $\mathcal{O}_I$) appears with the same power in (4.15) and (4.16). Thus, given the known bulk response in (4.4) and (4.5), one may form the λ_I- and T-independent Wilson ratio

$$R_W\equiv\frac{\chi_{\rm imp}/\chi_{\rm bulk}}{C_{\rm imp}/C_{\rm bulk}}=\frac{4}{3}\left(1+\frac{v_s}{v_c}\right)=\frac{8}{3}\left\{1-\frac{g}{v_F}+O\left[\left(\frac{g}{v_F}\right)^2\right]\right\}. \tag{4.17}$$

In the $g\rightarrow 0$ $(v_c,v_s\rightarrow v_F)$ limit we recover the universal number 8/3 obtained by Affleck and Ludwig[19] for the two-channel Kondo effect.

In the alternative charge-spin-flavor scheme for the two-channel Kondo effect, the flavor sector also contributes a dimension-3/2 boundary operator. This produces a second term in $C_{\rm imp}$, proportional to the square of a flavor scaling field $\lambda_{\rm flavor}$. To obtain the Wilson ratio $R_W=8/3$, one has to resort to the argument sketched above, suggesting that $\lambda_{\rm flavor}\rightarrow 0$ as one approaches the fixed point. In contrast, the reduction to *one* scaling field λ_I is automatic in our approach.

The results in (4.15) and (4.16) also apply to the associated spin-chain problem, with two neighboring sites coupled antiferromagnetically with equal strength J_0 to a spin-1/2 impurity (cf. Fig. 2 and the discussion in Sec. II). This corroborates the result in Ref. 33, and shows that the closing of an open chain with its two ends coupled symmetrically to the same impurity does not affect the scaling behavior: both the

open and closed chains exhibit the same "two-channel Kondo physics."[14] This, from a naive point of view, is somewhat surprising: The two spins at the end points of the open chain are locked into an $S=1$ state via the interaction with the impurity. This causes overscreening of the impurity spin, resulting in the two-channel Kondo behavior. Upon closing the chain the end point spins get mutually coupled via an antiferromagnetic exchange, call it J. As J is increased above J_0 one may have guessed that the local $S=1$ state gets destabilized, thus inhibiting overscreening, and leading to a decoupling of the impurity spin ("Curie behavior"). This does not happen however. The results in (4.15) and (4.16) are valid independent of the relative strengths of the bare couplings J_0 and J: The low-temperature fixed point is *stable*, attracting a flow for arbitrary initial (bare) values of the couplings J and J_0. What *does* change, however, is the temperature scale at which one reaches this fixed point. For example, if the ratio J_0/J is very small, the Kondo temperature will also be small, and one has to go to very low temperatures to see the "two-channel behavior."

D. Symmetry breaking perturbations

It is instructive to study the stability of the forward scattering fixed point against symmetry breaking perturbations. Consider first an asymmetric electron-impurity interaction,

$$\mathscr{H}_F' = \lambda_L \boldsymbol{J}_L(0)\cdot\boldsymbol{S} + \lambda_R \boldsymbol{J}_R(0)\cdot\boldsymbol{S}, \tag{4.18}$$

with λ_L and λ_R arbitrary. This may be expressed as $\mathscr{H}_F' = \mathscr{H}_F + \mathscr{H}_{\rm pert}$ where $\mathscr{H}_F$ is the symmetric electron-impurity interaction [with $\lambda=(1/2)(\lambda_L+\lambda_R)$] studied above, and

$$\mathscr{H}_{\rm pert} = \frac{1}{2}(\lambda_L-\lambda_R)[\boldsymbol{J}_L^1(0)-\boldsymbol{J}_L^2(0)]\cdot\boldsymbol{S}. \tag{4.19}$$

$\mathscr{H}_{\rm pert}$ breaks invariance under channel exchange, and using (4.13) one has

$$\mathscr{H}_{\rm pert} \sim [\boldsymbol{\phi}(0)\times\boldsymbol{\epsilon}(0)]\cdot\boldsymbol{S}. \tag{4.20}$$

It follows that $\boldsymbol{\epsilon}$ enters as an allowed boundary operator, being contained in the spectrum (4.11). Having dimension $\Delta=1/2$, it is relevant, and destabilizes the symmetric forward scattering fixed point.[51] This is similar to adding a flavor anisotropic perturbation to the two-channel Kondo interaction, using a charge-spin-flavor scheme.[26] In this case the flavor sector contributes a dimension-1/2 operator which takes the system to a one-channel (Fermi-liquid-like) fixed point. Again, equivalence between the two schemes requires $\boldsymbol{\epsilon}$ to perform the same way. An interesting question is how the scaling region about this one-channel fixed point is influenced by the electron-electron interaction. However, we shall not pursue this problem here.

Instead, let us study the effect of adding electron backscattering on the impurity, that is, adding $\mathscr{H}_B$ in (1.8) to $\mathscr{H}_F$. $\mathscr{H}_B$ breaks both chiral SU(2) *and* U(1) invariance. In particular, this means that the numbers of left- and right-moving particles are not conserved separately. As a consequence, $\Delta Q \equiv Q_L - Q_R$ is no longer restricted to zero, and the charge sector can now make nontrivial contributions to the spectrum of boundary operators. The lowest-dimension operator with $\Delta Q \neq 0$ allowed by the selection rule in (3.65) is obtained from

$$(Q,\Delta Q,j,\phi) = (0,\pm 2,0,1), \tag{4.21}$$

and has dimension $\Delta=(1/2)e^{-2\theta}\leqslant 1/2$. Any $|\Delta Q|>2$ will produce an irrelevant operator, and hence the composite primary operator from (4.21) is the unique relevant operator generated by breaking chiral U(1) invariance. [Note that the breaking of chiral SU(2) invariance is already effected by $\mathscr{H}_F$.] As expected, $\mathscr{H}_B$ is thus a *relevant perturbation* that pulls the system away from the forward scattering fixed point, towards a new fixed point. The scaling behavior about this new fixed point is the topic of the next section.

V. KONDO INTERACTION

The problem we have studied so far (with only forward electron scattering off the impurity) is a model problem: we do not expect it to be observed in the laboratory. The reason is simply that by inserting a spinful (and chargeless) impurity into a, say, quantum wire, one necessarily produces a sharp scattering potential, the core of the interaction being a localized spin exchange. In contrast, the exclusion of electron back scattering off the impurity, i.e., the exclusion of *large* momentum transfers, requires an effective scattering potential that varies slowly in space. To have an experimentally relevant model we must therefore incorporate the back scattering term $\mathscr{H}_B$, as in Eq. (1.8), and study the *full* electron-impurity (Kondo) interaction $\mathscr{H}_F+\mathscr{H}_B$ in the strong-coupling, low-temperature regime. As we shall see, the insights gained from the treatment of the forward-scattering problem will turn out to be crucial in attacking this more difficult problem.

Before plunging into the analysis, we briefly comment upon the recent work by Furusaki and Nagaosa[13] who exploit a version of "poor-man's scaling" to analyze the problem. As for the ordinary Kondo problem,[52] a set of scaling equations is derived perturbatively in the limit of weak electron-impurity coupling, and tentatively extended into the strong-coupling regime. This procedure suggests that the coupling increases indefinitely, implying a ground state where the impurity spin locks into a singlet with the conduction electrons, causing the chain to break. A drawback of this method is that the scaling equations are formally valid only for weak couplings, and may miss out on possible intermediate-coupling fixed points, as in the two-channel Kondo effect. However, considering the analogy with the ordinary (one-channel) Kondo problem, it is quite likely that the result that ensues is in fact valid. Nonetheless, it is important to make an *independent* test of the result, using a method that is internally consistent. This is the aim of the following analysis.

A. The noninteracting problem

As a preliminary, let us look at the case of 1D *free* electrons coupled to a localized spin ($S=1/2$) by the Kondo interaction $\mathscr{H}_K=\mathscr{H}_F+\mathscr{H}_B$. The total Hamiltonian $\mathscr{H}=\mathscr{H}_0+\mathscr{H}_K$ is given by

$$\mathscr{H}_0=\frac{v_F}{2\pi}\int dx\Big[:\psi^\dagger_{L,\sigma}(x)i\frac{d}{dx}\psi_{L,\sigma}(x):$$
$$-:\psi^\dagger_{R,\sigma}(x)i\frac{d}{dx}\psi_{R,\sigma}(x):\Big], \tag{5.1}$$

$$\mathscr{H}_K=\lambda_K:[\psi^\dagger_{L,\sigma}(0)+\psi^\dagger_{R,\sigma}(0)]\ \tfrac{1}{2}\boldsymbol{\sigma}_{\sigma\mu}[\psi_{L,\mu}(0)$$
$$+\psi_{R,\mu}(0)]:\cdot\boldsymbol{S}. \tag{5.2}$$

Noting that only the even-parity part of the electron field couples to the impurity, it is convenient to pass to a *Weyl basis* via the canonical transformation

$$\psi_{\pm,\sigma}(x)=\frac{1}{\sqrt{2}}[\psi_{L,\sigma}(x)\pm\psi_{R,\sigma}(-x)]. \tag{5.3}$$

From this construction it follows that both $\psi_{+,\sigma}$ and $\psi_{-,\sigma}$ are chiral, left-moving fields with definite parity P under $x\to -x$:

$$P\psi_{\pm,\sigma}(x)=\pm\psi_{\pm,\sigma}(x). \tag{5.4}$$

In this basis the total Hamiltonian takes the form

$$\mathscr{H}_0+\mathscr{H}_K=\frac{v_F}{2\pi}\sum_{r=+,-}\int dx:\psi^\dagger_{r,\sigma}(x)i\frac{d}{dx}\psi_{r,\sigma}(x):$$
$$+2\lambda:\psi^\dagger_{+,\sigma}(0)\ \tfrac{1}{2}\boldsymbol{\sigma}_{\sigma\mu}\psi_{+,\mu}(0):\cdot\boldsymbol{S}. \tag{5.5}$$

We recognize (5.5) as *identical* to the Hamiltonian representing three-dimensional free electrons in two channels ("+" and "−"), coupled to a Kondo impurity in the "+" channel only. This leads to a one-channel Kondo fixed point with a $\pi/2$ phase shift of the single-electron levels in the "+" channel, the "−" channel being unaffected. [In fact, as pointed out already by Nozières and Blandin[3] (see also Ref. 26), the same conclusion holds for any channel-anisotropic Kondo interaction: the screening of the impurity is fully attained by the electrons in the more strongly coupled channel.]

The low-temperature impurity thermodynamics of the one-channel problem is that of a local Fermi liquid.[2] In particular, the impurity specific heat $C_{\rm imp}$ and magnetic susceptibility $\chi_{\rm imp}$ scale as

$$C_{\rm imp}\sim T+O(T^2),\quad \chi_{\rm imp}\sim T^0+O(T). \tag{5.6}$$

From (5.5), this result also holds for *one-dimensional* free electrons coupled to a spin-1/2 Kondo impurity.

B. The interacting problem

The construction above is no longer useful when the electron-electron interaction is included, that is, when

$$\mathscr{H}_{\rm int}=\frac{1}{2\pi}\int dx$$
$$\times\Big\{\frac{g}{2}\sum_{r,s=L,R}:\psi^\dagger_{r,\sigma}(x)\psi_{r,\sigma}(x)::\psi^\dagger_{s,-\sigma}(x)\psi_{s,-\sigma}(x):$$
$$+g:\psi^\dagger_{R,\sigma}(x)\psi_{L,\sigma}(x)\psi^\dagger_{L,-\sigma}(x)\psi_{R,-\sigma}(x):\Big\} \tag{5.7}$$

is added to $\mathscr{H}_0+\mathscr{H}_K$: The interaction in (5.7) mixes left- and right-moving fields, and hence becomes nonlocal in the Weyl basis.

To make progress we must take a less direct route. We shall here exploit the expectation (see Sec. I and Ref. 20) that the full Kondo interaction $\mathscr{H}_K$ can be described as a renormalized boundary condition on $\mathscr{H}^*_{\rm TL}$, analogous to the case of the "forward" interaction $\mathscr{H}_F$ studied in the previous sections. This is indeed a well-founded assumption: In the noninteracting limit ($g=0$) a canonical transformation on the even-parity spin current removes the impurity from the Hamiltonian in the Weyl basis (cf. Sec. V A). This, "by construction," automatically leads to a change of boundary condition on the critical bulk theory. Turning on the bulk interaction, this boundary condition must still be present, although its effect (coded in the new selection rule for combining conformal towers) may change with a variation of the bulk coupling. However — as we have just seen — when $g\neq 0$, $\mathscr{H}_K$ cannot be reformulated in terms of spin currents without violating locality of the electron-electron interaction. Therefore, we cannot identify the correct selection rule (boundary condition) by a redefinition of the spin current, as we did for $\mathscr{H}_F$ in Sec. III. In fact, we should not even expect that the new selection rule is simply related to the old one by a recombination of conformal towers in the spin sector only. To the contrary: Since the Kondo interaction $\mathscr{H}_K$ carries a charge component in the chiral basis, the charge sector is affected too. In other words, the selection rule for combining the two charge conformal towers may also change. This implies that the boundary operators may be composites of nontrivial operators from the spin *and* charge sectors. The fact that these are described by *two distinct conformal field theories* signals the novel aspect of the problem.

To obtain a sufficiently general framework for this new situation, we introduce a notation that does not make an implicit relation between the two charge towers, and denote a general combination of conformal towers by

$$(C_1,D_1;C_2,D_2;j;\phi). \tag{5.8}$$

Here (C_i,D_i) labels the U(1) channel-i tower ($i=1,2$), while j and ϕ, as before, denote the $\mathrm{SU}(2)_2$ and Ising towers. The two U(1) towers are now treated as *independent*. In the description of $\mathscr{H}^*_{\rm TL}+\mathscr{H}_F$, as well as that of $\mathscr{H}^*_{\rm TL}$ with a trivial boundary, $C_1=C_2=Q$ and $D_1=D_2=\Delta Q$. These identities are the above-mentioned implicit relations between the charge towers and may be interpreted as part of the corresponding selection rule, whereas more general boundary conditions, associated with other types of impurity interactions, like $\mathscr{H}_K$, may require $C_1\neq C_2$ and $D_1\neq D_2$. Note that we still require $C_i\pm D_i$ to be even for $i=1,2$, so as to preserve the spectrum *within* each conformal tower.

The next step is to find the selection rule, and we first focus on the effect of $\mathscr{H}_K$ on the charge sector. As the quantum numbers C_i and D_i can take any integer values, we assume that any selection rule for combining the U(1) conformal towers can be expressed by a linear relation:

$$C_1=\alpha C_2+\beta D_2+\eta, \tag{5.9a}$$

$$D_1=\gamma C_2+\delta D_2+\zeta, \tag{5.9b}$$

with $\alpha,\beta,\ldots,\zeta$ integers satisfying $\alpha\delta-\beta\gamma\neq 0$. As we shall see, a few symmetry constraints severely limit the number of possibilities and leave us with only two permissible selection rules. Given these two rules for the U(1)×U(1) charge sector, we then consider all possible combinations of conformal towers from the SU(2)$_2$× Ising sector. Each combination of states corresponds to a (composite) boundary operator, and those with scaling dimensions $\geqslant 1$ are candidates for being the *leading correction-to-scaling-boundary operator (LCBO)* that governs the critical behavior at the ''Kondo fixed point'' in a Luttinger liquid. As the Kondo interaction (5.2) breaks the chiral U(1) [as well as the SU(2)] invariance of $\mathscr{H}^*_{\rm TL}$, C_i and D_i of the LCBO may now take nonzero values. However, *global* U(1) [as well as SU(2)] remains a symmetry, imposing other, weaker, constraints on these quantum numbers. [The conditions on the quantum numbers in the SU(2)$_2$ and Ising sectors are as before.] The list of candidate operators is then restricted by requiring that any LCBO respects the symmetry of the original Hamiltonian (including $\mathscr{H}_K$), *and* that the Fermi liquid scaling in (5.6) is correctly reproduced as $g\to 0$. (Note that a selection rule defines a *boundary* fixed point, and is valid for all values of the marginal bulk coupling g. Hence, given a selection rule, Fermi liquid scaling must emerge in the limit $g\to 0$.)

With these preliminaries, let us make a first list of candidate LCBO scaling dimensions at the $\mathscr{H}_{\rm TL}+\mathscr{H}_K$ fixed point. From (3.46), (3.56), and (3.57), treating the two U(1) towers in (3.46) as independent, we have

$$\Delta=\Delta_c+\Delta_s, \tag{5.10}$$

where

$$\Delta_c=\frac{1}{4}\{(q_1)^2+(q_2)^2\}+N_c, \quad N_c\in\mathbb{N}, \tag{5.11}$$

with

$$q_1=\tfrac{1}{2}C_1e^{\theta}+\tfrac{1}{2}D_1e^{-\theta},$$
$$q_2=\tfrac{1}{2}C_2e^{\theta}-\tfrac{1}{2}D_2e^{-\theta} \tag{5.12}$$

[replacing the former q^i of (3.32)] and

$$\Delta_s=\begin{Bmatrix}0\\ \frac{3}{16}\\ \frac{1}{2}\end{Bmatrix}+\begin{Bmatrix}0\\ \frac{1}{16}\\ \frac{1}{2}\end{Bmatrix}+N_s, \quad N_s\in\mathbb{N}. \tag{5.13}$$

To cut down the list of possible scaling dimensions in (5.10), we next study the constraints imposed by the symmetries of the model.

C. Symmetries and selection rules

All states in the charge sector are combinations of states from the two U(1) conformal towers labeled by (C_1,D_1) and (C_2,D_2), or q_1 and q_2 for short. Similarly, we label the Kac-Moody primary states of these conformal towers $|q_1\rangle$ and $|q_2\rangle$, respectively. They transform under independent U(1) transformations as

$$|q_j\rangle\to e^{iq_j\phi_j}|q_j\rangle. \tag{5.14}$$

It is convenient to introduce the linear combinations

$$q\equiv q_1+q_2=\tfrac{1}{2}(C_1+C_2)e^{\theta}+\tfrac{1}{2}(D_1-D_2)e^{-\theta},$$
$$\Delta q\equiv q_1-q_2=\tfrac{1}{2}(C_1-C_2)e^{\theta}+\tfrac{1}{2}(D_1+D_2)e^{-\theta} \tag{5.15}$$

and

$$\phi\equiv\tfrac{1}{2}(\phi_1+\phi_2),$$
$$\Delta\phi\equiv\tfrac{1}{2}(\phi_1-\phi_2), \tag{5.16}$$

so that we get a general combination of U(1) transformations as

$$|q_1\rangle\otimes|q_2\rangle\equiv|q_1,q_2\rangle\to e^{i\varphi}|q_1,q_2\rangle \tag{5.17}$$

with

$$\varphi=q_1\phi_1+q_2\phi_2=q\phi+\Delta q\Delta\phi. \tag{5.18}$$

In terms of these phase factors, global U(1) is generated by ϕ and chiral U(1) by ϕ and $\Delta\phi$. Hence, $|q_1,q_2\rangle$ is global U(1) invariant if $q=0$ and chiral U(1) invariant if $q=\Delta q=0$. This is consistent with our previous notion of global and chiral U(1) invariance in terms of Q and ΔQ, because $q=Qe^{\theta}$ and $\Delta q=\Delta Qe^{-\theta}$ for the (bulk) Luttinger liquid *and* forward scattering problem, where $Q=C_1=C_2$ and $\Delta Q=D_1=D_2$.

Consider now the original fixed-point theory $\mathscr{H}^*_{\rm TL}$ without impurity. Adding $\mathscr{H}_K$ breaks chiral, but not global, U(1) invariance. As $\mathscr{H}^*_{\rm TL}$ satisfies the Luttinger liquid selection rule, we see that the effect of adding the perturbation $\mathscr{H}_K$ is to break $\Delta\phi$ invariance, i.e., to remove the constraint $\Delta q=0$. Under renormalization the theory flows to a new fixed point, also associated with $\mathscr{H}^*_{\rm TL}$, but with a different selection rule, where, possibly, the two U(1) towers are decoupled. Whereas Q and ΔQ can then no longer be used to label the U(1) sector, q and Δq remain well defined. Hence, at the new Kondo fixed point, the signature of adding backward scattering against the impurity is to allow operators with $\Delta q\neq 0$. The same arguments lead to the requirement $q=0$, in order to preserve the global U(1) invariance of $\mathscr{H}_K$. A LCBO must therefore not only be compliant with the selection rules (5.9), but also with

$$C_1=-C_2 \quad \text{and} \quad D_1=D_2. \tag{5.19}$$

Next we consider the effect of a discrete symmetry. Although $\mathscr{H}_K$ breaks chiral U(1) symmetry, it is invariant under channel exchange $\mathscr{E}:1\leftrightarrow 2$, or equivalently $L\leftrightarrow R$ in (1.8). This translates into invariance under $(q,\Delta q)\to(q,-\Delta q)$, which is equivalent to invariance under

$$\mathscr{E}: C_1 \leftrightarrow C_2 \quad \text{and} \quad D_1 \leftrightarrow -D_2. \tag{5.20}$$

Any candidate LCBO must respect this symmetry. In case $q=\Delta q=0$, this is trivially fulfilled and $\Delta_c=0$. We will return to this special case later and now focus on $\Delta q\neq 0$. Invariance under (5.20) then implies that a LCBO must be a symmetric combination of operators with opposite signs of Δq. To force the coexistence of two such operators, we have to constrain the selection rules (5.9) to be invariant under (5.20), i.e., they must satisfy

$$C_2=\alpha C_1-\beta D_1+\eta, \tag{5.21a}$$

$$-D_2=\gamma C_1-\delta D_1+\zeta. \tag{5.21b}$$

If we now require at least one boundary operator, there must be a solution to (5.9), (5.19), and (5.21). Inserting (5.19) in (5.9a) and (5.21a) yields $(1+\alpha)C_1=\beta D_1+\eta$ and $(1+\alpha)C_1=\beta D_1-\eta$, respectively, i.e., $\eta=0$. Using (5.9b) and (5.21b), we similarly get $\zeta=0$. We may therefore conclude that $\eta=\zeta=0$ is a necessary condition for the selection rules.

By demanding full consistency between (5.9) and (5.21) we may further reduce the list of possible selection rules: For instance, combining (5.9a) and (5.21a) requires $(1-\alpha^2)C_1=-\alpha\beta D_1+\beta D_2$. Let us first consider the case $\alpha^2\neq 1$. Then $\alpha\beta=0$, as otherwise C_1 would be a function of D_1. (Remember that selection rules only give relations between conformal towers and should not pose constraints within.) $\beta=0$ implies $C_1=0$, which is not allowed by the same reason. The other possibility, $\alpha=0$, implies $C_1=\beta D_2$. However, as C_1 may be any integer, we can only allow $\beta=\pm 1$. Inserting this relation in (5.21b) yields $(\beta^{-1}+\gamma)C_1=\delta D_1$. To avoid constraints within a conformal tower, we must then require $\beta^{-1}+\gamma=\delta=0$. Hence, the only solution for $\alpha^2\neq 1$ is $\alpha=\delta=0$ and $\beta=-\gamma=\pm 1$. The next case, $\alpha=1$, implies $\beta D_1=\beta D_2$. If we assume $\beta\neq 0$, then $D_1=D_2$, which gives $C_1=C_2+\beta D_1$ using (5.9a). But then $\beta=0$, as C_1 cannot be a function of D_1, i.e., $\beta\neq 0$ leads to a contradiction. The only possibility for $\alpha=1$ therefore is to require $\beta=0$, which does not lead to a contradiction. The same result holds for $\alpha=-1$.

Analogous treatment of (5.9b) and (5.21b) implies that either $\alpha=\delta=0$ and $\beta=-\gamma=\pm 1$ or else $\delta=\pm 1$ and $\gamma=0$. Hence, in total there are only six possible selection rules in the charge sector:

$$C_1=C_2 \quad \text{and} \quad D_1=D_2, \tag{5.22a}$$

$$C_1=C_2 \quad \text{and} \quad D_1=-D_2, \tag{5.22b}$$

$$C_1=-C_2 \quad \text{and} \quad D_1=D_2, \tag{5.22c}$$

$$C_1=-C_2 \quad \text{and} \quad D_1=-D_2, \tag{5.22d}$$

and

$$C_1=D_2 \quad \text{and} \quad D_1=-C_2, \tag{5.23a}$$

$$C_1=-D_2 \quad \text{and} \quad D_1=C_2. \tag{5.23b}$$

The first selection rule (5.22a) is the Luttinger liquid selection rule that we start off with before we include Kondo scattering off the impurity. The effect of including $\mathscr{H}_K$ is to move us to a new fixed point, which may be described by any of the above six selection rules. In analogy to changing a fixed point in the case of forward scattering off the impurity, or any other quantum impurity problem, we shall call such a transformation a *fusion* in the charge sector. It is a prescription for how the conformal towers are recombined when we change fixed points. From (5.22) and (5.23) it follows that there are six possible fusion rules that can be applied to the Luttinger liquid selection rule, and one of these should correspond to adding $\mathscr{H}_K$ to the fixed-point Hamiltonian. For instance, the fusion rule $(C_2,D_2)\rightarrow(-C_2,D_2)$ changes (5.22a) to (5.22c) and $(C_2,D_2)\rightarrow(D_2,-C_2)$ changes (5.22a) to (5.23a). Applying the correct fusion rule *once* to (5.22a) gives the new Kondo fixed point, and the selection rule can be used to extract the finite-size energy spectrum. Furthermore, we expect that applying the same fusion rule *twice* should give us the selection rule that determines the boundary scaling dimensions. It is easy to check that any fusion rule that takes (5.22a) to any of the selection rules in (5.22) gives (5.22a) back after double fusion, whereas the fusion rules that take (5.22a) to any of (5.23) give (5.22d) after double fusion. We therefore conclude that the only possible selection rules for the boundary scaling dimensions are (5.22a) and (5.22d).

We may now apply the symmetry constraint (5.19) to extract a "short list" of boundary scaling dimensions from the charge sector. The first rule, (5.22a), together with (5.19), requires a LCBO to have $C_1=C_2=0$ and $D_1=D_2$ an even integer. Using (5.11) and (5.12),

$$\Delta_c=\tfrac{1}{2}p^2e^{-2\theta}+N_c, \quad p,N_c\in\mathbb{N}. \tag{5.24}$$

Similarly, the second selection rule, (5.22d), combined with (5.19), requires $D_1=D_2=0$ and $C_1=-C_2$ an even integer, i.e.,

$$\Delta_c=\tfrac{1}{2}p^2e^{2\theta}+N_c, \quad p,N_c\in\mathbb{N}. \tag{5.25}$$

The full boundary dimensions are obtained by coupling the $\mathrm{SU}(2)_2$ and Ising conformal towers to the pairs of U(1) towers in (5.22a) and (5.22d), respectively. Starting with the $\mathrm{SU}(2)_2$ sector, the $j=1/2$ tower is expelled by global $\mathrm{SU}(2)_2$ invariance: Spin rotational invariance of the Hamiltonian $\mathscr{H}_{\mathrm{TL}}+\mathscr{H}_K$ implies that any LCBO must transform as a spin singlet, which, however, is missing from the $j=1/2$ tower. Turning to the $j=1$ tower, the primary operator $\boldsymbol{\phi}$ is expelled by the same reason. The lowest-dimension $\mathrm{SU}(2)_2$ singlet operator from this tower is $\boldsymbol{J}_{-1}\cdot\boldsymbol{\phi}$. However, this is the same operator that drives critical scaling in the forward scattering problem. In particular, it produces a diverging impurity susceptibility as $T\rightarrow 0$ [cf. (4.16)], in conflict with the known Fermi liquid scaling (5.6) in the $g\rightarrow 0$ limit of the present problem.[53] Assigning evenness to the $j=1$ conformal tower under channel exchange (as in the forward scattering problem) implies that $\boldsymbol{J}_{-1}\cdot\boldsymbol{\phi}$ is allowed by symmetry, and hence any selection rule must suppress this tower. The reverse assignment of parity under channel exchange instead implies that the $j=1$ tower is suppressed by symmetry. Summarizing, the only possible contributions to a LCBO from the $\mathrm{SU}(2)_2$ sector, consistent with established results for $g=0$, are the identity operator and its descendants. We are

thus left with the problem of gluing together the pairs of U(1) towers in (5.22a) and (5.22d) with the towers in the Ising sector.

Let us start with (5.22a). Putting $D_1=D_2=0$, the resulting identity towers can be combined only with the identity tower in the Ising sector [together with that of $SU(2)_2$]. This is so, since the presence of a $\phi=\sigma$ or ϵ tower would lead to a relevant boundary operator: the primary operators σ and ϵ both have dimensions <1. However, at $g=0$ the fixed point is known to be stable (cf. Sec. V A), excluding the presence of a relevant operator. In fact, this conclusion may be extended to $g\neq 0$: To remain at an unstable fixed point (that is, to maintain criticality) requires fine tuning of some parameter in the bare Hamiltonian. As g is the only tunable parameter in $\mathscr{H}_{\rm TL}^{*}$ (with a renormalized boundary condition replacing H_K), an unstable fixed point would imply noncriticality for all values of $g\neq 0$. In other words, the total scaling dimension $\Delta=\Delta_c+\Delta_s$ of any boundary operator must be >1. It is here important to stress that — by the *raison d'etre* of renormalization — *any* boundary operator allowed by symmetry will also appear at the fixed point. Therefore, one cannot argue that a LCBO with $\Delta>1$ can be obtained by forming descendants in the $\phi=\sigma$ or ϵ towers. If any of these towers were present, the corresponding primary operators of dimension $\Delta=1/16$ and 1/2, respectively, would be present as well, implying an unstable fixed point.

The lowest-dimension operators emerging from the identity towers are the first Kac-Moody descendants in the U(1) sectors, with $\Delta=1$: The marginal boundary operators $\mathscr{O}^{1,2}(w)=j_L^{1,2}(w)$ always appear in the charge sector, as the particle-hole symmetry of the original lattice model in (2.2) is broken away from half-filling.[54] Upholding particle-hole symmetry, the lowest dimension operators would instead be the second *Virasoro descendants* $L_{-2}\mathbb{1}$ in respective sectors, of dimensions $\Delta=2$. The next choice of U(1) quantum numbers, $D_1=D_2=2$ [$p=1$ in (5.24)], leads to a relevant boundary operator for any combination of Ising towers, and is therefore not allowed. In contrast, $D_1=D_2\geq 4$ [$p\geq 2$ in (5.24)] yields operators with $\Delta\geq 1$ when combined with any Ising tower. Summarizing, the possible couplings of Ising conformal towers to the U(1) towers selected by (5.22a) yield the following candidate LCBO dimensions:

$$\Delta_{\rm LCBO}=1,\ \tfrac{1}{2}p^2e^{-2\theta}+\{0,\tfrac{1}{16},\tfrac{1}{2}\},\quad p\in\mathbb{N}+2. \tag{5.26}$$

Turning to the second selection rule for the U(1)×U(1) sector, (5.22d), employing the same reasoning as above, one finds a second class of possible LCBO dimensions:

$$\Delta_{\rm LCBO}=1,\ \tfrac{1}{2}e^{2\theta}+\tfrac{1}{2},\ \tfrac{1}{2}p^2e^{2\theta}+\{0,\tfrac{1}{16},\tfrac{1}{2}\},\quad p\in\mathbb{N}+2. \tag{5.27}$$

Before exploring the critical behavior implied by the various dimensions in (5.26) and (5.27), two comments may be in order. First, note that no scaling dimensions of descendant operators — other than those of the identity — appear in (5.26) or (5.27). This is so, since a LCBO is the (composite) boundary operator with *lowest* dimension ≥ 1, given a particular combination of conformal towers. Only if one, or several, of the nontrivial primary operators are expelled by symmetry can a descendant operator enter the stage (as in the forward scattering problem). This is not the case here. Second, the appearance of the ϵ conformal tower requires the reverse assignment of parity under channel exchange, as compared to the forward scattering problem. That is, a consistent representation now forces the ϵ tower to be invariant under channel exchange, while ϕ and its descendants (already suppressed by the selection rules) change sign. (Note that there is no contradiction with the forward scattering case, as the argument in Sec. IV B for the other assignment of parity no longer applies.)

D. Impurity specific heat

Analogous to the forward scattering problem, an effective scaling Hamiltonian $\mathscr{H}$ is obtained by adding a boundary term to the fixed-point theory of $\mathscr{H}_{\rm TL}=\mathscr{H}_0+\mathscr{H}_{\rm int}$:

$$\mathscr{H}=\mathscr{H}_{\rm TL}^{*}+\lambda_I\mathscr{O}(0), \tag{5.28}$$

where $\mathscr{O}(0)$ is a LCBO with conjugate scaling field λ_I. By mapping the half-plane $\mathbb{C}^+=\{{\rm Im}z>0\}$ for zero temperature onto the finite-T geometry (Fig. 5), $\Gamma^+=\{w=v\tau+ix=(v\beta/\pi)\arctan(z)\}$ [with $v=v_c(v_s)$ for a LCBO from the charge (spin) sector[55]], the partition function in zero magnetic field is written

$$e^{-\beta F(\beta,\lambda_I)}=e^{-\beta F(\beta,0)}\left\langle \exp\left(\lambda_I\int_{-\beta/2}^{\beta/2}d\tau\,\tilde{\mathscr{O}}(\tau,0)\right)\right\rangle_T, \tag{5.29}$$

so that $\delta f_{\rm imp}(\beta,\lambda_I)\equiv f_{\rm imp}(\beta,\lambda_I)-f_{\rm imp}(\beta,0)$ satisfies

$$e^{-\beta\delta f_{\rm imp}(\beta,\lambda_I)}=\left\langle \exp\left(\lambda_I\int_{-\beta/2}^{\beta/2}d\tau\,\tilde{\mathscr{O}}(\tau,0)\right)\right\rangle_T. \tag{5.30}$$

We have here used the decomposition in (4.1) and passed to a Lagrangian formalism, "tilde" and $\langle\ \rangle_T$ referring to Γ^+. By a linked cluster expansion,

$$\begin{aligned}\delta f_{\rm imp}=&-\frac{\lambda_I}{\beta}\int_{-\beta/2}^{\beta/2}d\tau\langle\tilde{\mathscr{O}}(\tau,0)\rangle_T\\ &-\frac{\lambda_I^2}{2\beta}\int\int_{-\beta/2}^{\beta/2}d\tau_1\,d\tau_2\langle\tilde{\mathscr{O}}(\tau_1,0)\tilde{\mathscr{O}}(\tau_2,0)\rangle_{T,c}\\ &+O(\lambda_I^3),\end{aligned} \tag{5.31}$$

with $\langle\ \rangle_{T,c}$ denoting a cumulant in Γ^+.

Here two cases must be distinguished: (i) $\mathscr{O}$ is a Virasoro descendant of $\mathbb{1}$. Then $\langle\tilde{\mathscr{O}}(\tau,0)\rangle_T$ may be nonzero and hence the leading contribution to $\delta f_{\rm imp}$ will be linear in λ_I. (ii) In any sector of the theory, $\mathscr{O}$ is Virasoro primary or a Virasoro descendant of an operator other than $\mathbb{1}$. Then $\langle\tilde{\mathscr{O}}(\tau,0)\rangle_T=0$ and $\delta f_{\rm imp}$ is quadratic in λ_I. To see how this comes about, consider a chiral (say, left-moving) Virasoro primary operator $\mathscr{A}(z)\neq\mathbb{1}$ with dimension Δ ($\neq 0$) in the half-plane $\mathbb{C}^+$. The expectation value of a chiral operator in a half-plane is the same as in the full plane, as translational invariance in one direction implies this in all directions. (Uniqueness of analytic functions implies that the expectation value is constant everywhere.) The scale transformation $z\to z/a$, where we later choose $a=z$, implies

$$\langle \mathscr{A}(z)\rangle = z^{-\Delta}\langle \mathscr{A}(1)\rangle, \tag{5.32}$$

and we conclude that $\langle \mathscr{A}(z)\rangle = 0$. (Note that the argument crucially depends on $\mathscr{A}$ being chiral: a nonchiral operator may pick up a nonzero expectation value in the presence of a boundary.) Any Virasoro descendant operator $L_{-n_1}L_{-n_2}\cdots L_{-n_M}\mathscr{A}(z)$ $(n_j>0)$ has a vanishing expectation value as well, since $\langle L_{-n_1}\cdots L_{-n_M}\mathscr{A}(z)\rangle = \mathscr{L}_{-n_1}\cdots\mathscr{L}_{-n_M}\langle A(z)\rangle = 0$, $\mathscr{L}_{-n_j}$ $(j=1,\cdots,n_M)$ being differential operators.[56] Now map $\mathbb{C}^+$ onto Γ^+. Again, by conformal invariance, $\langle \tilde{\mathscr{A}}(w)\rangle_T = (dw/dz)^{-\Delta}\langle A(z)\rangle = 0$. The transformation law for a Virasoro descendant is more complicated and relates one descendant in Γ^+ with a sum of operators from the same Virasoro tower in $\mathbb{C}^+$. However, as the expectation values of these are zero, it follows that the expectation value of a Virasoro descendant of $\mathscr{A}$ vanishes in Γ^+ as well. In contrast to this, descendants of the unit operator $\mathbb{1}$ may acquire nonzero expectation values in Γ^+. As an example, the energy-momentum tensor is a descendant of $\mathbb{1}$ and satisfies $\langle \tilde{T}(w)\rangle_T = (c/12)\{z,w\}$, with c the conformal anomaly number and $\{z,w\} = d^3z/dw^3/dz/dw - (3/2)(d^2z/dw^2/dz/dw)^2$ the Schwarzian derivative of the map $\mathbb{C}^+\to\Gamma^+$.

Let us first study case (i) where $\mathscr{O}$ is a descendant of $\mathbb{1}$. Since $L_{-1}\mathbb{1} = d\mathbb{1}/dz = 0$, the energy momentum tensor $T(z)\equiv L_{-2}\mathbb{1}$ is the leading Virasoro descendant of $\mathbb{1}$ and has $\Delta = 2$. In the present problem, each sector contributes its own energy momentum tensor, $T_1(z) = \frac{1}{4}{:}j_L^1(z)j_L^1(z){:}$, $T_2(z) = \frac{1}{4}{:}j_L^2(z)j_L^2(z){:}$, $T_3(z) = \frac{1}{4}{:}\boldsymbol{J}(z)\cdot\boldsymbol{J}(z){:}$ and $T_4(z) = T_{\mathrm{Ising}}(z)$, all with $\Delta = 2$ and satisfying $\langle T_j(z)\rangle = 0$ in $\mathbb{C}^+$. In other words, there are *four* degenerate LCBO's for this case. Passing to Γ^+, their contribution to the impurity specific heat is given by

$$\delta f_{\mathrm{imp}} = -\sum_{j=1}^{4}\frac{\lambda_j}{\beta}\int_{-\beta/2}^{\beta/2} d\tau \langle \tilde{T}_j(\tau,0)\rangle_T, \tag{5.33}$$

with

$$\langle \tilde{T}_j(w)\rangle_T = \frac{c_j}{12}\{z,w\} = \frac{c_j}{6}\left(\frac{\pi}{v_j\beta}\right)^2, \tag{5.34}$$

where $c_1 = c_2 = 1$, $c_3 = 3/2$, and $c_4 = 1/2$ are the conformal anomaly numbers of the different sectors, and $v_1 = v_2 = v_c$ and $v_3 = v_4 = v_s$ are the corresponding velocities. Inserting (5.34) into (5.33), integrating, and summing over the sectors, it follows that

$$\delta f_{\mathrm{imp}} = -\frac{\pi^2}{6}\sum_{j=1}^{4}\frac{\lambda_j c_j}{v_j^2}T^2, \tag{5.35}$$

producing a linear specific heat

$$C_{\mathrm{imp}} = -T\frac{\partial^2 f_{\mathrm{imp}}}{\partial T^2} = \frac{\pi^2}{3}\sum_{j=1}^{4}\frac{\lambda_j c_j}{v_j^2}T, \tag{5.36}$$

where we have used (4.10). This is the dominant contribution to C_{imp} that is linear in the scaling fields: higher-order descendants of $\mathbb{1}$ produce higher powers in temperature.

We now turn to the more interesting case (ii) where the candidate LCBO's are *not* Virasoro descendants of the unit operator. The one-point function in (5.31) then vanishes, and we are left with

$$\delta f_{\mathrm{imp}} = -\frac{\lambda_I^2}{2\beta}\int\int_{-\beta/2}^{\beta/2} d\tau_1\, d\tau_2 \langle \tilde{\mathscr{O}}(\tau_1,0)\tilde{\mathscr{O}}(\tau_2,0)\rangle_T + O(\lambda_I^3). \tag{5.37}$$

According to our symmetry analysis, any LCBO that is *not* a descendant of $\mathbb{1}$ must be a Virasoro primary operator.[57] It is therefore sufficient to consider the case when $\mathscr{O}$ is Virasoro primary, for which the two-point function in $\mathbb{C}^+$ takes the familiar form

$$\langle \mathscr{O}(z_1)\mathscr{O}(z_2)\rangle = \frac{A}{(z_1-z_2)^{2\Delta}}. \tag{5.38}$$

Here Δ $(\neq 0)$ is the scaling dimension of $\mathscr{O}$, and A is a normalization constant (to be determined). Using $z\to w = v\tau + ix = (v\beta/\pi)$ arctanz and the transformation rule for a Virasoro primary operator, we can get $\langle \tilde{\mathscr{O}}(w_1)\tilde{\mathscr{O}}(w_2)\rangle_T$. However, as $\mathscr{O}$ may be composed of operators from sectors with different velocities v, we must perform the transformation in each sector independently. The expression simplifies somewhat on the boundary $x = 0$ and becomes

$$\langle \tilde{\mathscr{O}}(\tau_1,0)\tilde{\mathscr{O}}(\tau_2,0)\rangle_T = \frac{A}{(v_c^{\Delta_c}v_s^{\Delta_s})^2\left|\frac{\beta}{\pi}\sin\left[\frac{\pi}{\beta}(\tau_1-\tau_2)\right]\right|^{2\Delta}}, \tag{5.39}$$

where $\Delta = \Delta_c + \Delta_s$, and the subscripts refer to the charge and spin sectors, respectively. The integrand (5.39) of (5.37) is even and periodic, with period β, and can be replaced by a single integral over $\tau = \tau_1 - \tau_2$. Putting $u \equiv \tan(\pi/\beta)$, and inserting a short-time cutoff $\tau_0 = \epsilon\beta/\pi$, yields the expression

$$\delta f_{\mathrm{imp}} = -\frac{\lambda_I^2 A}{(v_c^{\Delta_c}v_s^{\Delta_s})^2}\left(\frac{\pi}{\beta}\right)^{2\Delta-1}\int_{\tan\epsilon}^{\infty} du\,\frac{(1+u^2)^{\Delta-1}}{u^{2\Delta}}. \tag{5.40}$$

Let us first study the case when $\Delta \geqslant 1$ is an integer. We can then make the expansion $(1+u^2)^{\Delta-1} = 1 + (\Delta-1)u^2 + \cdots + u^{2(\Delta-1)}$, which yields

$$I \equiv \int_{\tan\epsilon}^{\infty} du\,\frac{(1+u^2)^{\Delta-1}}{u^{2\Delta}} = \frac{1}{2\Delta-1}\frac{1}{(\tan\epsilon)^{2\Delta-1}} \times\left(1 + \frac{\Delta-1}{2\Delta-3}(2\Delta-1)\tan^2\epsilon + O(\epsilon^4)\right). \tag{5.41}$$

In the limit of small ϵ, i.e., $\tau_0 T\to 0$, Eqs. (5.40) and (5.41) give

$$C_{\mathrm{imp}} = -T\frac{\partial^2 f_{\mathrm{imp}}}{\partial T^2} = \frac{\lambda_I^2 A}{(v_c^{\Delta_c}v_s^{\Delta_s})^2}\frac{2\Delta}{3(2\Delta-3)} \times \pi^2\tau_0^{3-2\Delta}T(1+O[(\tau_0 T)^2]). \tag{5.42}$$

The case of noninteger dimension $\Delta>1$ requires a more lengthy analysis. One partial integration of I yields

$$I=\frac{1}{2\Delta-1}\left(\frac{\beta}{\pi\tau_0}\right)^{2\Delta-1}\left\{1+\frac{\Delta-2}{3}\left(\frac{\pi\tau_0}{\beta}\right)^2+O\left[\left(\frac{\pi\tau_0}{\beta}\right)^4\right]\right\}+\frac{2(\Delta-1)}{2\Delta-1}I_1 \tag{5.43}$$

with I_1 defined by

$$I_j=\int_{\tan\epsilon}^{\infty}du\,\frac{(1+u^2)^{\Delta-j-1}}{u^{2(\Delta-j)}},\quad j\in\mathbb{N}. \tag{5.44}$$

When $\Delta<3/2$, I_1 is finite as $\epsilon\to0$, while $\Delta=3/2$ produces a logarithmic singularity:

$$I_1\to\begin{cases}\frac{1}{2}B\left(\frac{3}{2}-\Delta,\frac{1}{2}\right), & \Delta<\frac{3}{2},\\ \ln\frac{1}{\tau_0T}+\text{const}, & \Delta=\frac{3}{2},\end{cases}\quad\text{as}\quad\tau_0T\to0 \tag{5.45}$$

with $B(p,q)=\Gamma(p)\Gamma(q)/\Gamma(p+q)$ the beta function.

For $\Delta>3/2$, I_1 diverges algebraically with the cutoff, and we must perform a second partial integration to identify the rate of divergence. This leads to

$$I_1=\frac{1}{2\Delta-3}\left(\frac{\beta}{\pi\tau_0}\right)^{2\Delta-3}\left\{1+O\left[\left(\frac{\pi\tau_0}{\beta}\right)^2\right]\right\}+\frac{2(\Delta-2)}{2\Delta-3}I_2 \tag{5.46}$$

with I_2 defined as in (5.44). In the interval $3/2<\Delta<5/2$, I_2 is finite as $\epsilon\to0$, while for $\Delta=5/2$ one gets the same logarithmic singularity as for $\Delta=3/2$, i.e., $I_2(\Delta=5/2)=I_1(\Delta=3/2)$. The procedure is now iterated, I_{j+1} being the remainder from partially integrating I_j. However, the terms in T thus generated are always *subleading* compared to those coming from the partial integrations of I and I_1 in (5.43) and (5.45), respectively. Thus, collecting the results, we obtain for the impurity specific heat

$$C_{\text{imp}}=\frac{\lambda_I^2A}{(v_c^{\Delta_c}v_s^{\Delta_s})^2}\times\begin{cases}\frac{2\Delta}{3(2\Delta-3)}\pi^2\tau_0^{3-2\Delta}T+\cdots, & \Delta=1\ \text{ or }\ \Delta>\frac{3}{2},\\ 2(\Delta-1)^2\pi^{2\Delta-1}B\left(\frac{3}{2}-\Delta,\frac{1}{2}\right)T^{2\Delta-2}+\cdots, & 1<\Delta<\frac{3}{2},\\ \pi^2T\ln\left(\frac{1}{\tau_0T}\right)+\cdots, & \Delta=\frac{3}{2},\end{cases} \tag{5.47}$$

where "$\cdots$" denotes subleading corrections.

Combining (5.36) and (5.47) with the result from the symmetry analysis, (5.26) and (5.27), we find that there are only two distinct possibilities for critical scaling:

$$\text{(i)}\quad C_{\text{imp}}=O(T), \tag{5.48}$$

and

$$\text{(ii)}\quad C_{\text{imp}}=\frac{\lambda_I^2A\pi^{1/K_\rho}}{2v_sv_c^{1/K_\rho}}\left(\frac{1}{K_\rho}-1\right)^2B\left(1-\frac{1}{2K_\rho},\frac{1}{2}\right)T^{1/K_\rho-1}+O(T), \tag{5.49}$$

with $K_\rho\equiv e^{-2\theta}=(1+2g/v_F)^{-1/2}$ the Luttinger liquid charge parameter. The first case, (5.48), is implied when the LCBO carries dimension $\Delta=1$ or $\Delta>3/2$. In contrast, the leading term in the second case, (5.49), is driven by a composite LCBO of dimension $\Delta=\Delta_c+\Delta_s=(1/2)(e^{2\theta}+1)$, corresponding to the second entry in (5.27).[58] In terms of quantum numbers $(C_1,D_1;C_2,D_2;j;\phi)$, the operator is given by the sum of

$$(2,0;-2,0;0;\epsilon)\quad\text{and}\quad(-2,0;2,0;0;\epsilon). \tag{5.50}$$

Hence, the charge sector contributes a channel-symmetric combination of primary operators with $q=0$ and $\Delta q=\pm2e^{\theta}$ [cf. discussion after (5.20)], which can be explicitly expressed in terms of vertex operators of free boson fields. The contribution to the scaling dimension is $\Delta_c=(1/2)e^{2\theta}=1/2K_\rho$. Only by combining these operators with the Ising energy density ϵ does one obtain a boundary operator of dimension $\Delta\geqslant1$, as required for a LCBO. The linear term in (5.49) comes from subleading terms generated by the same operator, as well as from leading terms due to the marginal operators $j^1(z)$ and $j^2(z)$. As we have seen, the latter operators are always present, in case (i) as well, due to the breaking of particle-hole symmetry.

The specific heat in (5.49) exhibits the same anomalous scaling in temperature as found by Furusaki and Nagaosa.[13] Also, the way the anomalous terms vanish as $K_\rho\to1(g\to0)$ are identical.[59] Although more work is needed to firmly establish which of the two cases applies, (5.48) (Fermi liquid) or (5.49) (non-Fermi liquid), the second, non-Fermi liquid case is clearly favored considering its emergence in an independent analysis. However, a caveat is advisable. In a recent study, Schiller and Ingersent treat a simplified model of a magnetic impurity in a "reduced" Luttinger liquid, composed of right-moving spin-up electrons and left-moving spin-down electrons.[60] This problem, with only two branches of electrons (compared to the four branches of the full problem: two chiralities with two spin

projections each) is mapped exactly onto the Kondo effect in a Fermi liquid, with $C_{\rm imp}$ as in (5.48). Although the relevance of this simplified model to the Kondo effect in a full Luttinger liquid remains unclear to us, the result may be taken — as argued by the authors in Ref. 60 — to give some indirect support to a Fermi liquid scenario. In any event, it is reassuring that our exact analysis gives room *only* to those two scenarios that have been conjectured in the literature.

Before closing this section, let us point out that the impurity specific heat (4.15) for the forward scattering problem follows from the third case in (5.47) by inserting $\Delta_c=0$, $\Delta_s=3/2$, and the value of the normalization constant for this case, $A=\langle \boldsymbol{J}_{-1}\cdot\boldsymbol{\phi}|\boldsymbol{J}_{-1}\cdot\boldsymbol{\phi}\rangle=9$.[61] The amplitude in (5.49) can similarly be calculated by evaluating the norm of the corresponding LCBO. However, as we shall find in the next section, the favored case, with the LCBO in (5.50), does not lead to a universal Wilson ratio when $g\neq 0$. For this reason we here leave the amplitudes undetermined.

E. Impurity susceptibility

In the presence of a magnetic field, the partition function takes the form

$$e^{-\beta F(\beta,h,\lambda_I)}=e^{-\beta F(T,0,0)}\left\langle \exp\left[\int_{-\beta/2}^{\beta/2}\left(\lambda_I d\tau\ \tilde{\mathscr{O}}(\tau,0)+\frac{h}{2\pi}\int_{-\infty}^{\infty}dx\ \tilde{J}^z(\tau,x)\right)\right]\right\rangle_T . \tag{5.51}$$

The shift of the magnetic susceptibility due to the impurity,

$$\chi_{\rm imp}=-\left.\frac{\partial^2 f_{\rm imp}}{\partial h^2}\right|_{h=0}, \tag{5.52}$$

may thus be expanded to second order in λ_I as

$$\chi_{\rm imp}=\frac{\lambda_I}{4\pi^2\beta}\int\int_{-\infty}^{\infty}dx_1\,dx_2\int\cdots\int_{-\beta/2}^{\beta/2}d\tau_1\cdots d\tau_3\langle\tilde{J}^z(\tau_1,x_1)\tilde{J}^z(\tau_2,x_2)\tilde{\mathscr{O}}(\tau_3,0)\rangle_{T,c}$$
$$+\frac{\lambda_I^2}{8\pi^2\beta}\int\int_{-\infty}^{\infty}dx_1\,dx_2\int\cdots\int_{-\beta/2}^{\beta/2}d\tau_1\cdots d\tau_4\langle\tilde{J}^z(\tau_1,x_1)\tilde{J}^z(\tau_1,x_2)\tilde{\mathscr{O}}(\tau_3,0)\tilde{\mathscr{O}}(\tau_4,0)\rangle_{T,c}. \tag{5.53}$$

Let us consider the non-Fermi liquid scenario with $\mathscr{O}$ given by (5.50). The U(1), $SU(2)_2$, and Ising sectors are decoupled, and hence there are no dynamical correlations between operators belonging to different sectors. As $\mathscr{O}$ contains only the identity as a $SU(2)_2$ factor, it follows that

$$\langle\tilde{J}^z(\tau_1,x_1)\cdots\tilde{J}^z(\tau_m,x_m)\tilde{\mathscr{O}}(\tau,0)\rangle_{T,c}=0, \tag{5.54}$$

using that the LCBO is Virasoro primary, i.e., $\langle\tilde{\mathscr{O}}(w)\rangle_T=0$. The expectation value of $\tilde{J}^z$ vanishes by the same reason, and we may decompose

$$\langle J_1^zJ_2^z\mathscr{O}_3\mathscr{O}_4\rangle_c=\langle J_1^zJ_2^z\mathscr{O}_3\mathscr{O}_4\rangle-\langle J_1^zJ_2^z\rangle\langle\mathscr{O}_3\mathscr{O}_4\rangle$$
$$-\langle J_1^z\mathscr{O}_3\rangle\langle J_2^z\mathscr{O}_4\rangle-\langle J_1^z\mathscr{O}_4\rangle\langle J_2^z\mathscr{O}_3\rangle=0. \tag{5.55}$$

Hence, we infer that $\mathscr{O}$ does not give any contribution to $\chi_{\rm imp}$ to $O(\lambda_I^2)$. The same conclusion holds for *any* candidate LCBO obtained in Sec. V C, as all of them are Virasoro primary and contain only the identity as a $SU(2)_2$ factor. Higher order terms in an expansion of $\chi_{\rm imp}$ in λ_I can similarly be shown to vanish. The leading contribution to $\chi_{\rm imp}$ is instead given by the lowest-dimension boundary operator that has a *nontrivial* $SU(2)_2$ factor. By our symmetry analysis in Sec. V C, this is given by the $SU(2)_2$ energy-momentum tensor $T_3(z)=1/4{:}\boldsymbol{J}(z)\cdot\boldsymbol{J}(z){:}$, of dimension $\Delta=2$. Analogous to the ordinary Kondo problem,[20] $T_3(z)$ produces a finite impurity susceptibility (to first order in λ_I),

$$\chi_{\rm imp}\sim T^0+O(T). \tag{5.56}$$

Thus, comparing with the free case (5.6), the electron-electron interaction is seen *not* to influence $\chi_{\rm imp}$: the impurity remains completely screened, in agreement with the result of Furusaki and Nagaosa.[13] Also note that, by (5.49) and (5.56), the favored non-Fermi liquid scenario implies a Wilson ratio that is nonuniversal, and depends on temperature.

VI. SUMMARY

We have studied the low-temperature properties of a spin-1/2 magnetic impurity coupled to a one-dimensional interacting electron system. By turning the problem into a boundary critical phenomenon and using conformal field theory we have reached the important conclusion that the symmetry of the problem admits only one of two possible fixed points describing the local electron-impurity composite: *Either* the theory remains a local Fermi liquid in the presence of electron-electron interaction (as for the ordinary Kondo problem with free electrons) *or* electron correlations drive the system to a new fixed point with an anomalous specific heat, identical to that proposed recently by Furusaki and Nagaosa.[13] We have also shown that the suppression of back scattering off the impurity destabilizes *both* fixed points and produces an impurity critical behavior identical to that of the two-channel Kondo model, but with a new Wilson ratio.

The non-Fermi liquid fixed point is distinguished by the presence of a leading correction-to-scaling operator of dimension $\Delta=\Delta_c+\Delta_s=(1/2)e^{2\theta}+1/2$, corresponding to the energy level E', with[62]

$$E'-E_0=\frac{\pi v_c}{\ell}\frac{1}{2}e^{2\theta}+\frac{\pi v_s}{\ell}\frac{1}{2}. \tag{6.1}$$

By numerically computing the finite-size energy spectrum of the two-impurity auxiliary problem and checking for the presence of E', one has, in principle, a diagnostic tool for deciding which scenario is realized: Fermi or non-Fermi liquid. Unfortunately, with our approach we cannot derive a unique and complete finite-size spectrum at the non-Fermi liquid fixed point as the selection rule in (5.22d) only applies to the charge sector; there are still a multitude of ways of coupling the two U(1) conformal towers to those in the $SU(2)_2$ and Ising sectors. When deriving possible LCBO dimensions we were helped by symmetry constraints, which, however, are no longer applicable when considering the full spectrum. This fact makes the check against numerics more difficult, as one is essentially restricted to search for the single level E'.

To make contact with future experiments clearly requires a more complete description of the system, as well as the inclusion of potential scattering off the impurity (see Sec. I). In the case of a magnetic defect implanted in a quantum wire, the observable of greatest interest is the shift of the average conductance due to the impurity, as this is the quantity most easily accessible in the laboratory. Other important characteristics include the local spin and charge Green's functions, the scattering matrix, and the residual entropy. Considering the success of conformal field theory techniques for obtaining these quantities in the multichannel Kondo problem,[19] we judge that the approach as presented in this paper will be equally powerful. We hope to return to these, and related issues in a future publication.

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[48] Note that we omit the irrelevant bulk fields in (4.1). This can safely be done: *If* there is a correlation between irrelevant bulk and boundary operators, the corresponding term in a perturbative expansion of f_{imp} in T can be shown to produce a subleading contribution compared to that coming from the leading irrelevant boundary operator alone. On the other hand, a bulk irrelevant field may produce nonanalytic corrections to f_{bulk}, but this is of no concern to us here.

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[51] Note that the spin-1 field $\boldsymbol{\phi}$ is odd under time reversal ($\mathscr{T}$), and hence cannot appear alone, but only as a piece of a composite operator that respects $\mathscr{T}$. As can be seen from (4.13), $\boldsymbol{\epsilon}$ is even under $\mathscr{T}$, and is thus allowed. Also note that exactly *at* the fixed point, the scaling field conjugate to $\boldsymbol{\epsilon}$ vanishes, and the invariance under $\boldsymbol{\epsilon}\to-\boldsymbol{\epsilon}$ of the critical Ising model is recovered *(Kramers-Wannier duality)*.

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[55] The finite-temperature geometry is *defined* in terms of τ and x by $\Gamma^+=\{-\beta/2\leqslant\tau\leqslant\beta/2, x\geqslant 0\}$ with periodic boundary condition in τ. When $v_c\neq v_s$, we *represent* it in complex notation by cylinders of different circumferences for charge and spin degrees of freedom, respectively. The only reason for introducing this complex notation is to simplify the derivation of correlation functions, whereas the calculation of the free energy is preferably done in terms of τ and x.

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[59] The charge parameter is defined as $K_\rho\equiv[(1-g_2/\pi v_F)/(1+g_2/\pi v_F)]^{1/2}$ in Ref. 13. For small g_2, and switching from the conventional normalization of Fermi fields, $\{\psi_{r,\sigma}(x),\psi_{s,\mu}^\dagger(y)\}=\delta_{rs}\delta_{\sigma\mu}\delta(x-y)$, to our normalization in (1.5), this is the same as our definition of K_ρ with $g_2=\pi g$.

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[62] Note that we here refer to the spectrum of an auxiliary problem with two impurities, one at each end of the interval $[0,\ell]$. Cf. the discussion on *double fusion* in the text after Eqs. (3.62) and (5.23b).