International Workshop and Seminar on
Quantum Information Concepts for Condensed Matter Problems
MPIPKS, Dresden, June 24, 2010

Two-impurity Kondo model:

Quantum criticality, entanglement, spin-orbit interactions, and all that...

in collaboration with
Erik Eriksson (UG) and **David Mross** (MIT)

UNIVERSITY OF GOTHENBURG

VETENSKAPSRÅDET
THE SWEDISH RESEARCH COUNCIL
supported by the
Swedish Research Council

Outline

Basics on two-impurity Kondo physics

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Entanglement at quantum criticality

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A new twist: Adding spin-orbit interactions

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A spin-orbit generated fixed point...?

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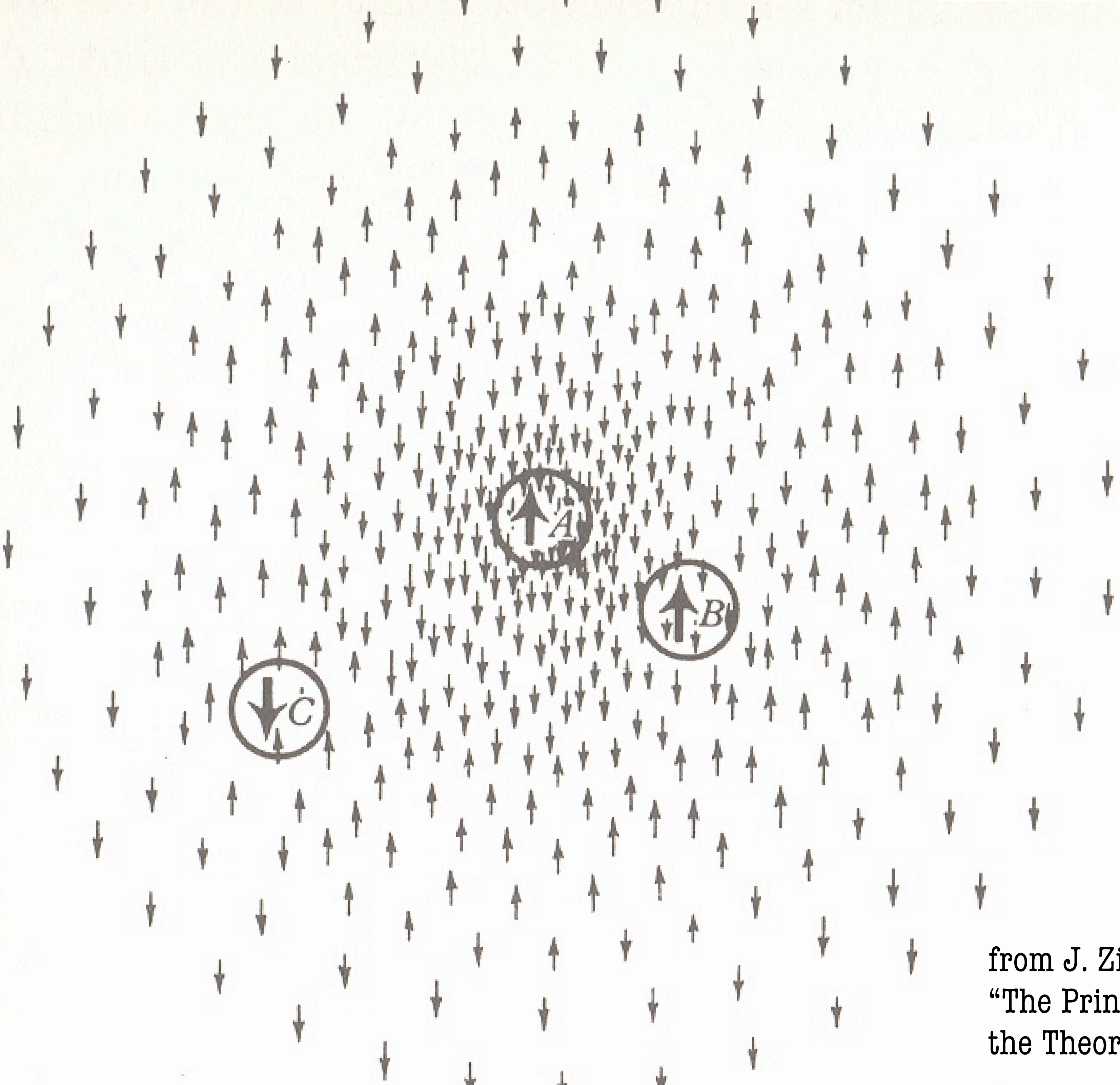
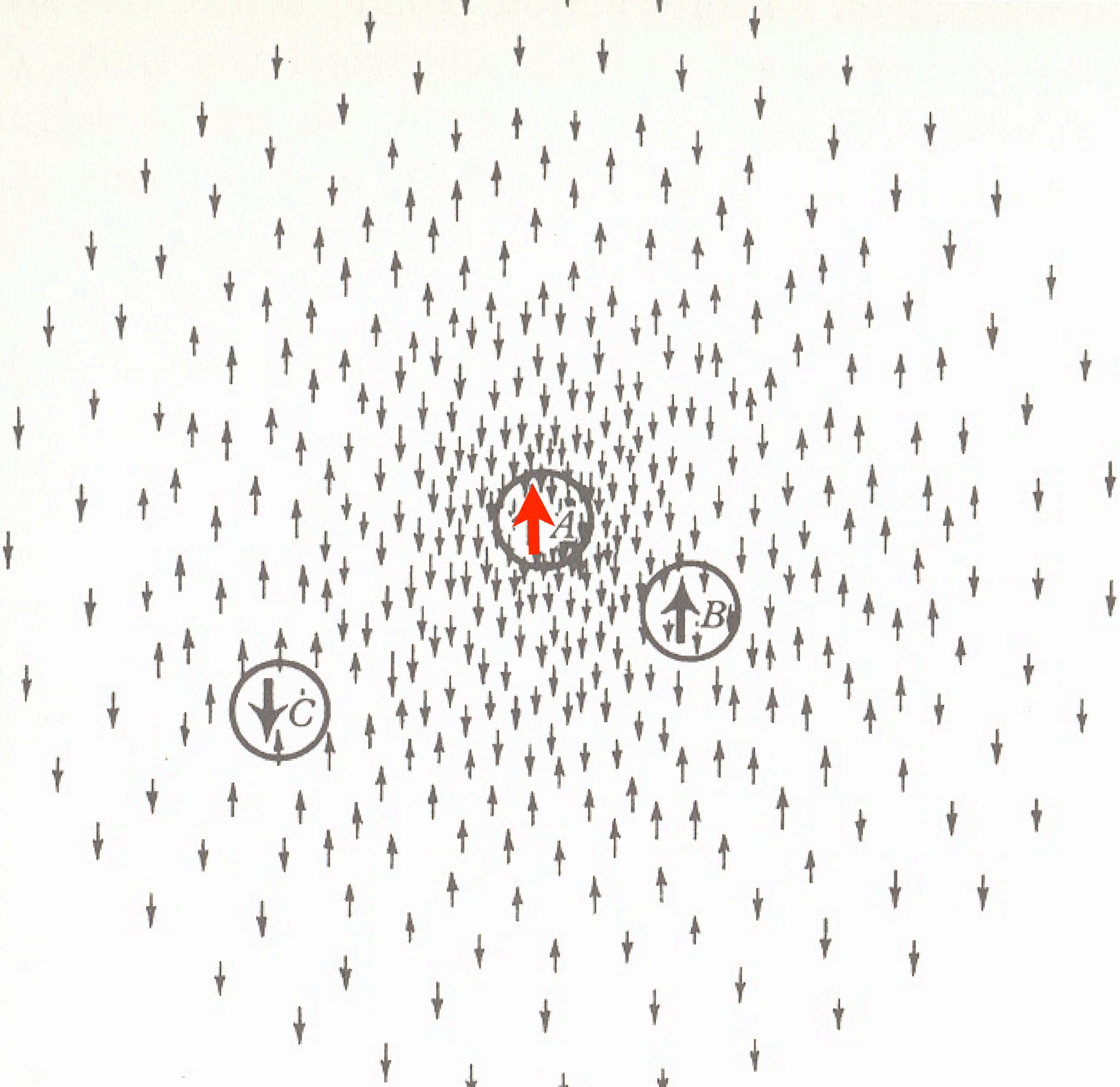
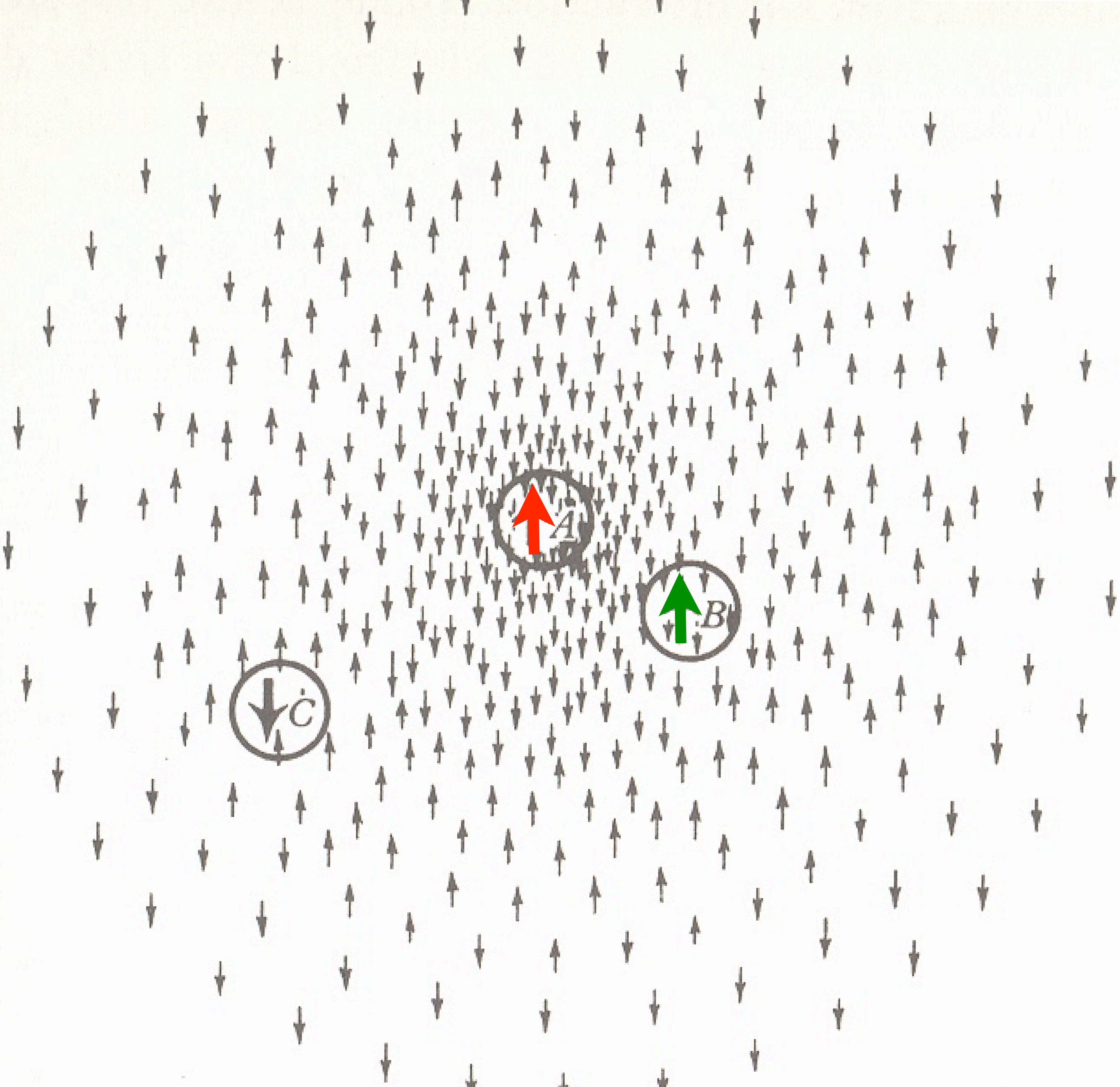
Summary



from J. Ziman,
“The Principles of
the Theory of Solids”


A
B
C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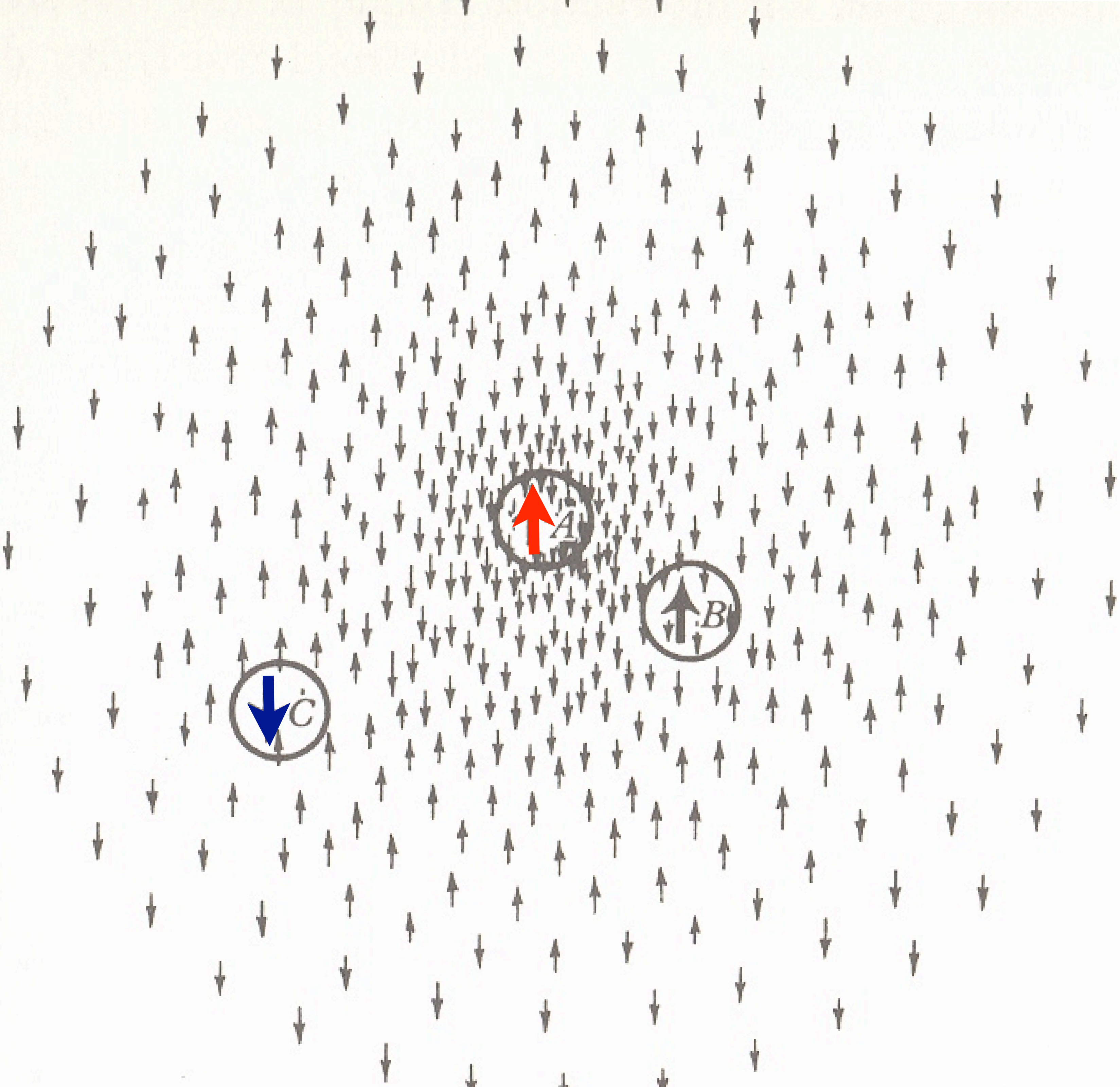
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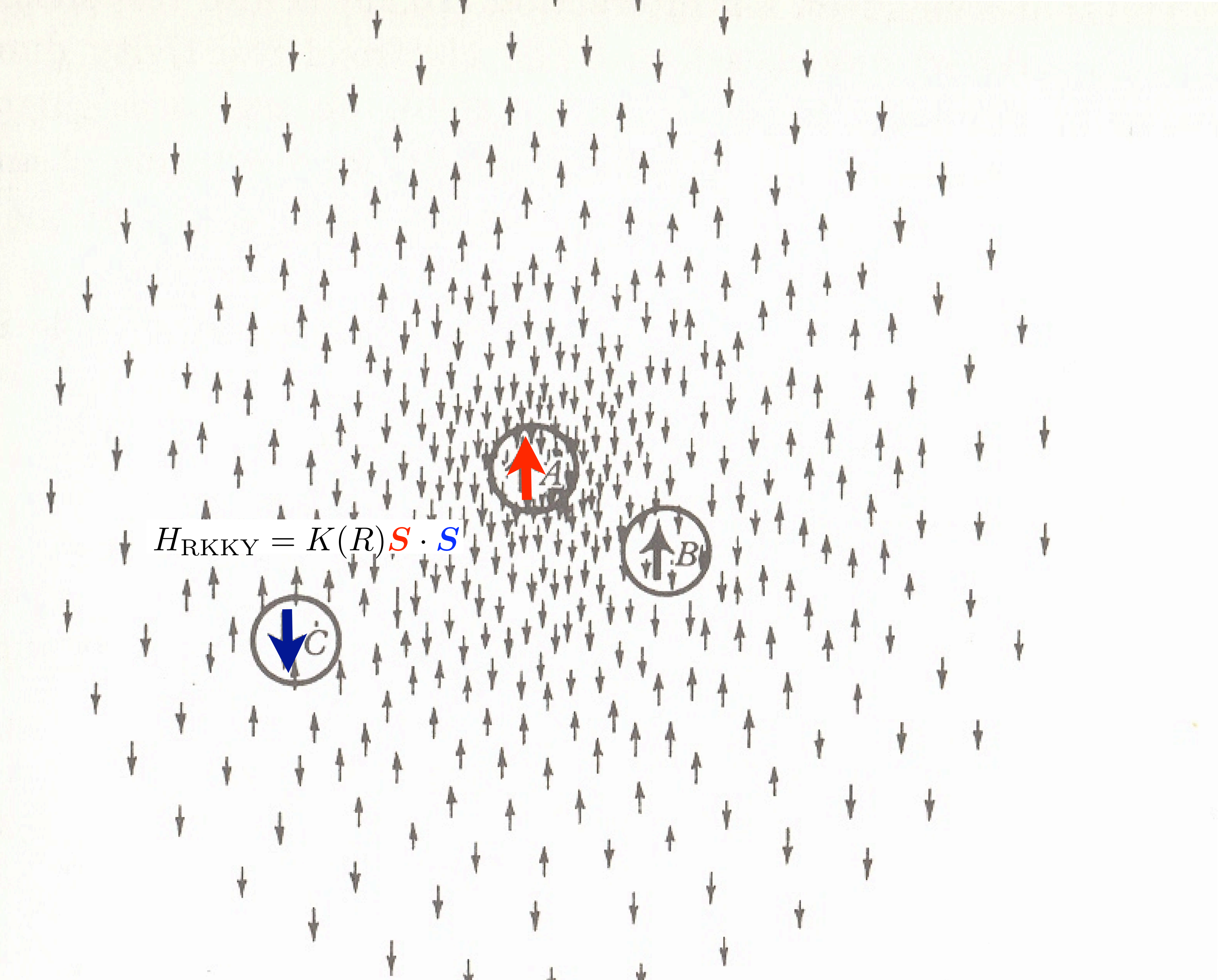
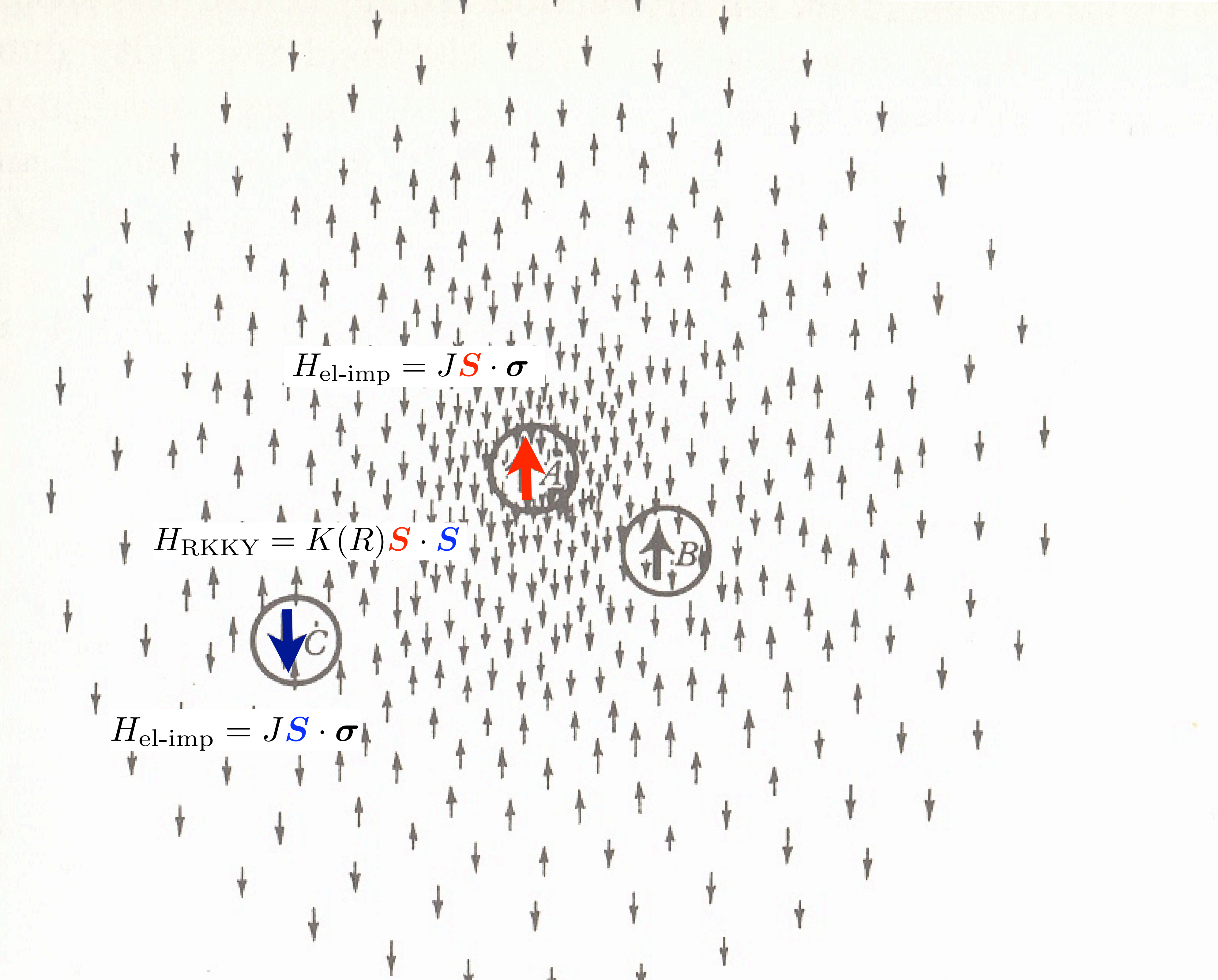


$H_{\mathrm{RKKY}} = K(R)\boldsymbol{S} \cdot \boldsymbol{S}$



$H_{\text{el-imp}} = J\boldsymbol{S} \cdot \boldsymbol{\sigma}$

$H_{\text{RKKY}} = K(R)\boldsymbol{S} \cdot \boldsymbol{S}$

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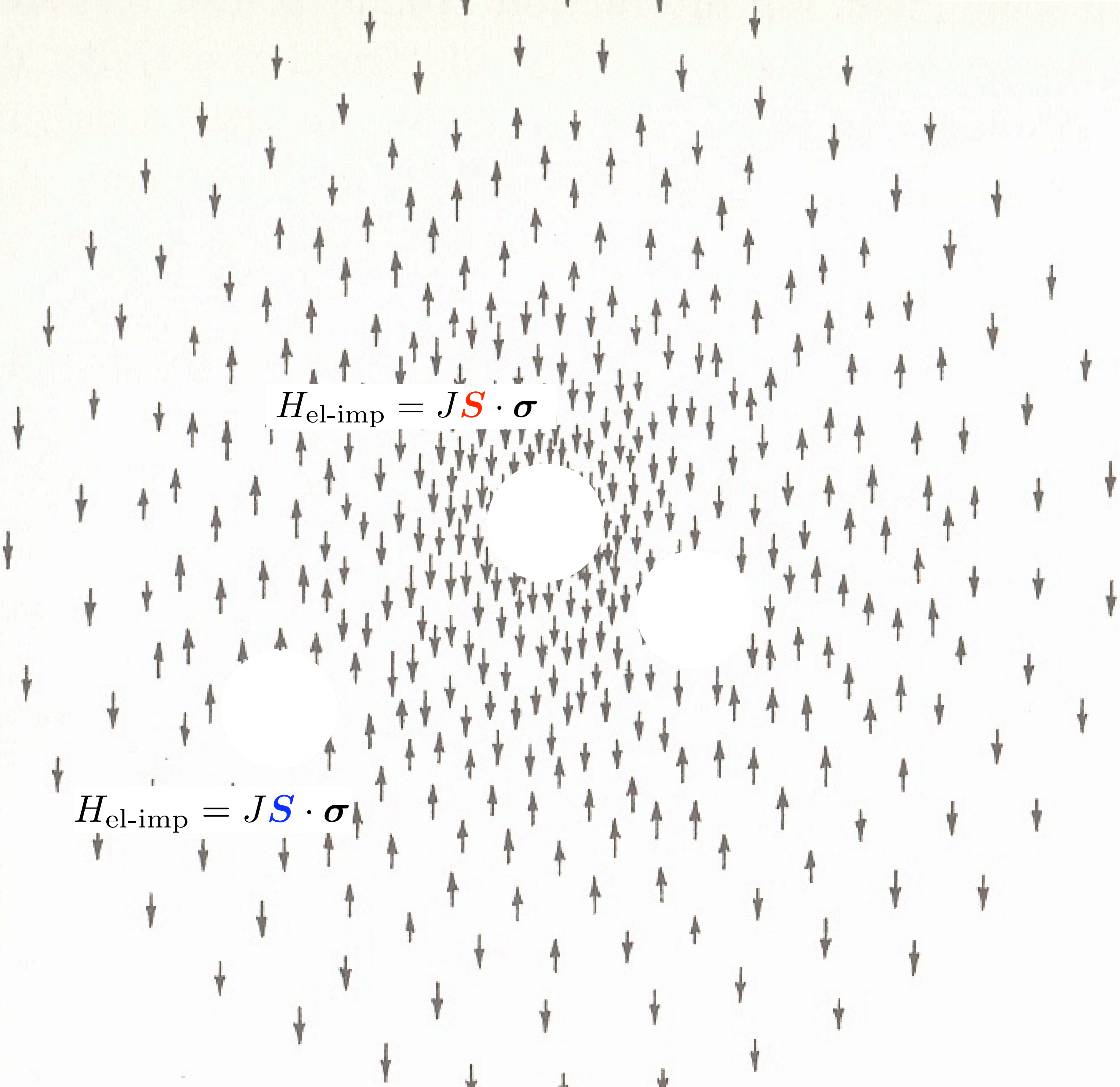

$$H_{\text{el-imp}} = J\boldsymbol{S}\cdot\boldsymbol{\sigma}$$

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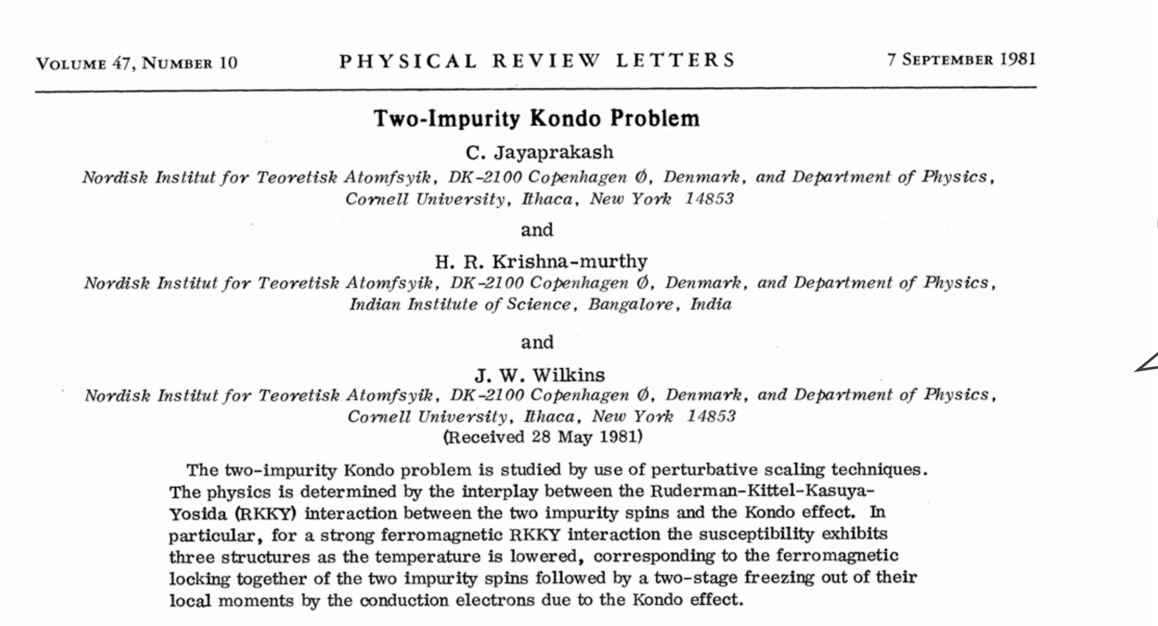
$$H = H_{\text{kin}} + J\boldsymbol{S}_1 \cdot \boldsymbol{\sigma} + J\boldsymbol{S}_2 \cdot \boldsymbol{\sigma} + K(R)\boldsymbol{S}_1 \cdot \boldsymbol{S}_2$$



VOLUME 47, NUMBER 10 PHYSICAL REVIEW LETTERS 7 SEPTEMBER 1981

Two-Impurity Kondo Problem

C. Jayaprakash
Nordisk Institut for Teoretisk Atomfsyik, DK-2100 Copenhagen Ø, Denmark, and Department of Physics, Cornell University, Ithaca, New York 14853

and

H. R. Krishna-murthy
Nordisk Institut for Teoretisk Atomfsyik, DK-2100 Copenhagen Ø, Denmark, and Department of Physics, Indian Institute of Science, Bangalore, India

and

J. W. Wilkins
Nordisk Institut for Teoretisk Atomfsyik, DK-2100 Copenhagen Ø, Denmark, and Department of Physics, Cornell University, Ithaca, New York 14853

(Received 28 May 1981)

The two-impurity Kondo problem is studied by use of perturbative scaling techniques. The physics is determined by the interplay between the Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction between the two impurity spins and the Kondo effect. In particular, for a strong ferromagnetic RKKY interaction the susceptibility exhibits three structures as the temperature is lowered, corresponding to the ferromagnetic locking together of the two impurity spins followed by a two-stage freezing out of their local moments by the conduction electrons due to the Kondo effect.

competition between RKKY-interaction and Kondo screening

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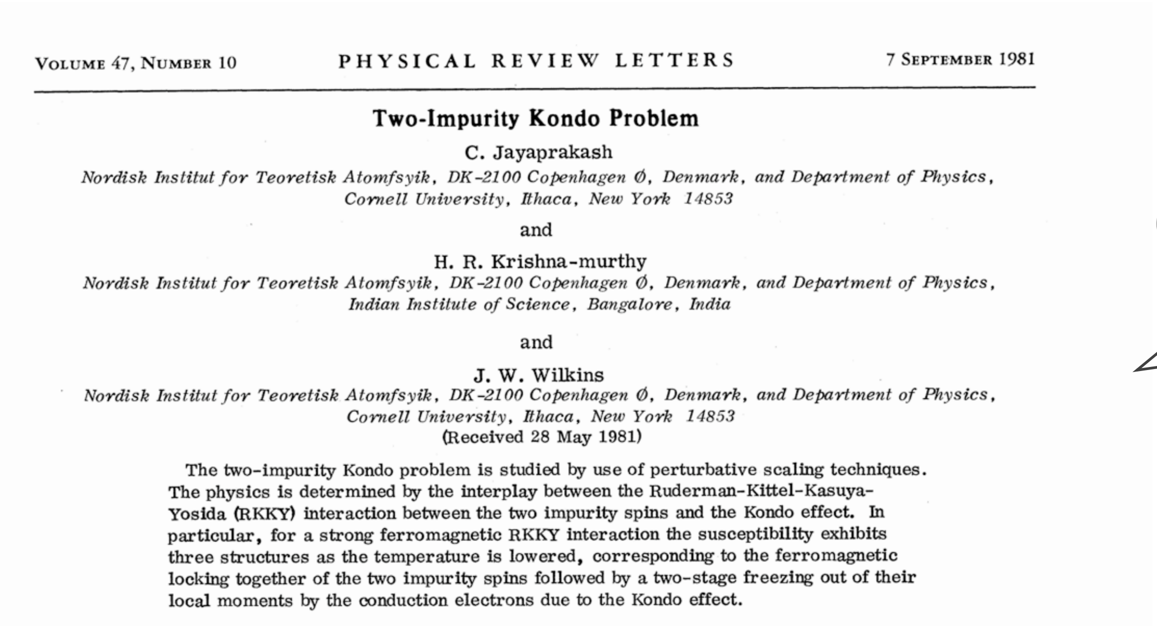

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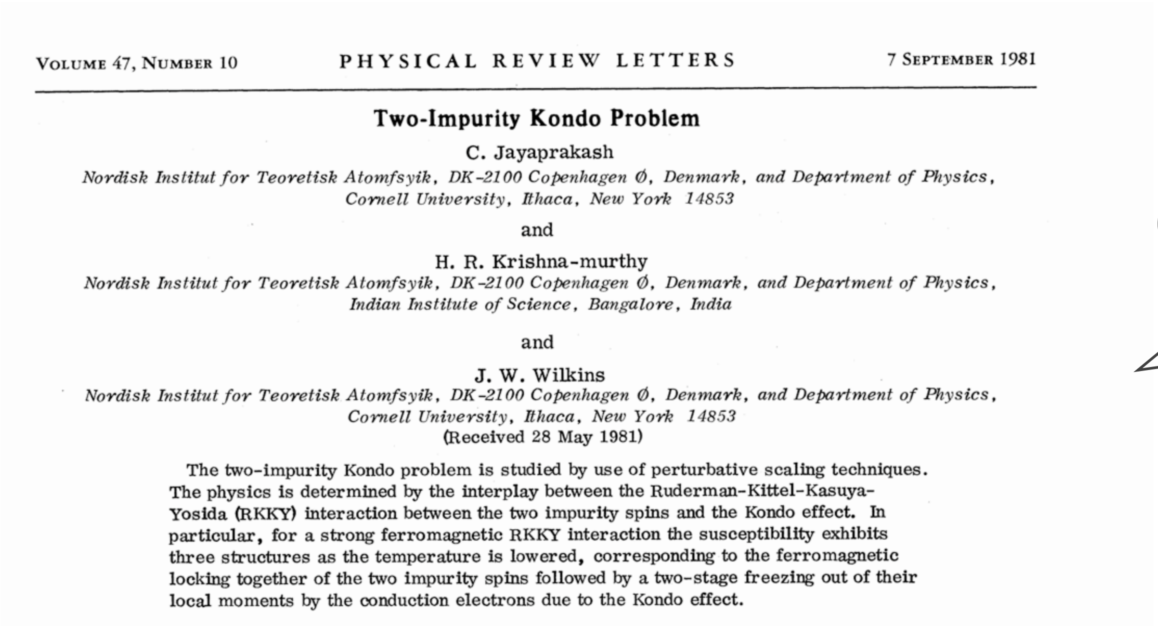
competition between RKKY-interaction and Kondo screening!

RKKY-coupled spin-singlet,
no Kondo screening

$K(R) \to -\infty$ $K(R) \to \infty$

$$H = H_{\rm kin} + J\boldsymbol{S}_1 \cdot \boldsymbol{\sigma} + J\boldsymbol{S}_2 \cdot \boldsymbol{\sigma} + K(R)\boldsymbol{S}_1 \cdot \boldsymbol{S}_2$$



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competition between RKKY-
interaction and Kondo screening!

RKKY-coupled spin-triplet,
Kondo screened by conduction electrons

RKKY-coupled spin-singlet,
no Kondo screening

$K(R) \to -\infty$

$K(R) \to \infty$

$$\delta = \pi/2$$

P. Nozières and A. Blandin,
J. Phys. (Paris) **41**, 193 (1980)

$$\delta = 0$$

RKKY-coupled spin-triplet,
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particle-hole symmetry → $\delta = 0$ *or* $\delta = \pi/2$

A. Millis et al.
Field Theories in Condensed Matter Physics
ed. Z. Tesanovic, 1990

$$\delta = \pi/2$$

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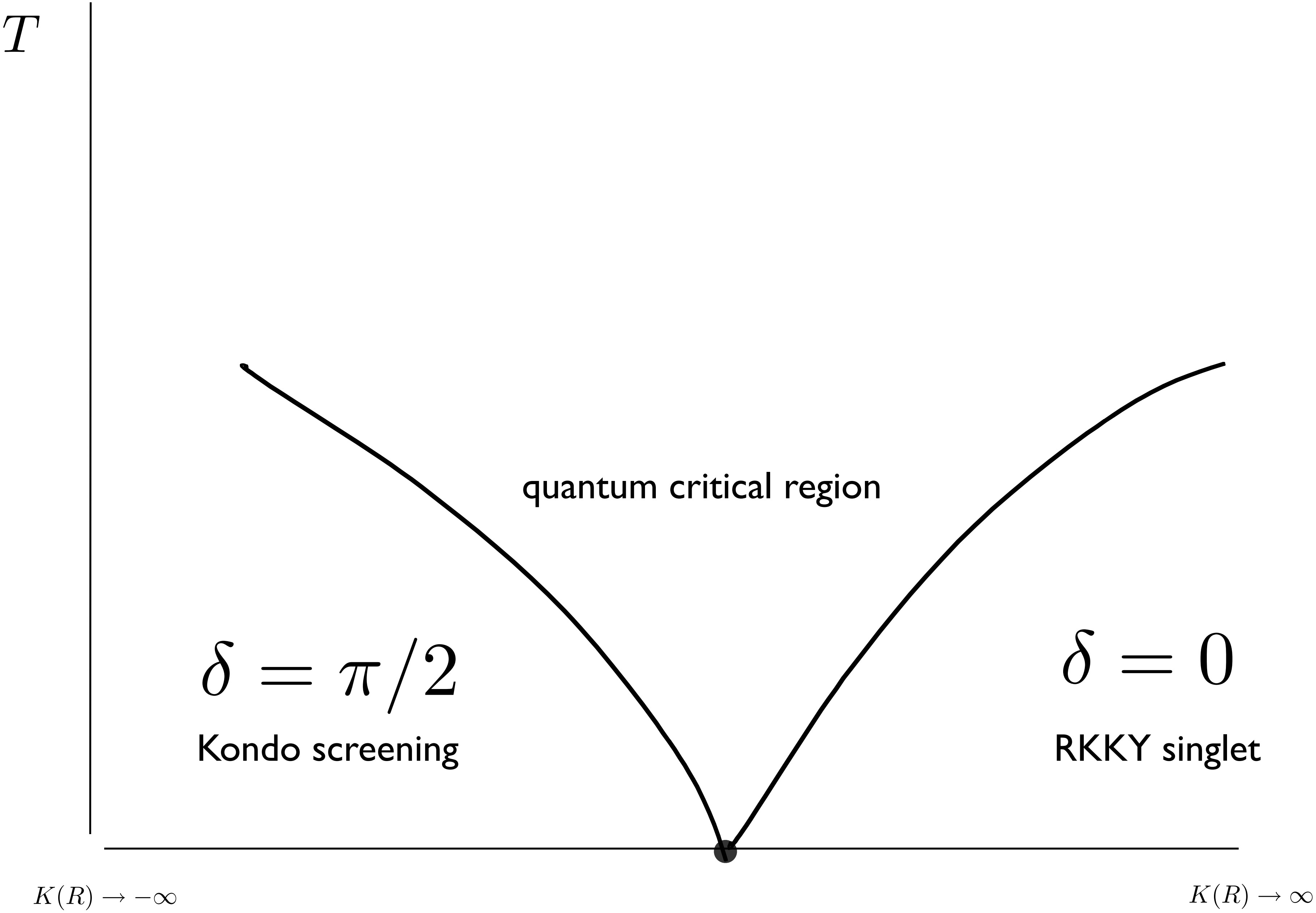


T

quantum critical region

$\delta = \pi/2$

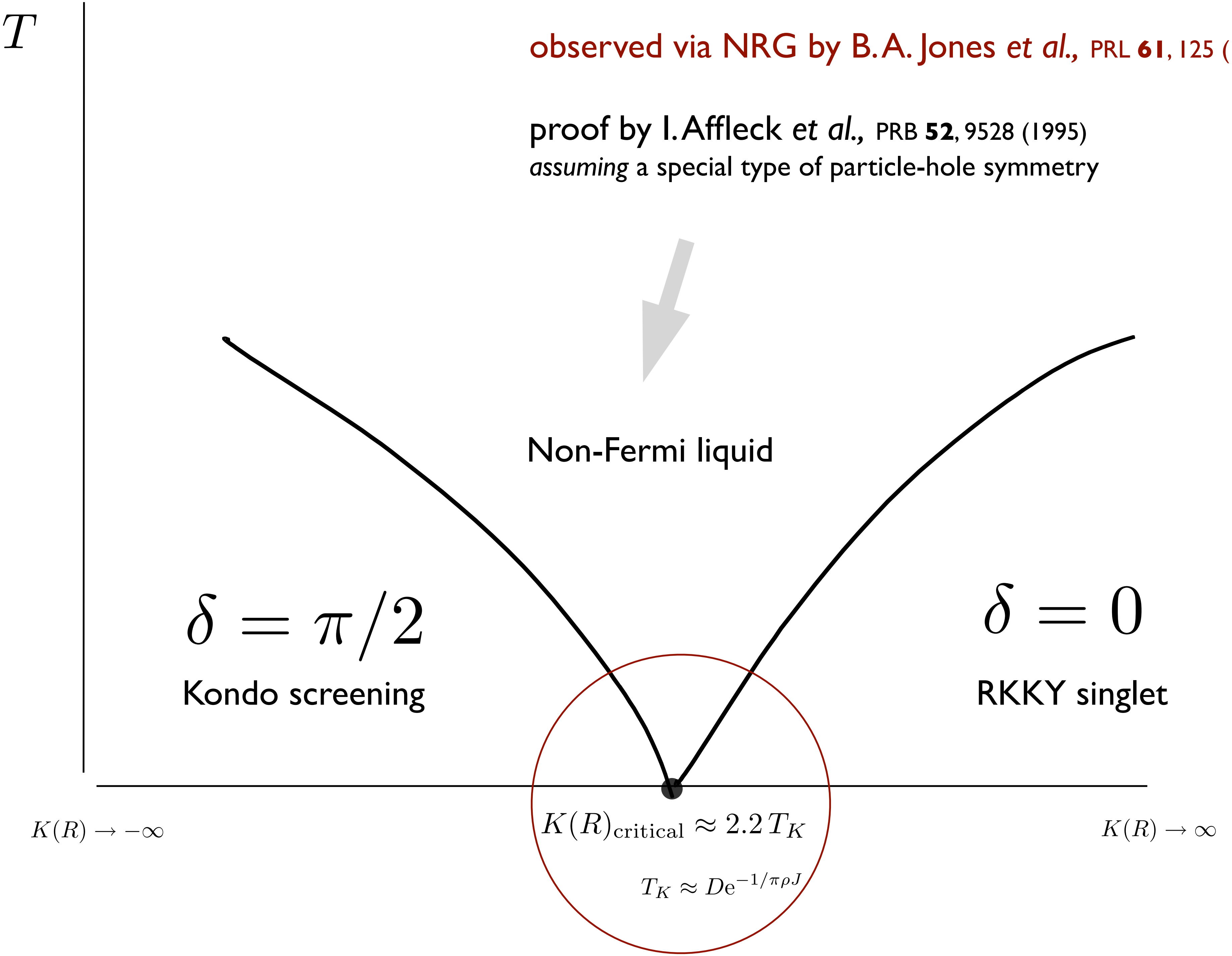
Kondo screening

$\delta = 0$

RKKY singlet

$K(R) \to -\infty$

$K(R) \to \infty$



observed via NRG by B.A. Jones *et al.*, PRL **61**, 125 (1988)

proof by I. Affleck *et al.*, PRB **52**, 9528 (1995)
assuming a special type of particle-hole symmetry

T

Non-Fermi liquid

$\delta = \pi/2$

Kondo screening

$\delta = 0$

RKKY singlet

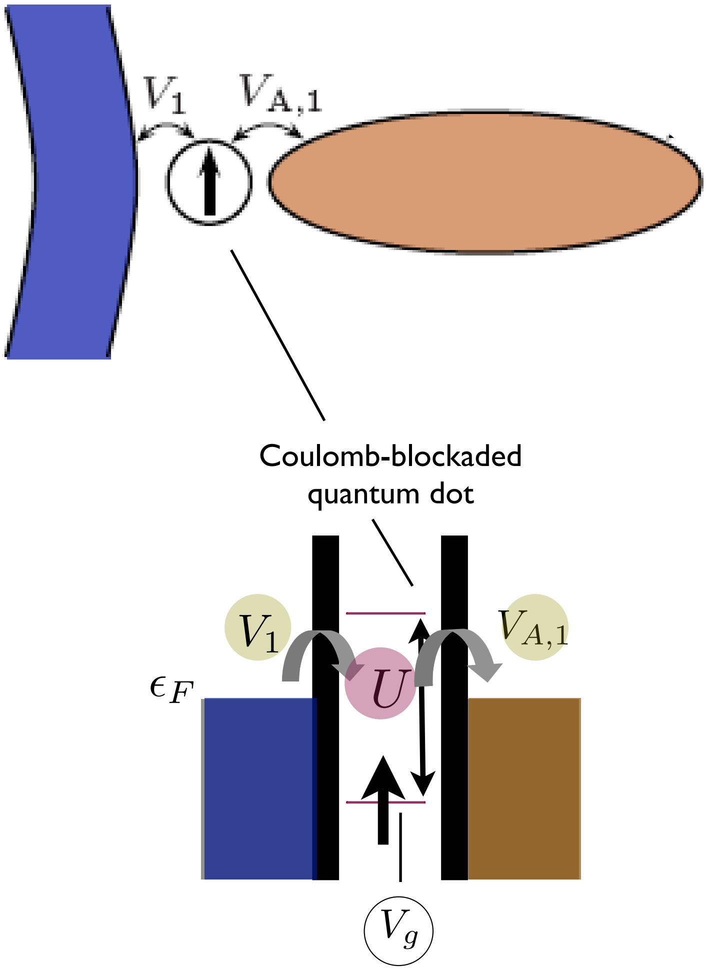
$K(R) \to -\infty$

$K(R)_{\text{critical}} \approx 2.2\, T_K$

$K(R) \to \infty$

$T_K \approx D e^{-1/\pi\rho J}$

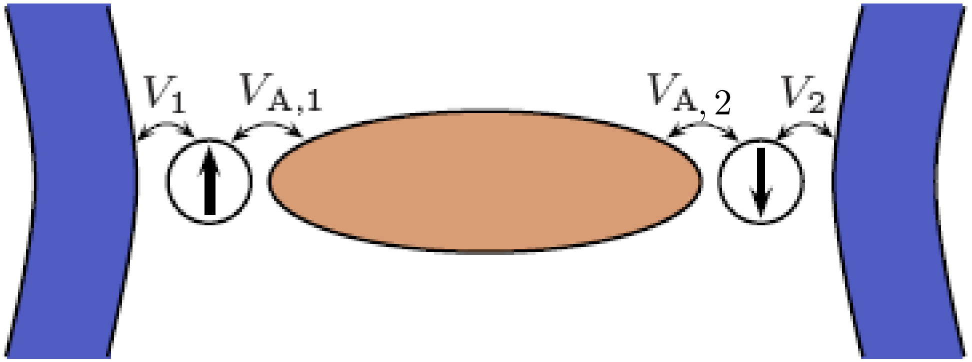
Realization in double quantum-dot systems



Coulomb-blockaded
quantum dot

spin exchange $J \propto V^2/U$

Realization in double quantum-dot systems



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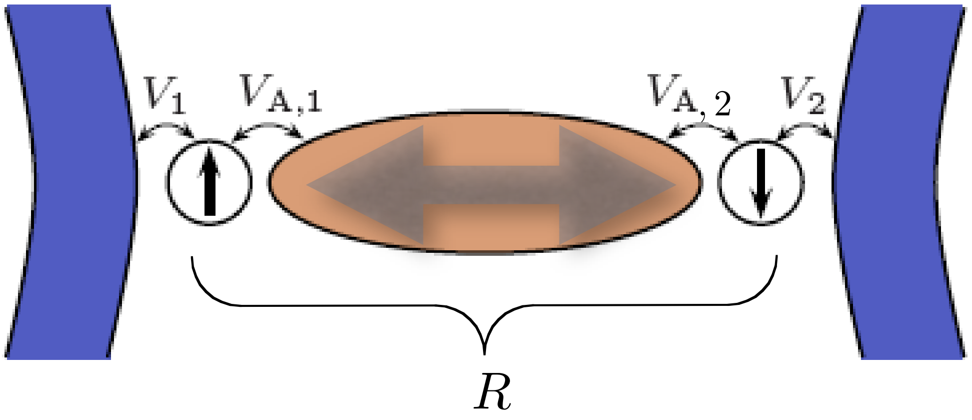
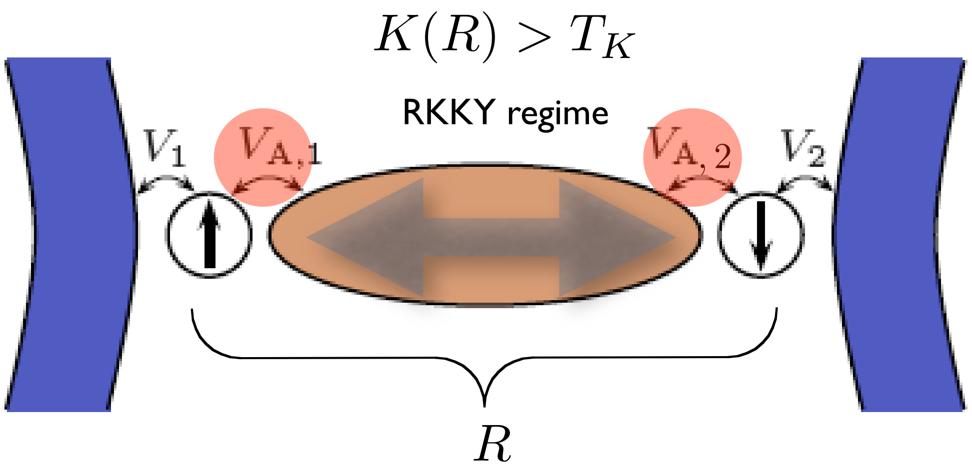


RKKY coupling $K(R) \propto (J^2/R^2)\cos(k_F R)$

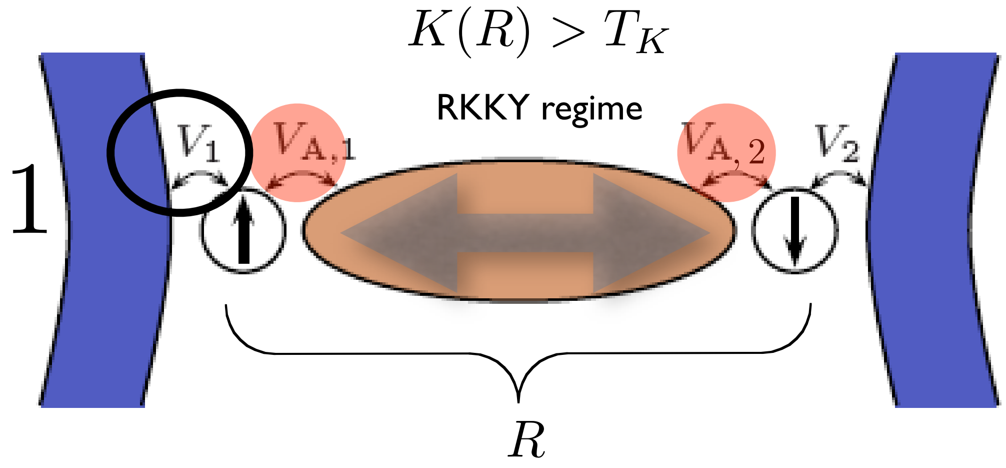
Realization in double quantum-dot systems



RKKY coupling $K(R) \propto (J^2/R^2)\cos(k_F R)$

Kondo temperature $T_K \propto D\exp(-1/\pi\rho J)$

Realization in double quantum-dot systems



$$H_{\text{int}} = J_1 \boldsymbol{S}_1 \cdot \boldsymbol{\sigma}_1 + J_2 \boldsymbol{S}_2 \cdot \boldsymbol{\sigma}_2 + K(R) \boldsymbol{S}_1 \cdot \boldsymbol{S}_2$$

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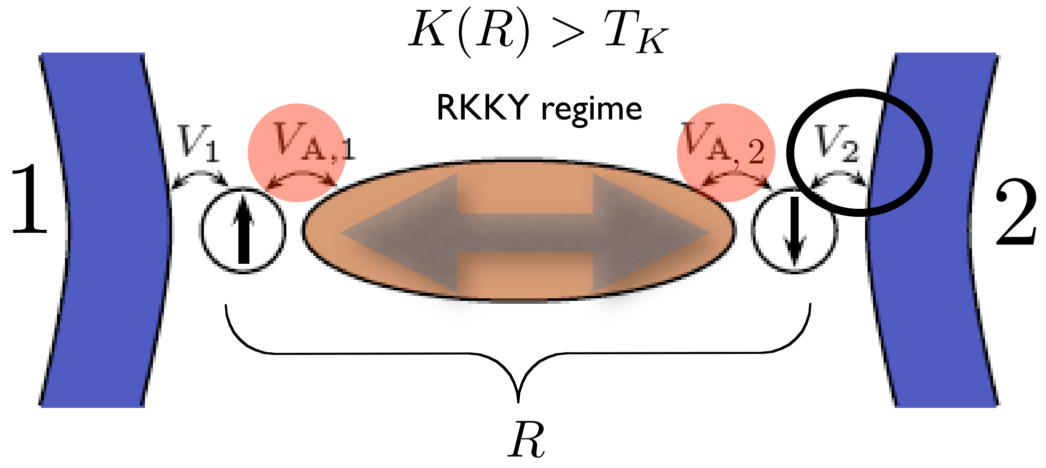

$$H_{\mathrm{int}} = J_1 \boldsymbol{S}_1 \cdot \boldsymbol{\sigma}_1 + \boxed{J_2 \boldsymbol{S}_2 \cdot \boldsymbol{\sigma}_2} + K(R) \boldsymbol{S}_1 \cdot \boldsymbol{S}_2$$

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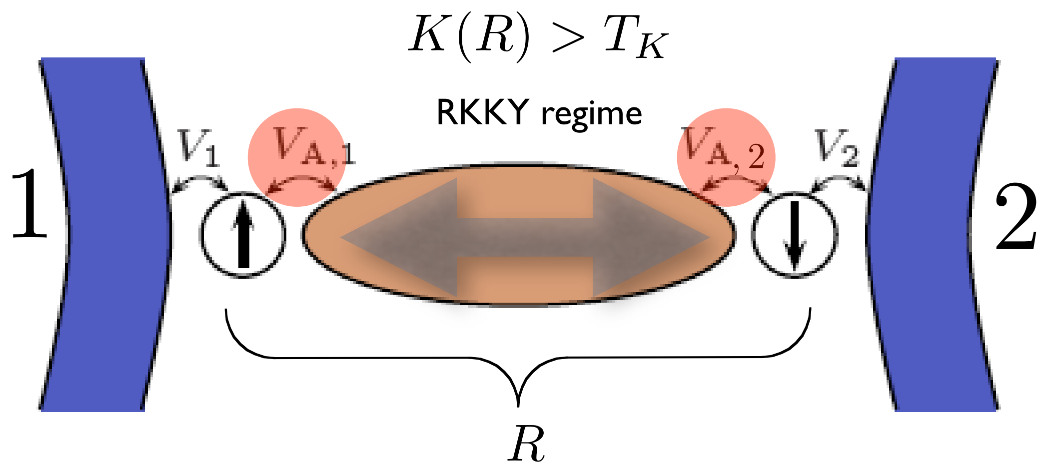

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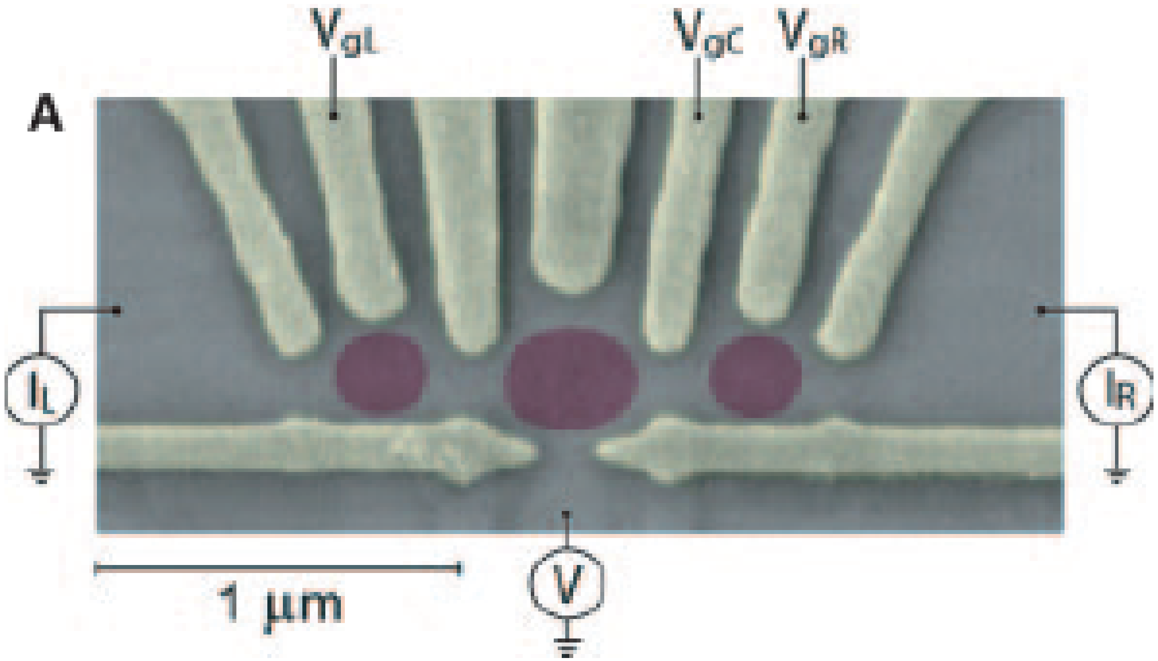
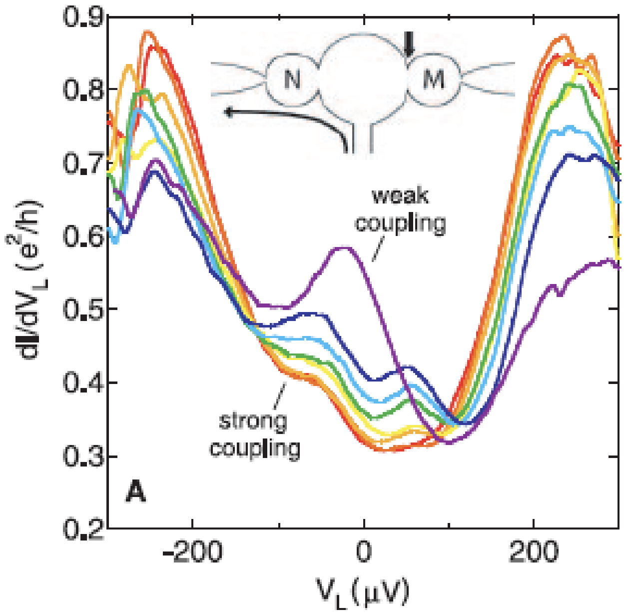
No transfer of electrons between 1 and 2:
quantum critical point $K_c \approx 2.2 T_K$ is stable
against electron-hole symmetry breaking
and breaking of parity

G. Zaránd *et al.*, PRL **97**, 166802 (2006)

Realization in double quantum-dot systems

N. J. Craig *et al.*, Science **304**, 565 (2004)





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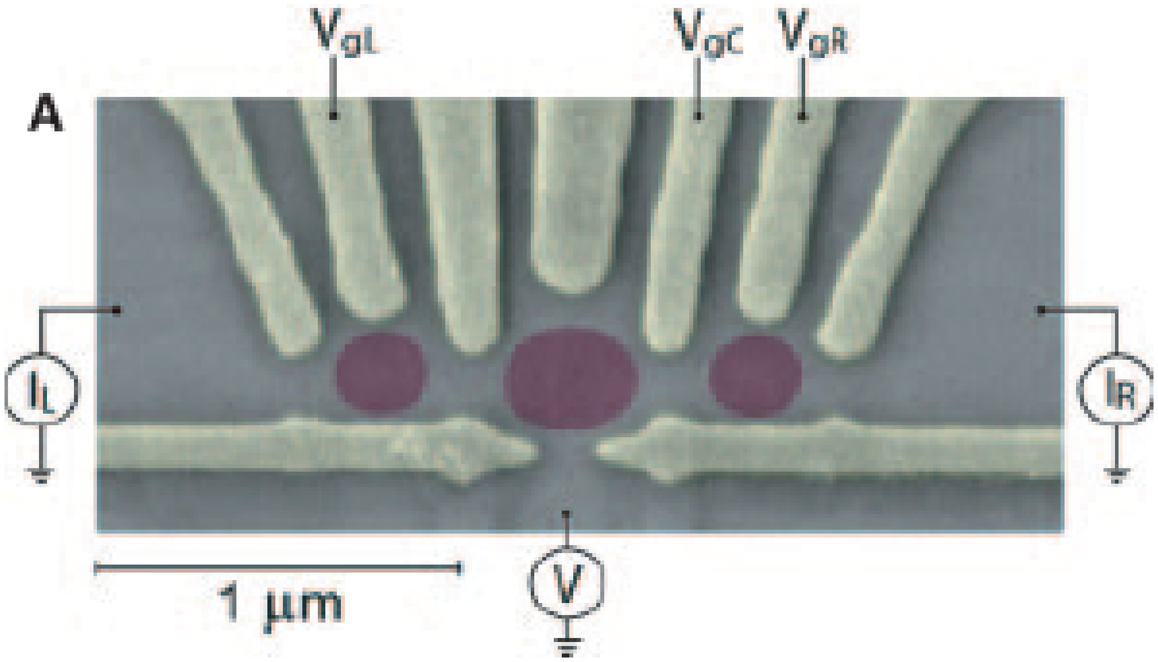
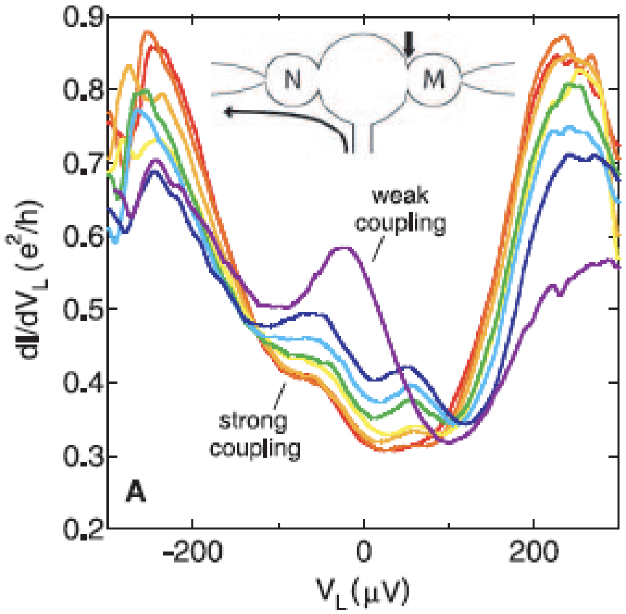
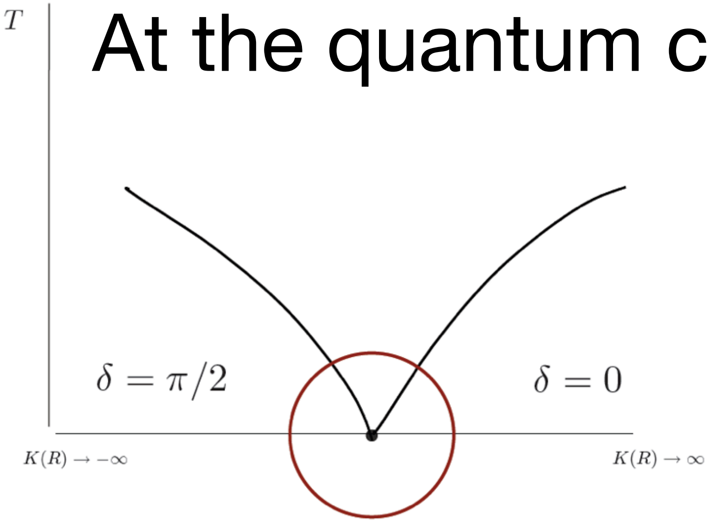




Nota Bene:
The central dot supports both RKKY *and* Kondo screening.
The experiment does *not* probe quantum criticality.

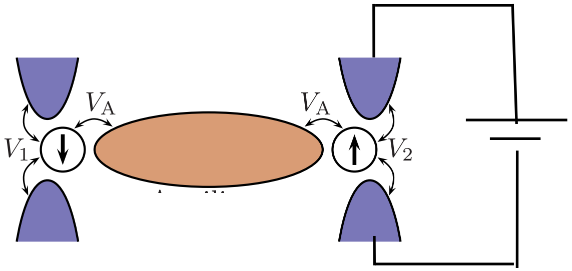
At the quantum critical point...



Transport

$$G \approx G_0(1 - \lambda_1 T^{1/2}), \quad T > T_1^*$$

G. Zaránd *et al.*, PRL **97**, 166802 (2006)



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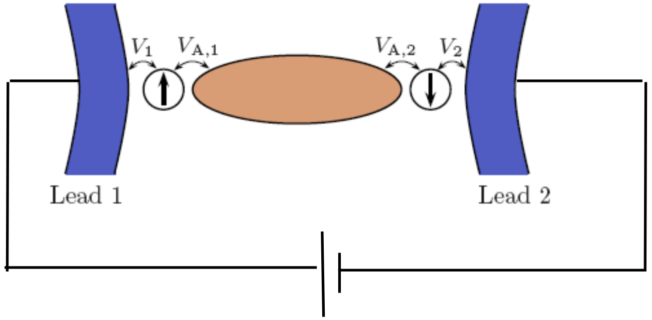
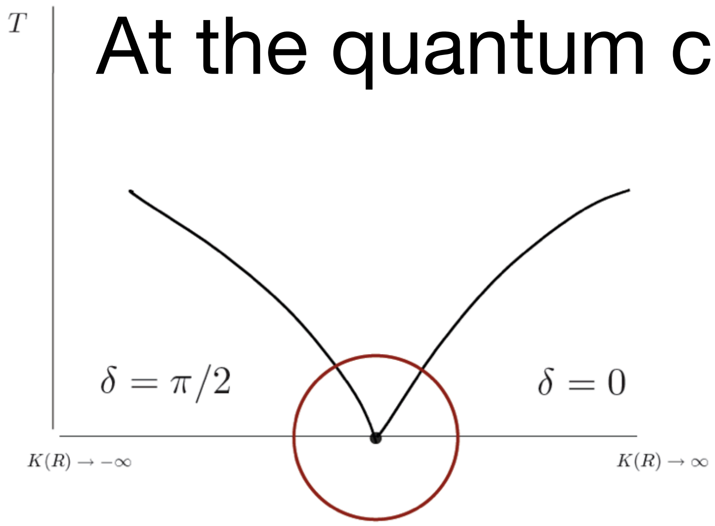
E. Sela and I. Affleck, PRL **102**, 047201(2009)





At the quantum critical point...

Thermodynamics (impurity contribution)

I. Affleck *et al.*, PRB **52**, 9528 (1995)

$$C_{\rm imp} \approx \text{constant} \times T$$

$$\chi_{\rm imp} = \text{constant}$$

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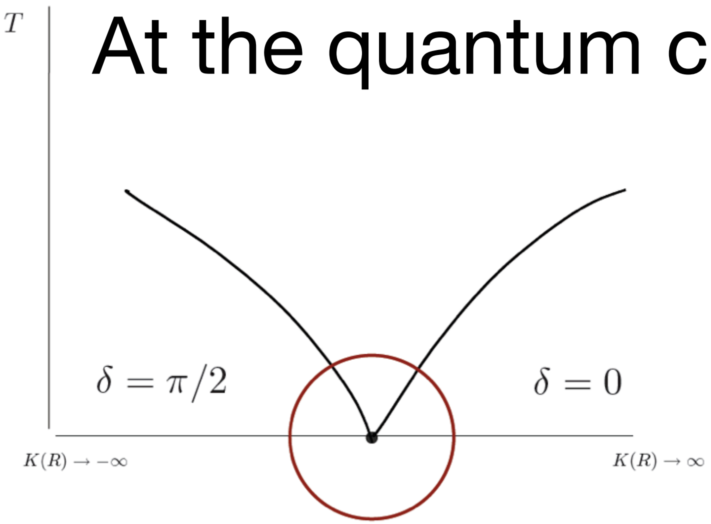

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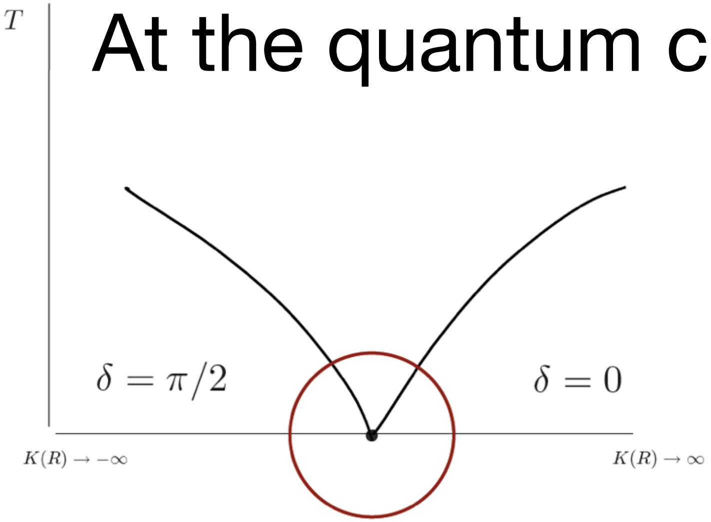

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"fractional ground state degeneracy" $g^A = \sqrt{2}$

At the quantum critical point...

Thermodynamic impurity entropy

$$S_{\rm imp} = \ln \sqrt{2}$$

determined by the *conformally invariant boundary condition*, "**A**" call it, that represents the impurity-electron interaction

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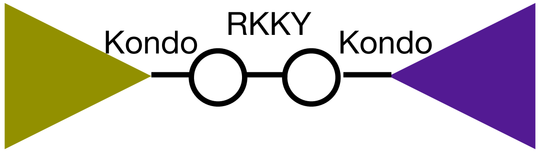

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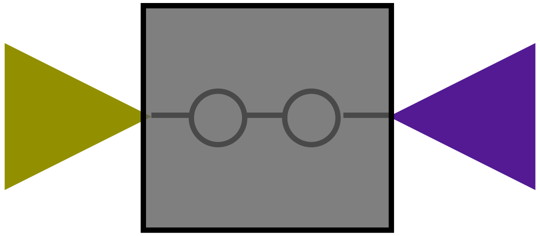

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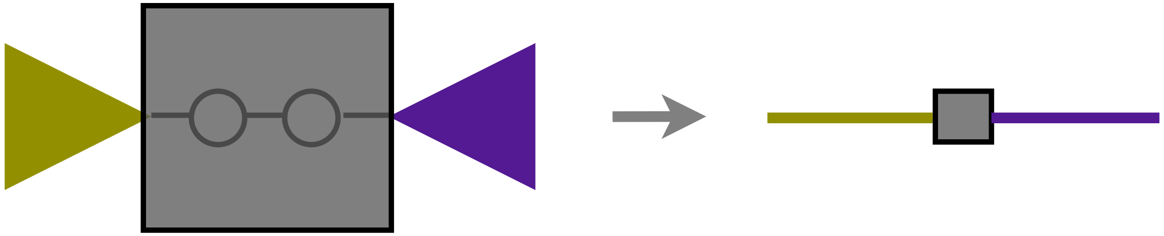

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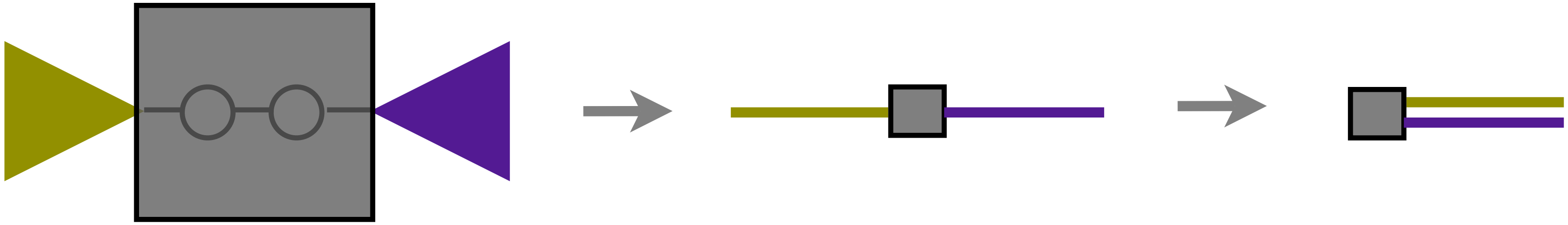

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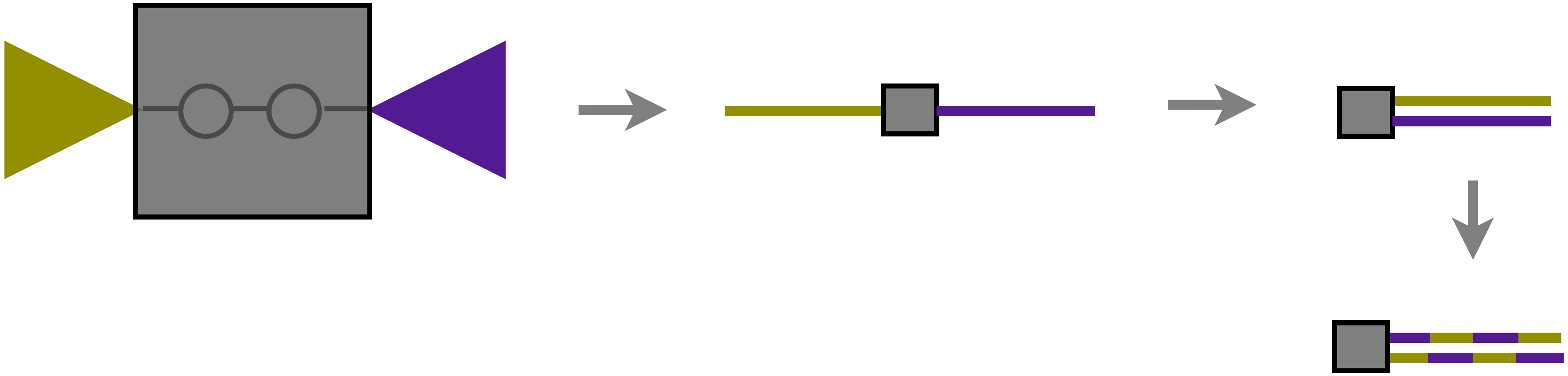

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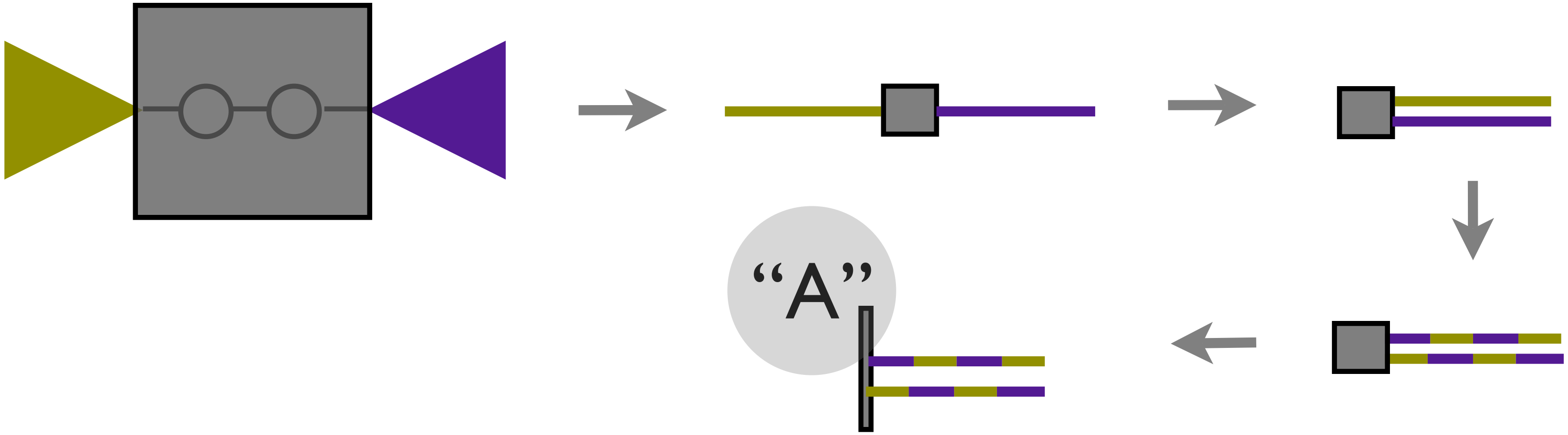

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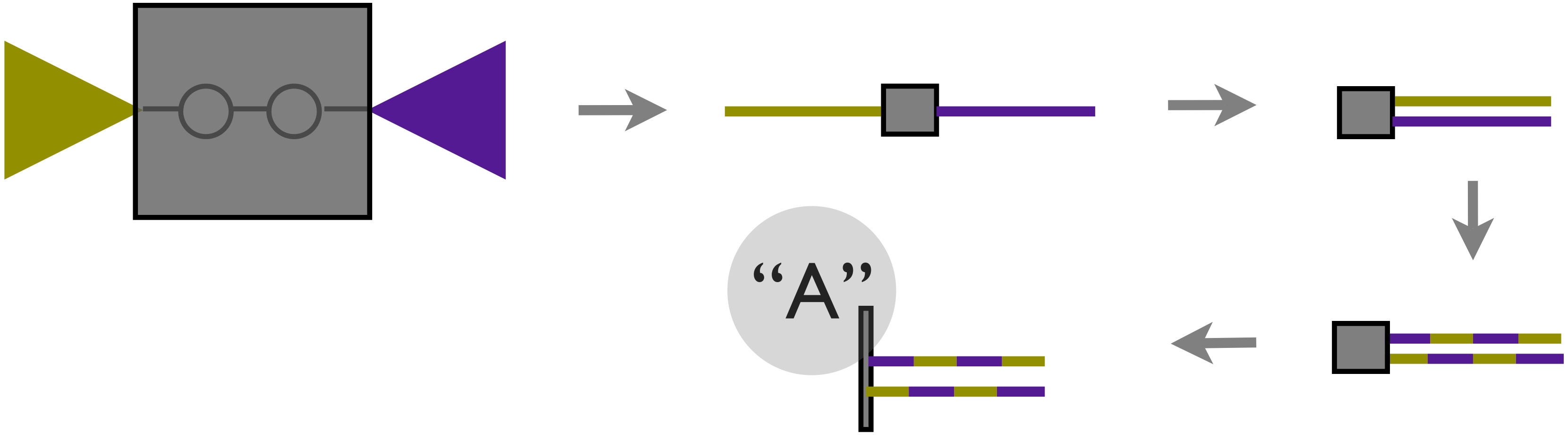

Symmetry

$$\mathrm{U}(1)\otimes\mathrm{U}(1)\otimes\mathrm{SU}(2)_1\otimes\mathrm{SU}(2)_1 \longrightarrow \mathrm{U}(1)\otimes\mathrm{U}(1)\otimes\mathrm{SU}(2)_2\otimes\mathrm{Ising}$$

coset construction

At the quantum critical point...

Thermodynamic impurity entropy

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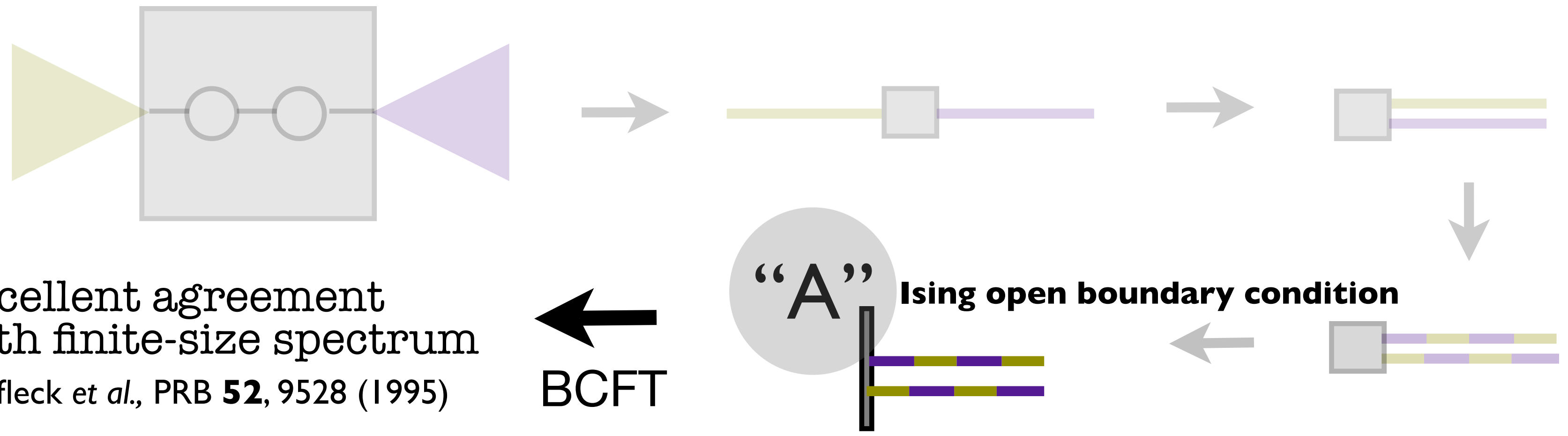

“A”

Ising open boundary condition

BCFT

excellent agreement
with finite-size spectrum

I. Affleck *et al.*, PRB **52**, 9528 (1995)

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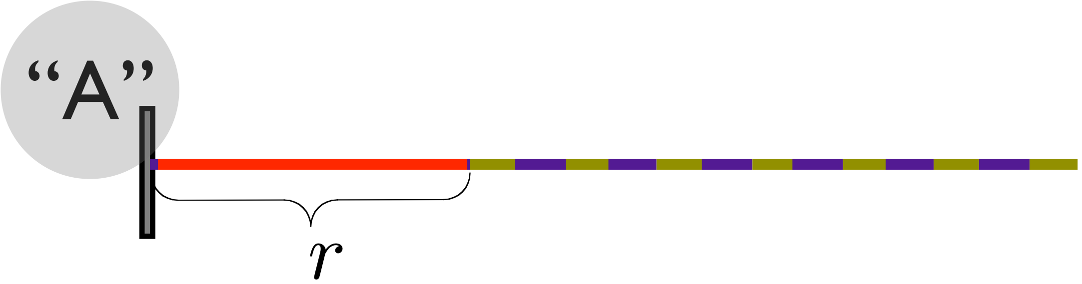
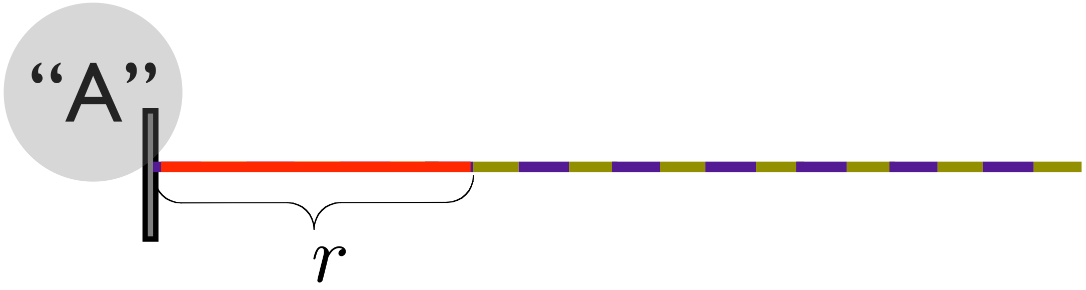


von Neumann entropy

$$S^A(r) = \frac{c}{6}\log(\frac{r}{a}) + c^A + \text{nonuniversal constant}$$

P. Calabrese and J. Cardy, J. Stat. Mech. P06002 (2004)

At the quantum critical point...



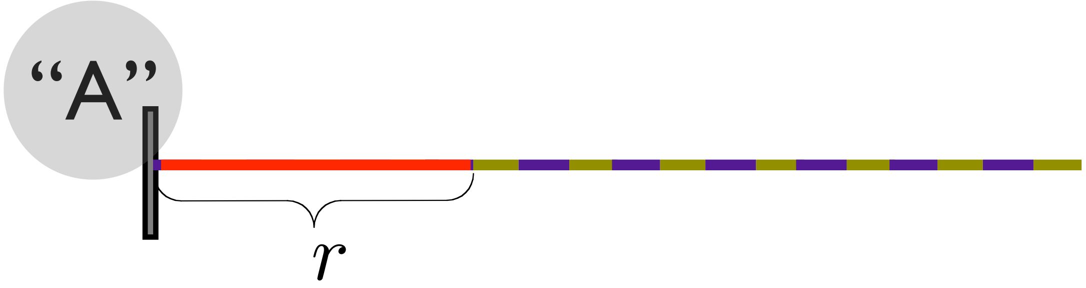
von Neumann entropy

$$S^A(r) = \frac{c}{6}\log(\frac{r}{a}) + c^A + \text{nonuniversal constant}$$

generalize to finite temperature $\beta \ll r$
compare to the thermodynamic entropy

identify $c^A = \ln\sqrt{2}$

At the quantum critical point...



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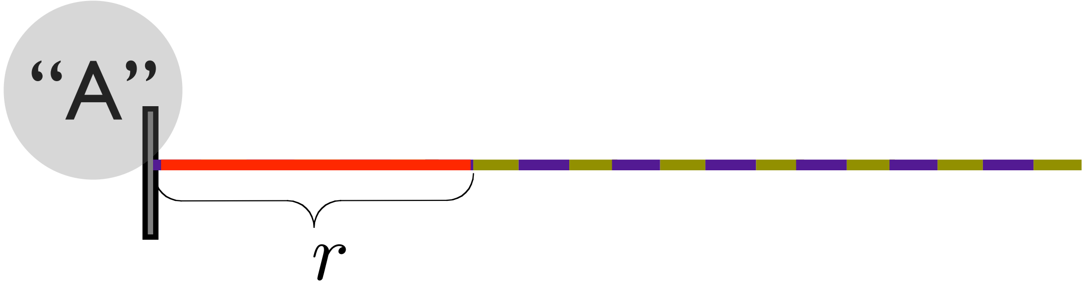
boundary entanglement

thermodynamic impurity entropy

At the quantum critical point...



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generalize to finite temperature $\beta \ll r$
compare to the thermodynamic entropy

identify $c^A = \ln\sqrt{2}$

More information about the critical behavior in the **scaling corrections** to the entanglement entropy!

cf. talk by P. Calabrese, last Friday

see also J. Cardy and P. Calabrese, J. Stat. Mech. P04023 (2010)

Scaling corrections to boundary entanglement

Single-impurity Kondo model

electron boundary
phase shift

$c^A = 0$ *at* the critical point

Scaling corrections to boundary entanglement

Single-impurity Kondo model

uniform part of c^A

$$c^A = 0 + \frac{\pi \xi_K}{12 r} + \ldots \qquad r \gg \xi_K = \frac{\hbar v_F}{T_K}$$

E. Sørensen *et al.*, J. Stat. Mech. P08003 (2007)

Scaling corrections to boundary entanglement

Single-impurity Kondo model

$$c^A = 0 + \frac{\pi\xi_K}{12r} + \ldots \qquad r \gg \xi_K = \frac{\hbar v_F}{T_K}$$

Two-impurity Kondo model

Ising open boundary condition

$c^A = \ln\sqrt{2}$ *at* the critical point

Scaling corrections to boundary entanglement

Single-impurity Kondo model

$$c^A = 0 + \frac{\pi \xi_K}{12 r} + \ldots \qquad r \gg \xi_K$$

Two-impurity Kondo model

$$c^A = \ln \sqrt{2} + \ldots ? \qquad r \gg \xi_K$$

Scaling corrections to boundary entanglement

$$c^A = \ln\sqrt{2} + \ldots? \qquad r \gg \xi_K$$

determined by the leading RG irrelevant boundary operator

$$H = H^* + \lambda_1 \mathcal{O}_1 + \ldots$$

$$S^A(r) = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}\rho_r^n = -\lim_{n\to 1}\frac{\partial}{\partial n}\frac{Z_{\mathcal{R}_n}}{Z^n}$$

P. Calabrese and J. Cardy, J. Stat. Mech. P06002 (2004)

$$Z_{\mathcal{R}_n} = Z^*_{\mathcal{R}_n} + \partial Z_{\mathcal{R}_n}$$

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for details, see E. Eriksson, **poster 12**, second floor

Scaling corrections to boundary entanglement

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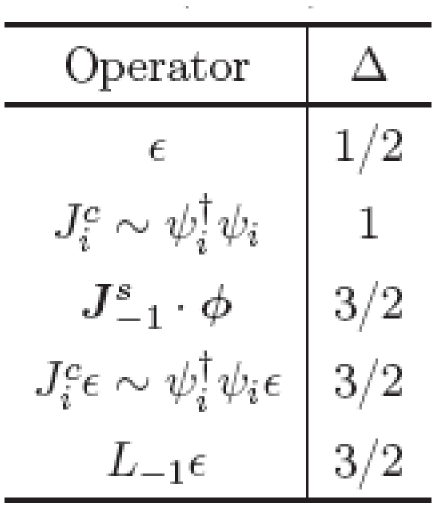

Operator	Δ
ϵ	$1/2$
$J_i^c \sim \psi_i^\dagger \psi_i$	1
$\boldsymbol{J}^s_{-1} \cdot \boldsymbol{\phi}$	$3/2$
$J_i^c \epsilon \sim \psi_i^\dagger \psi_i \epsilon$	$3/2$
$L_{-1}\epsilon$	$3/2$

D. F. Mross and HJ, PRB **78**, 035449 (2008)

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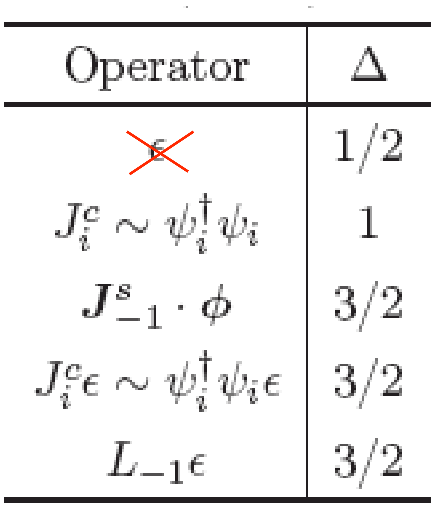

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fine tune to 0 to stay critical!

$$H = H^* + \lambda_1 \mathcal{O}_1 + \ldots$$

$$S^A(r) = -\lim_{n\to 1} \frac{\partial}{\partial n} \mathrm{Tr}\rho_r^n = -\lim_{n\to 1} \frac{\partial}{\partial n} \frac{Z_{\mathcal{R}_n}}{Z^n}$$

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for details, see E. Eriksson, **poster 12**, second floor

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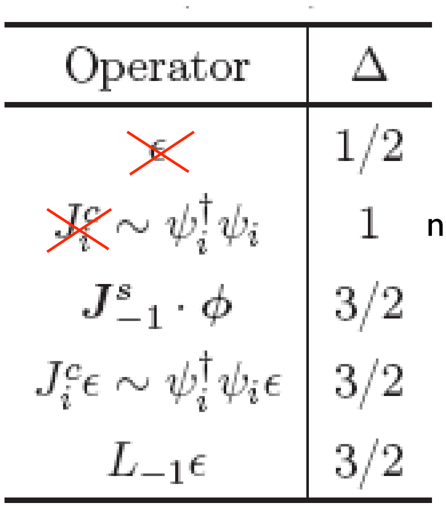

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no charge fluctuations!

$$H = H^* + \lambda_1 \mathcal{O}_1 + \ldots$$

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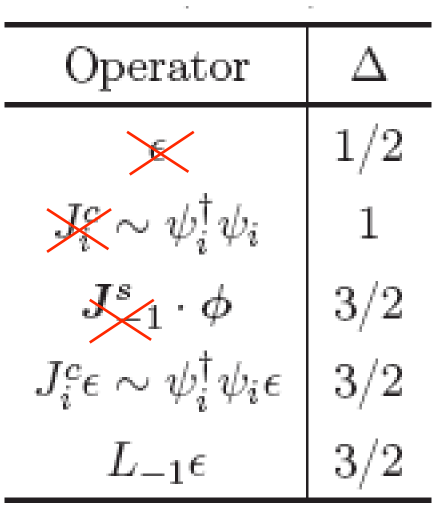

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vanishes for equal Kondo couplings J_1 and J_2

$$H = H^* + \lambda_1 \mathcal{O}_1 + \ldots$$

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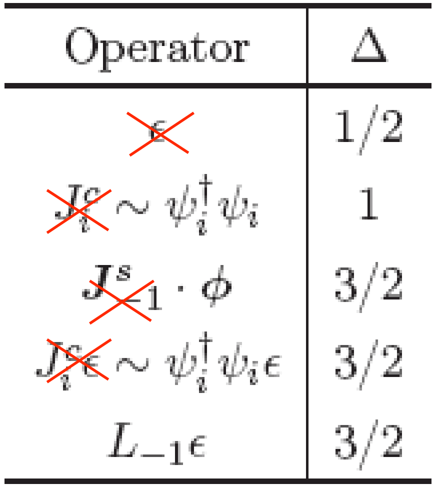

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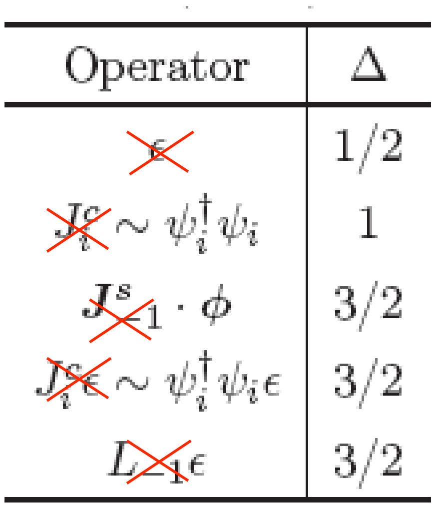

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$L_{\xcancel{-1}}\epsilon$	$3/2$

no contribution from Virasoro first descendants

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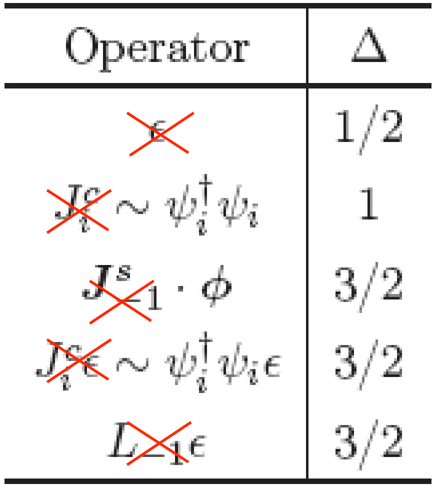

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leading correction-to-scaling operator given by the Ising energy-momentum tensor of ***dimension 2***

same leading scaling dimension as for the single-impurity Kondo model

for details, see E. Eriksson, **poster 12**, second floor

Scaling corrections to boundary entanglement

Two-impurity Kondo model

$$c^A = \ln\sqrt{2} + \frac{\pi\xi_K}{12r} + \dots \qquad r \gg \xi_K$$

same as for the single-impurity Kondo model

Breaking parity or allowing for charge fluctuations on the quantum dots (two-impurity Anderson model) → **different scaling corrections to the boundary entanglement**

E. Eriksson and HJ, in progress

Oscillating term in the entanglement entropy?

cf. N. Laflorencie et al., PRL **96**, 100603 (2006)

Boundary entanglement in the *integrable* TIKM?

cf. P. Schlottmann, PRL **80**, 4975 (1998); A.A. Zvyagin, PRB **65**, 214404 (2002)

Other entanglement probes of Kondo physics

Two-impurity concurrence

$C \longrightarrow 0$ at the TIKM critical point

S.Y. Cho and R. H. McKenzie, PRA **73**, 012109 (2006)

Negativity for Kondo "spin chain"

Measure of Kondo screening cloud

A. Bayat, P. Sodano, and S. Bose, PRB **81**, 064429 (2010)

Generalization to TIKM?

-
-
-

A new twist: **Adding spin-orbit interactions...**

Adding spin-orbit interactions...

relativistic correction in vacuum

$$H_{\mathrm{SO}} = \lambda_{\mathrm{vac}}(\nabla V \times \boldsymbol{k}) \cdot \boldsymbol{\sigma}$$

$$\lambda_{\mathrm{vac}} = \hbar^2/4m_0^2c^2 \approx 3.7 \times 10^{-6}\text{Å}^2$$

Adding spin-orbit interactions...

relativistic correction in vacuum

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relativistic correction in a semiconductor

$$H_{\text{SO}} = \lambda_{\text{crystal}}(\nabla V \times \boldsymbol{k}) \cdot \boldsymbol{\sigma}$$

$$\lambda_{\text{crystal}} \approx \hbar^2/4m^*E_g \approx 10^6\lambda_{\text{vac}}$$

bandgap

effective mass from periodic crystal potential

Adding spin-orbit interactions...

relativistic correction in vacuum

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aperiodic part of the total potential:
confinement, impurities, boundaries,
external electric fields,...

Adding spin-orbit interactions...

relativistic correction in a semiconductor

$$H_{\text{SO}} = \lambda_{\text{crystal}}(\nabla V \times \boldsymbol{k}) \cdot \boldsymbol{\sigma}$$

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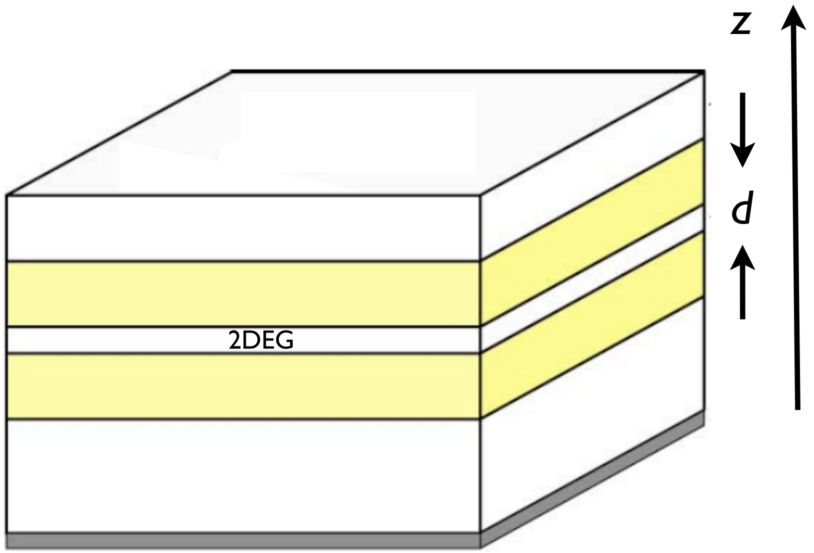
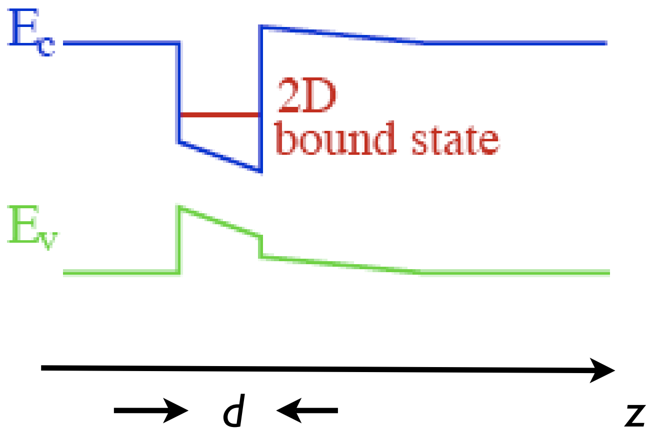


semiconductor heterostructure



Adding spin-orbit interactions...

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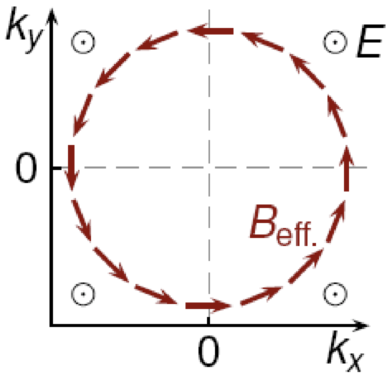
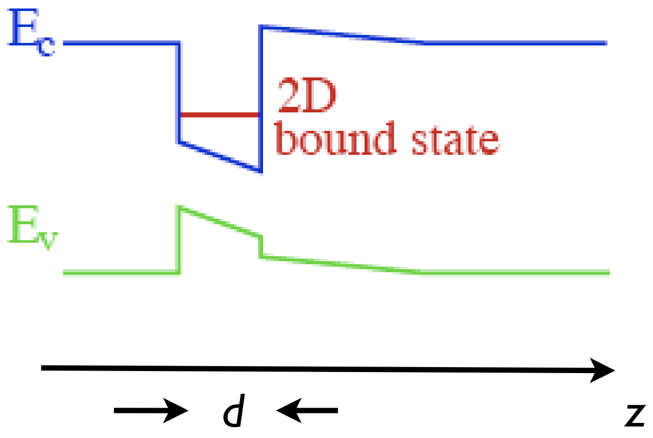


Rashba interaction

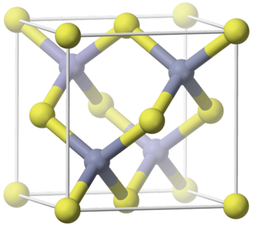
E. I. Rashba, Sov. Phys. Solid State **2**, 1109 (1960)

$$H_R = \alpha(k_x\sigma^y - k_y\sigma^x)$$

Spatial asymmetry of
band edges mimics an
E-field in the *z*-direction



Another type of spin-orbit interaction in 2D semiconductor heterostructures...



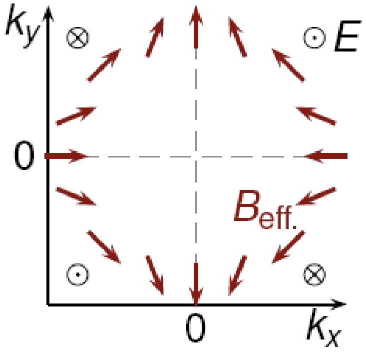
zincblende structures:
GaAs, InAs, HgTe,...

broken lattice
inversion symmetry

Dresselhaus interaction

G. Dresselhaus, Phys. Rev. **100**, 580 (1955)

$$H_D = \beta(k_x\sigma^x - k_y\sigma^y)$$



gate controllable

	$\beta k_F/\mathrm{meV}$	$\alpha k_F/\mathrm{meV}$
GaAs/AlGaAs	$0.01 - 0.1$	$0.01 - 0.1$
HgTe/CdTe	0.1	30

Dresselhaus interaction

$$H_D = \beta(k_x\sigma^x - k_y\sigma^y)$$

Rashba interaction

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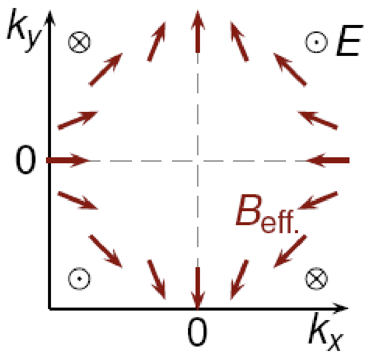
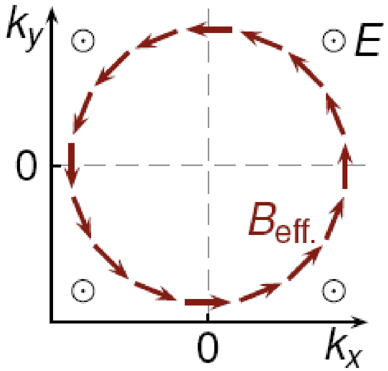




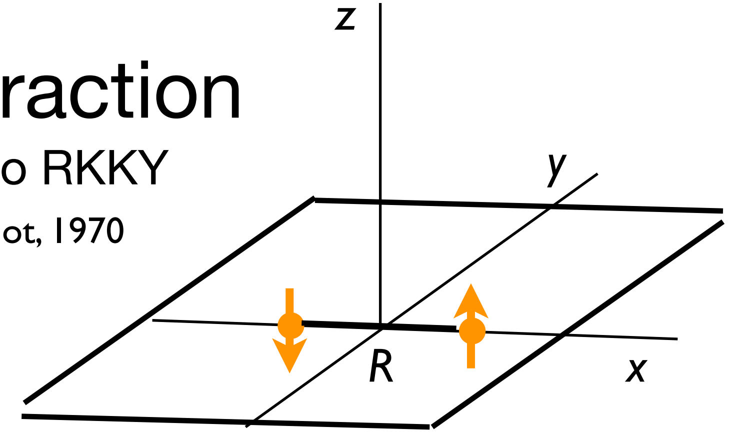
How do Rashba and Dresselhaus spin-orbit interactions influence two-impurity Kondo physics?

D. F. Mross and H. J., PRB **80**, 155302 (2009)

Spin-orbit effects on the RKKY interaction

from simple extension of standard perturbative approach to RKKY

Blackman and Elliot, 1970



$$H = \frac{\boldsymbol{k}^2}{2m} + \left[\begin{pmatrix} \beta & -\alpha \\ \alpha & -\beta \end{pmatrix}\begin{pmatrix} k_x \\ k_y \end{pmatrix}\right] \cdot \boldsymbol{\sigma}$$

$$G(\boldsymbol{k}, \omega) \equiv (\omega - H(\boldsymbol{k}))^{-1}$$

$$H_{\text{RKKY}} = -\frac{J_1 J_2}{\pi} \text{Im} \int_{-\infty}^{\omega_F} d\omega \ \text{Tr}\left[(\boldsymbol{S}_1 \cdot \boldsymbol{\sigma}) G(R, \omega + i0_+) \times (\boldsymbol{S}_2 \cdot \boldsymbol{\sigma}) G(-R, \omega + i0_+)\right]$$

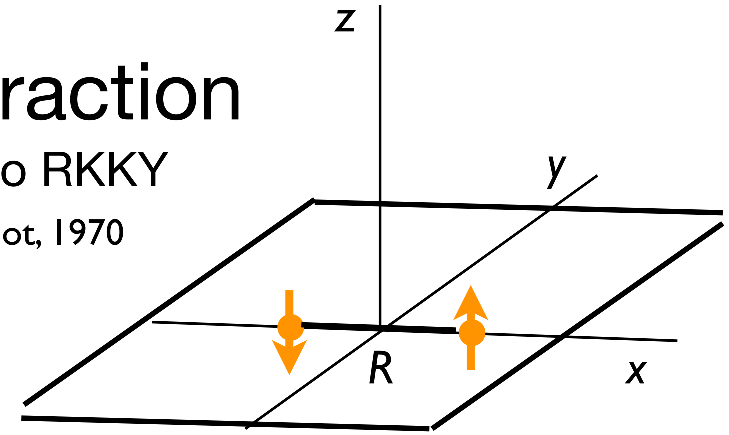
$$H_{\text{RKKY}}^{\text{SO}} = H_{\text{Heis.}} + H_{\text{Rashba}} + H_{\text{Dress.}} + H_{\text{interf.}}$$

$$\begin{aligned} H_{\text{Heis.}} &= F_0 \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 \\ H_{\text{Rashba}} &= \alpha F_1 \left(\boldsymbol{S}_1 \times \boldsymbol{S}_2\right)^y + \alpha^2 F_2 S_1^y S_2^y \\ H_{\text{Dress.}} &= \beta F_1 \left(\boldsymbol{S}_1 \times \boldsymbol{S}_2\right)^x + \beta^2 F_2 S_1^x S_2^x \\ H_{\text{interf.}} &= \alpha\beta F_2 \left(S_1^x S_2^y + S_1^y S_2^x\right). \end{aligned}$$

Spin-orbit effects on the RKKY interaction

from simple extension of standard perturbative approach to RKKY

Blackman and Elliot, 1970



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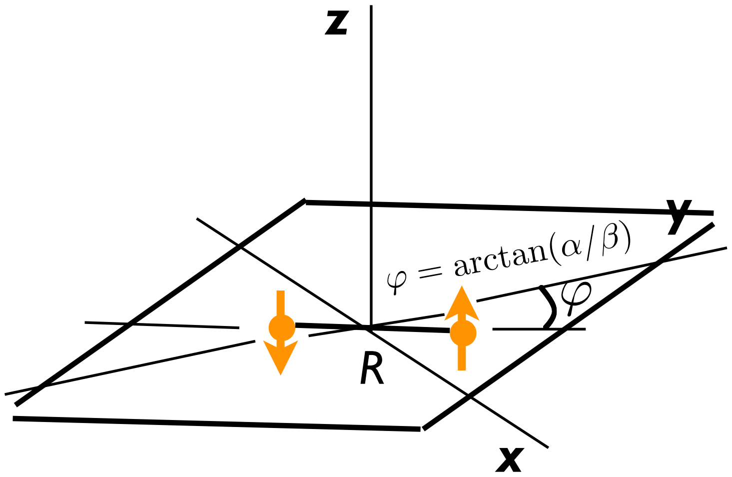
$$H^{\text{SO}}_{\text{RKKY}} = H_{\text{Heis.}} + H_{\text{Rashba}} + H_{\text{Dress.}} + H_{\text{interf.}}$$

spin-orbit effect

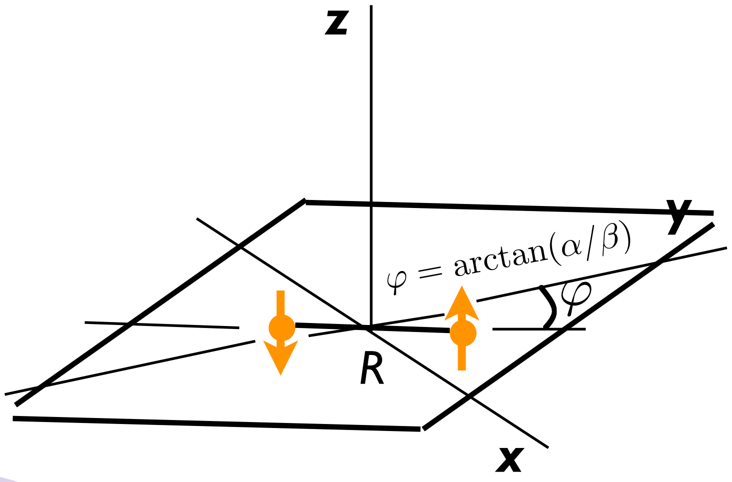
$$F_0(\alpha,0,R) \ = -\frac{J^2}{2\pi^2R^2}\frac{m}{\hbar^2}\sin 2R\sqrt{k_F^2+\frac{m^2\alpha^2}{\hbar^4}}$$

$$\begin{aligned}
H_{\text{Heis.}} &= F_0\boldsymbol{S}_1\cdot\boldsymbol{S}_2 \\
H_{\text{Rashba}} &= \alpha F_1\left(\boldsymbol{S}_1\times\boldsymbol{S}_2\right)^y + \alpha^2 F_2 S_1^y S_2^y \\
H_{\text{Dress.}} &= \beta F_1\left(\boldsymbol{S}_1\times\boldsymbol{S}_2\right)^x + \beta^2 F_2 S_1^x S_2^x \\
H_{\text{interf.}} &= \alpha\beta F_2\left(S_1^x S_2^y + S_1^y S_2^x\right).
\end{aligned}$$

More useful choice of coordinate system: rotate ***x***, ***y*** by $\pi/2 - \arctan(\alpha/\beta)$ around the ***z***-axis



$$H_{\rm RKKY}^{\rm SO} \;=\; K_{\rm H}\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + K_{\rm Ising} S_1^y S_2^y + K_{\rm DM}\left(\boldsymbol{S}_1 \times \boldsymbol{S}_2\right)^y$$



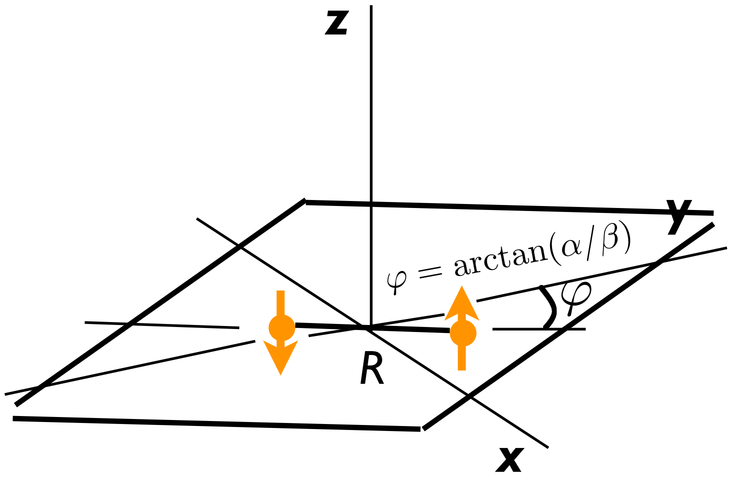
$$H^{\mathrm{SO}}_{\mathrm{RKKY}} = K_{\mathrm{H}} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + K_{\mathrm{Ising}} S_1^y S_2^y + K_{\mathrm{DM}} \left(\boldsymbol{S}_1 \times \boldsymbol{S}_2\right)^y$$

anisotropic and non-collinear interaction

Bad news for RKKY-control of two-qubit gating for spin-based quantum computing..?

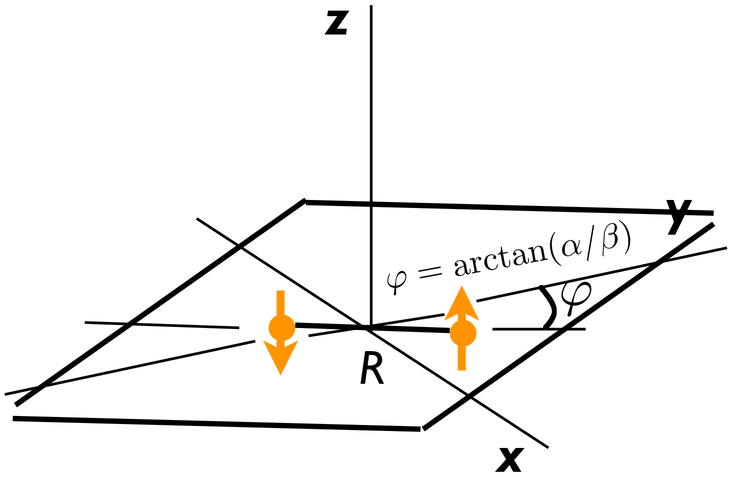
CNOT gate built from *isotropic spin exchange!*

D. Loss and D.P. DiVincenzo, PRA **57**, 120 (1998)



$$H^{\mathrm{SO}}_{\mathrm{RKKY}} = K_{\mathrm{H}} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + \cancel{K_{\mathrm{Ising}} S_1^y S_2^y} + \cancel{K_{\mathrm{DM}} (\boldsymbol{S}_1 \times \boldsymbol{S}_2)^y}$$

SU(2) symmetry recovered when $|\alpha|=|\beta|$!

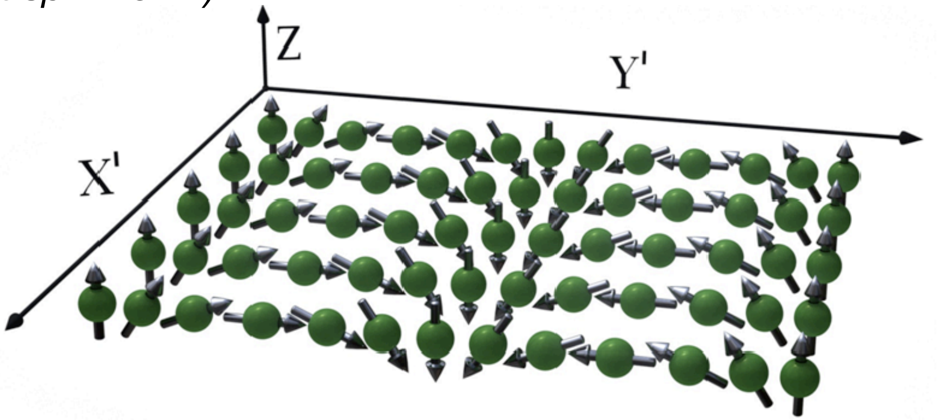


$$H_{\mathrm{RKKY}}^{\mathrm{SO}} = K_{\mathrm{H}} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + \xcancel{K_{\mathrm{Ising}} S_1^y} S_2^y + \xcancel{K_{\mathrm{DM}} (\boldsymbol{S}_1} \times \boldsymbol{S}_2)^y$$

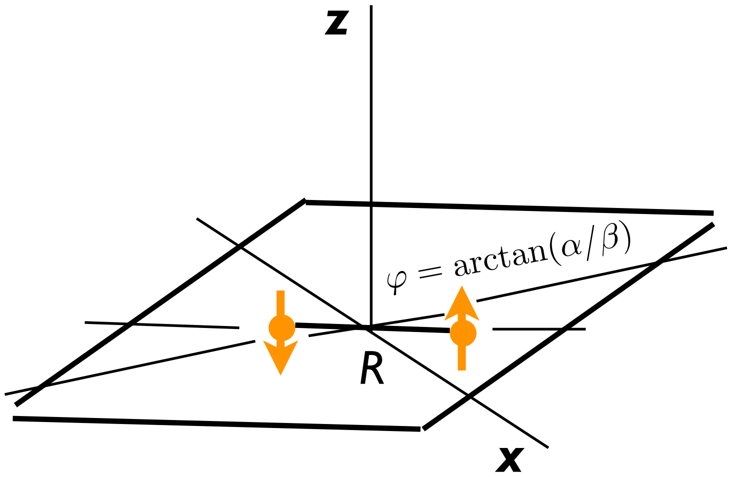
SU(2) symmetry recovered when $|\alpha| = |\beta|$!

Also predicted and observed in a 2DEG:
conservation of phase and amplitude of
a helical spin structure *("persistent spin helix")*

B. A. Bernevig *et al.*, PRL **97**, 236601 (2006)
J. D. Koralek *et al.*, Nature **458**, 610 (2009)



from J. D. Koralek *et al.*, Nature **458**, 610 (2009)



$$H_{\mathrm{RKKY}}^{\mathrm{SO}} \;=\; K_{\mathrm{H}} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + K_{\mathrm{Ising}} S_1^y S_2^y + K_{\mathrm{DM}} \left(\boldsymbol{S}_1 \times \boldsymbol{S}_2 \right)^y$$

$$K^y = K_{\mathrm{H}} + K_{\mathrm{Ising}}$$

$$e^{i\theta} K^{\perp} = K_{\mathrm{H}} + i K_{\mathrm{DM}}$$

$$\boldsymbol{S}_2' = e^{\mathrm{i}\theta S_2^y} \boldsymbol{S}_2 e^{-\mathrm{i}\theta S_2^y}$$

$$H_{\mathrm{RKKY}}^{\mathrm{SO}} = K^{\perp} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2' + (K^y - K^{\perp}) S_1^y S_2'^y .$$

effect of spin-orbit interactions: **twist** and **anisotropy**

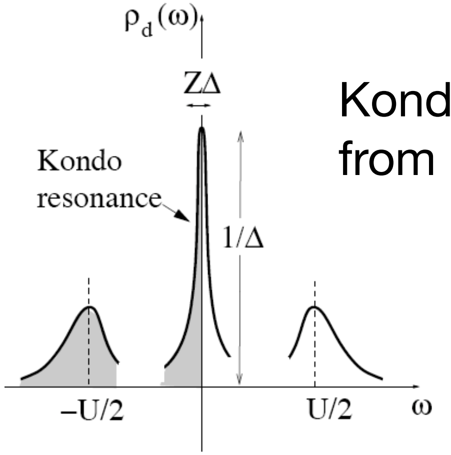
$K^y \neq K^{\perp}$ when Rashba and Dresselhaus are *both* present

By increasing the Kondo scale T_K , the twisted and anisotropic RKKY interaction gets competition from the Kondo effect...

...what happens at the TIKM critical point?

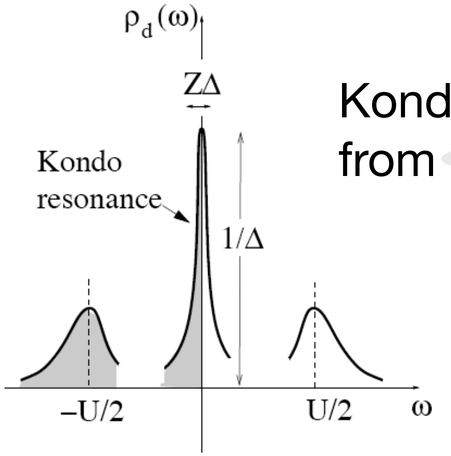
Effect from spin-orbit interactions on single-impurity Kondo effect?

Effect from spin-orbit interactions on single-impurity Kondo effect?



Kondo resonance at the Fermi level
from divergence in the electron self-energy

Effect from spin-orbit interactions on single-impurity Kondo effect?



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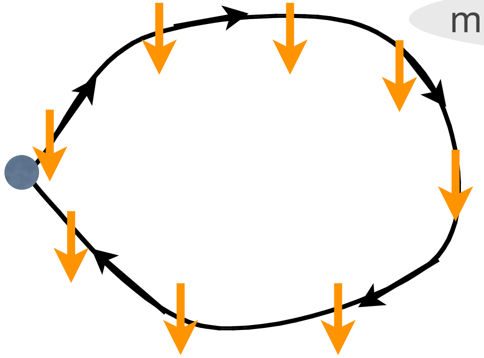
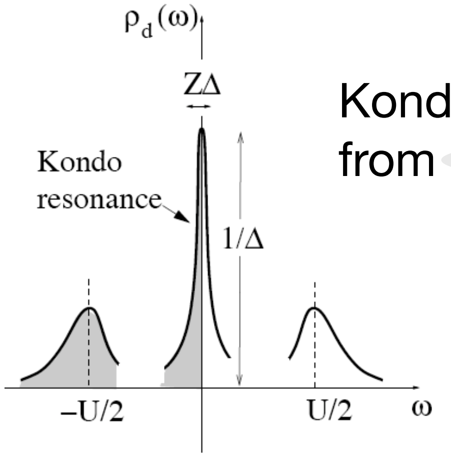


multiple electron-impurity scattering

Effect from spin-orbit interactions on single-impurity Kondo effect?



Kondo resonance at the Fermi level
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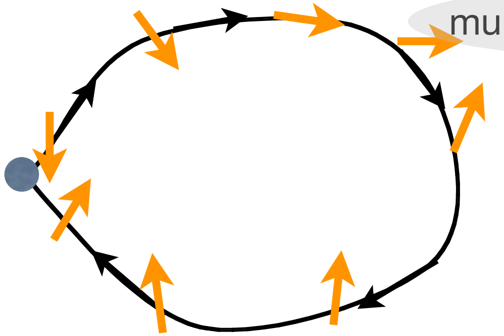
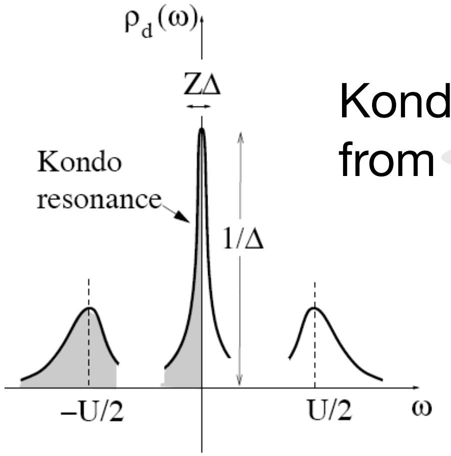


multiple electron-impurity scattering

non-conservation of electron spin in the presence of spin-orbit interactions

Effect from spin-orbit interactions on single-impurity Kondo effect?



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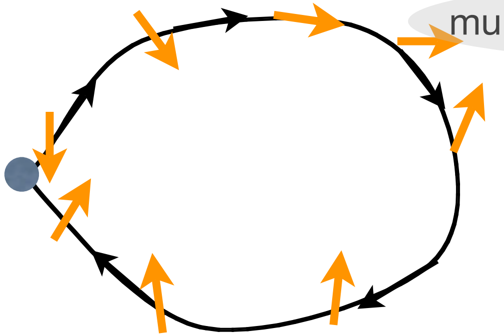


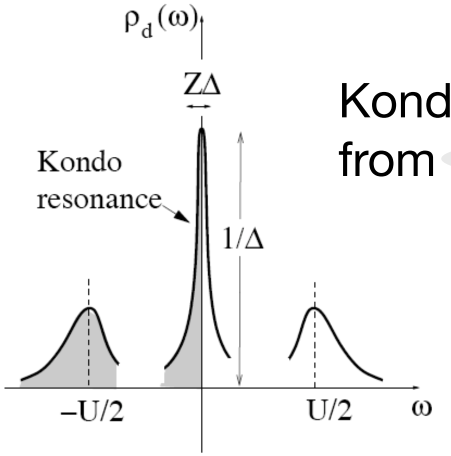
multiple electron-impurity scattering

non-conservation of electron spin in the presence of spin-orbit interactions!

But,... experiments show that the Kondo effect is insensitive to spin-orbit scattering...

G. Bergmann, PRL 57, 1460 (1986)

Effect from spin-orbit interactions on single-impurity Kondo effect?



Kondo resonance at the Fermi level
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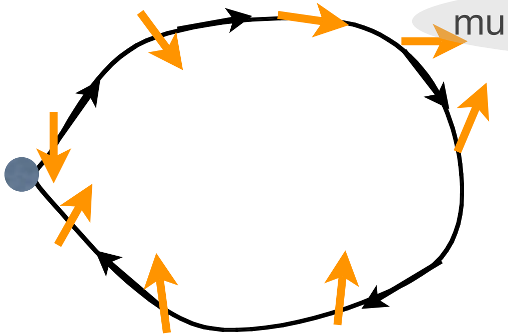
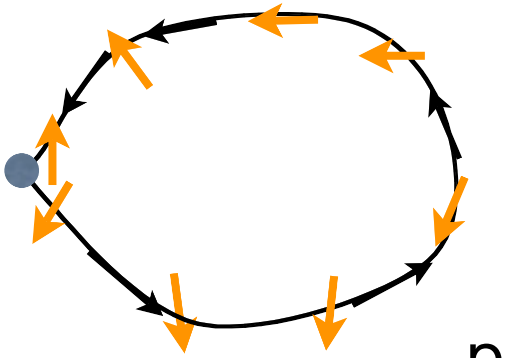
+



multiple electron-impurity scattering

non-conservation of electron spin in the presence of spin-orbit interactions!

-



= 0

But,... experiments show that the Kondo effect is insensitive to spin-orbit scattering...

G. Bergmann, PRL **57**, 1460 (1986)

...protected by time-reversal invariance

Y. Meir and N.S. Wingreen, PRB **50**, 4947 (1994)

Analysis of 2D single-impurity Kondo + Rashba model

J. Malecki, J. Stat. Phys. **129**, 741 (2007)

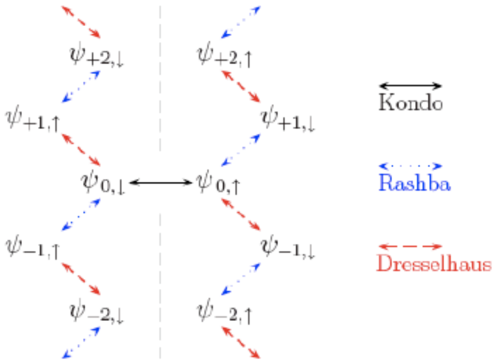
$$J \to J\sqrt{1 + m\alpha^2/2\epsilon_F}$$

Extension to Kondo + Dresselhaus *(only)* is straightforward...

$$J \to J\sqrt{1 + m\beta^2/2\epsilon_F}$$

Rashba + Dresselhaus couple an infinite number of orbital angular modes to the impurity

work in progress...



Summarizing spin-orbit effects:

rescaled coupling

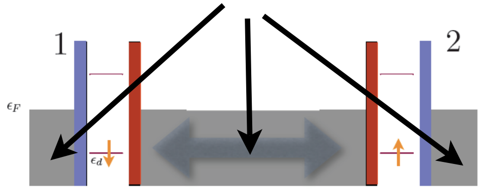
Kondo exchange: $H^{\mathrm{SO}}_{\text{el-imp}} = J\boldsymbol{S}\cdot\boldsymbol{\sigma}$

RKKY: $H^{\mathrm{SO}}_{\mathrm{RKKY}} = K^{\perp}\boldsymbol{S}_1\cdot\boldsymbol{S}'_2 + (K^y - K^{\perp})S^y_1 S'^y_2 .$

twist

anisotropy

TIKM with spin-orbit interactions



$$H^{\mathrm{SO}}_{\mathrm{TIKM}} = H_{\mathrm{kin}} + J_1\boldsymbol{S}_1\cdot\boldsymbol{\sigma}_1 + J_2\boldsymbol{S}'_2\cdot\boldsymbol{\sigma}_2 + K^{\perp}\boldsymbol{S}_1\cdot\boldsymbol{S}'_2 + (K^y - K^{\perp})S^y_1 S'^y_2$$

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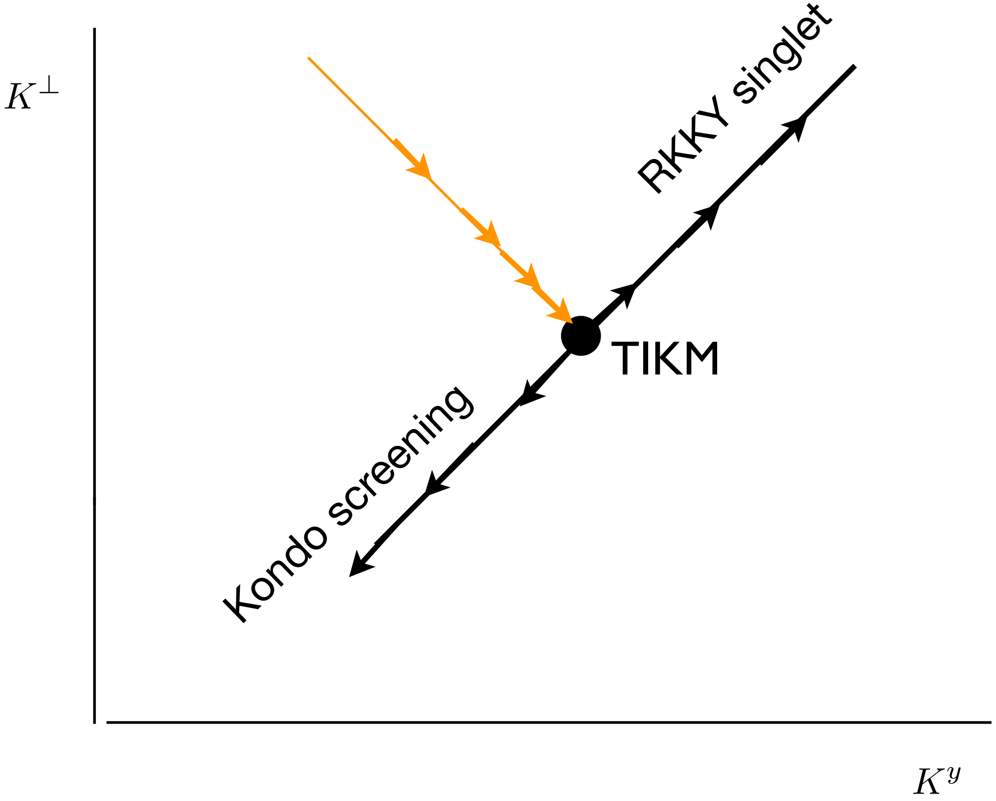
Critical behavior?

With fine-tuned $K^y, K^{\perp}$ criticality is still controlled by the isotropic TIKM fixed point. Else one flows towards one of the stable fixed points.



$$H_{\mathrm{TIKM}}^{\mathrm{SO}} = H_{\mathrm{kin}} + J_1 \boldsymbol{S}_1 \cdot \boldsymbol{\sigma}_1 + J_2 \boldsymbol{S}_2' \cdot \boldsymbol{\sigma}_2 + K^{\perp} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2' + (K^y - K^{\perp}) S_1^y S_2'^y$$

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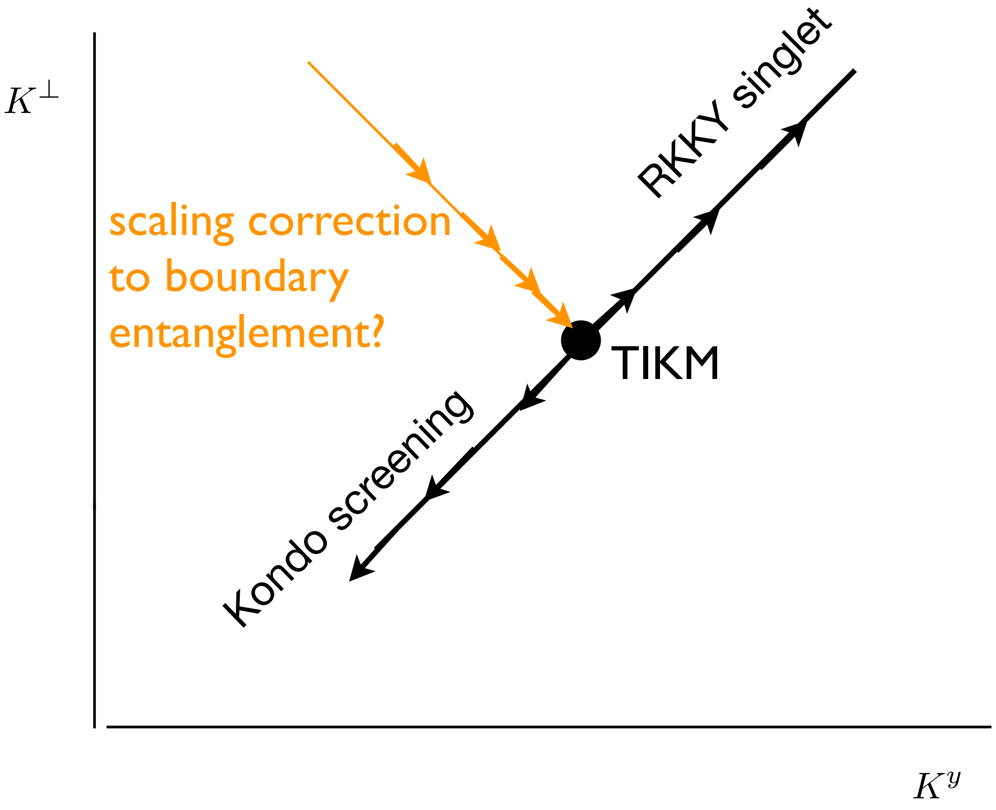

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Global RG flow?

$$H^{\rm SO}_{\rm TIKM} = H_{\rm kin} + J_1 \boldsymbol{S}_1 \cdot \boldsymbol{\sigma}_1 + J_2 \boldsymbol{S}'_2 \cdot \boldsymbol{\sigma}_2 + K^{\perp} \boldsymbol{S}_1 \cdot \boldsymbol{S}'_2 + (K^y - K^{\perp}) S^y_1 S'^y_2$$

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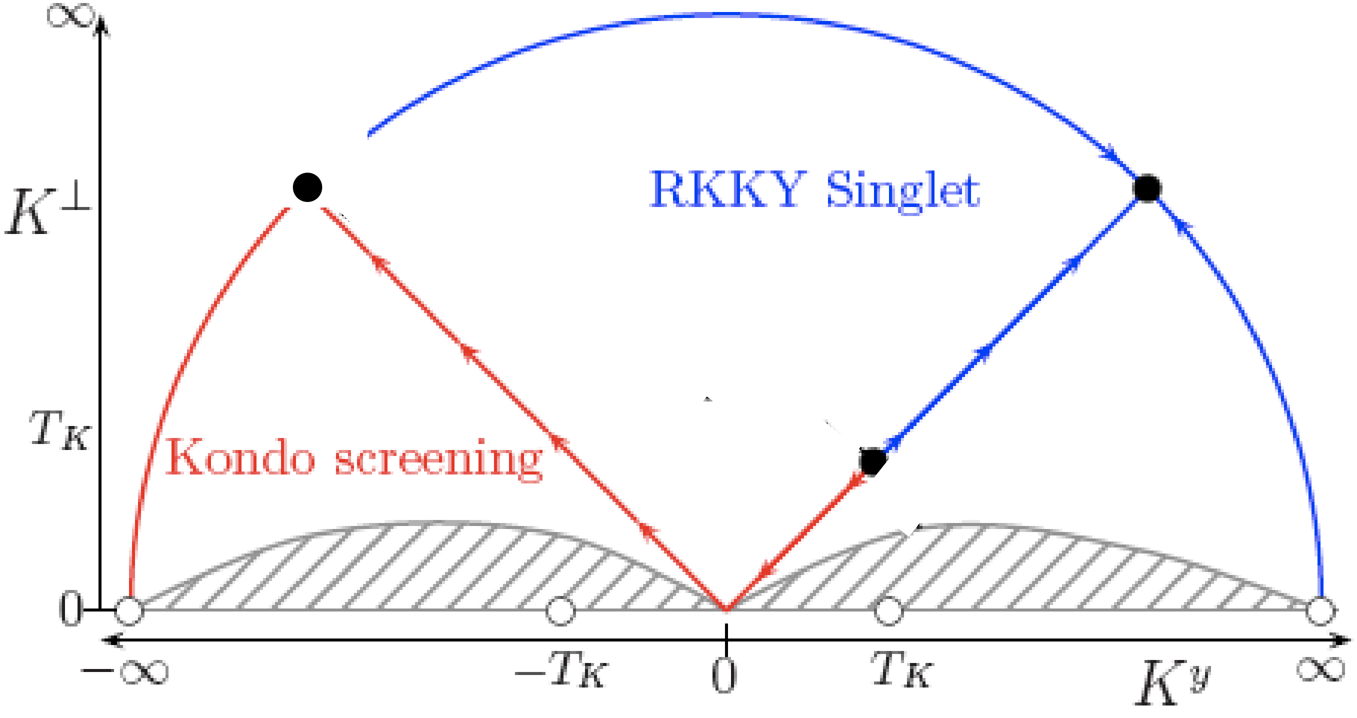

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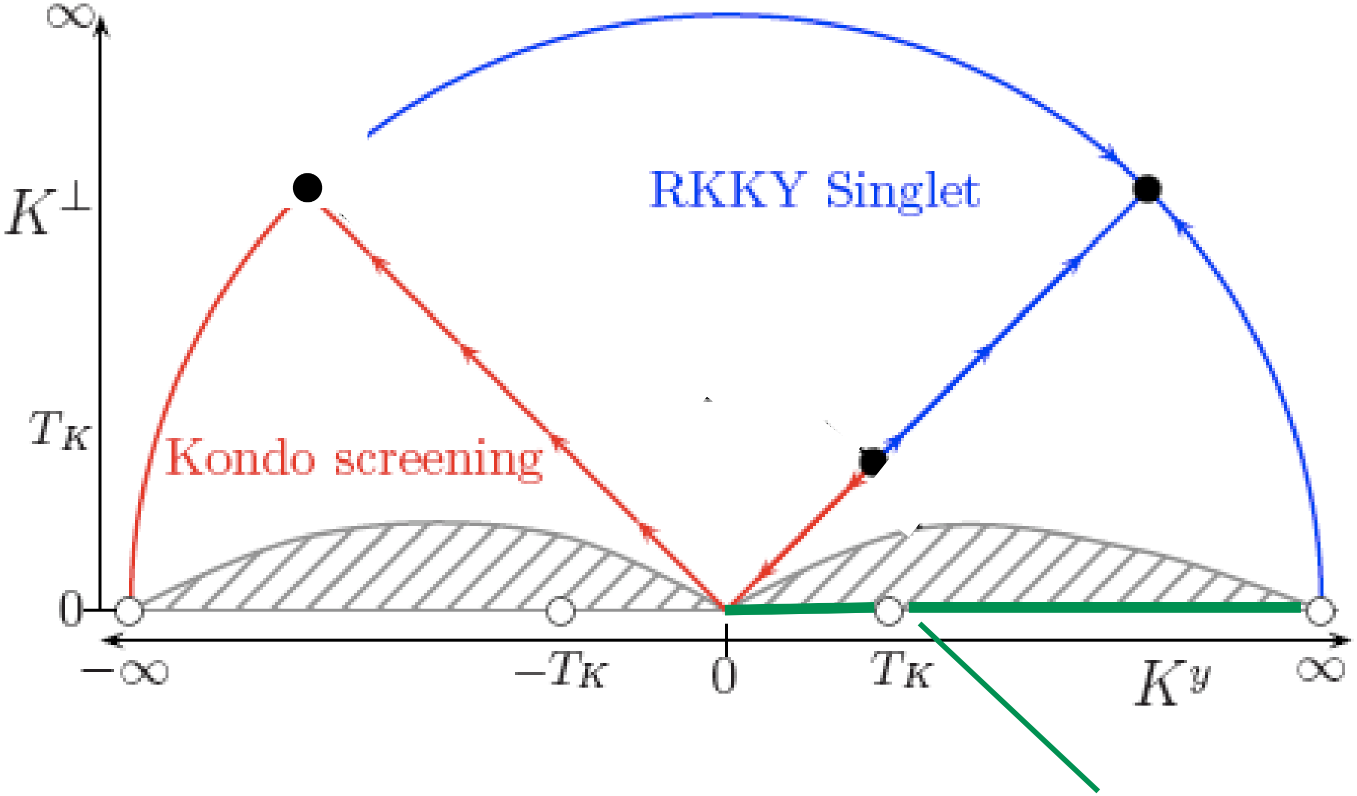

Ising-coupled impurities, quantum phase transition at $K^y \approx T_K$ with different behavior.

N. Andrei *et al.*, PRB **60**, 5125(R) (1999)

$$H_{\mathrm{TIKM}}^{\mathrm{SO}} = H_{\mathrm{kin}} + J_1 \boldsymbol{S}_1 \cdot \boldsymbol{\sigma}_1 + J_2 \boldsymbol{S}_2' \cdot \boldsymbol{\sigma}_2 + K^{\perp} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2' + (K^y - K^{\perp}) S_1^y S_2'^y$$

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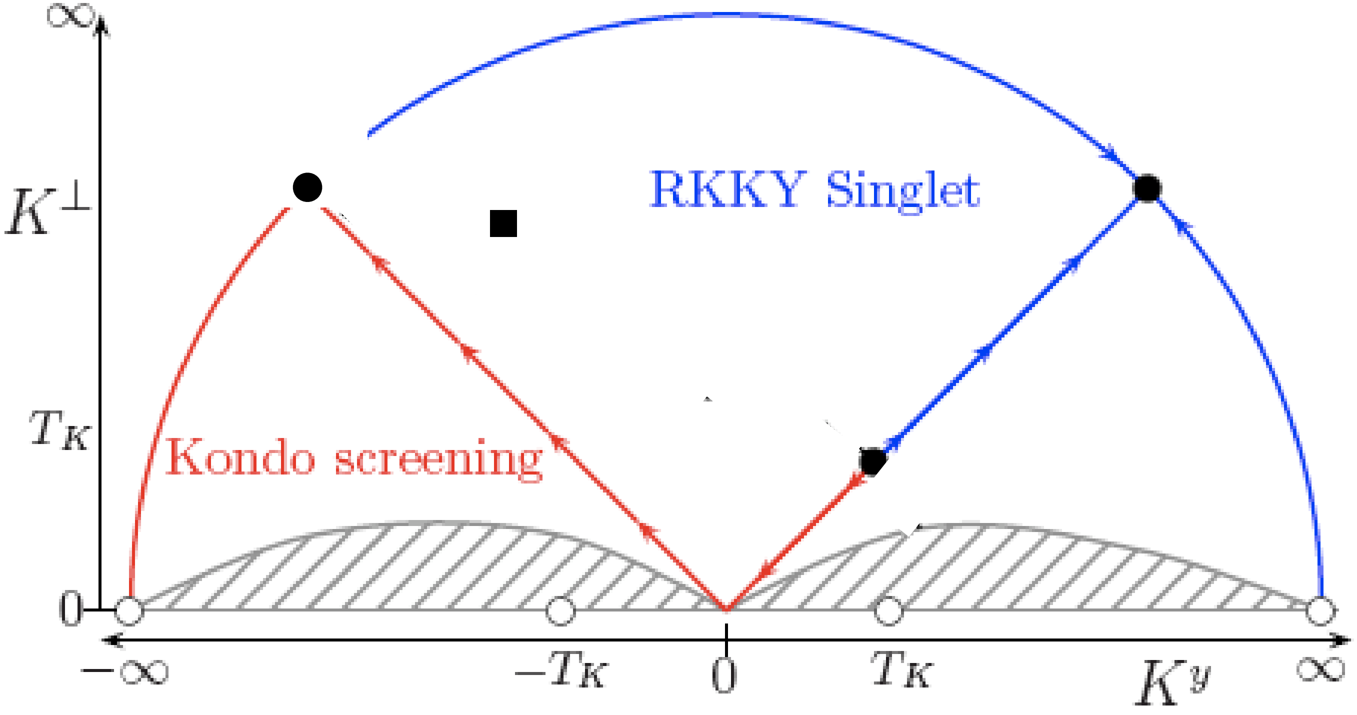

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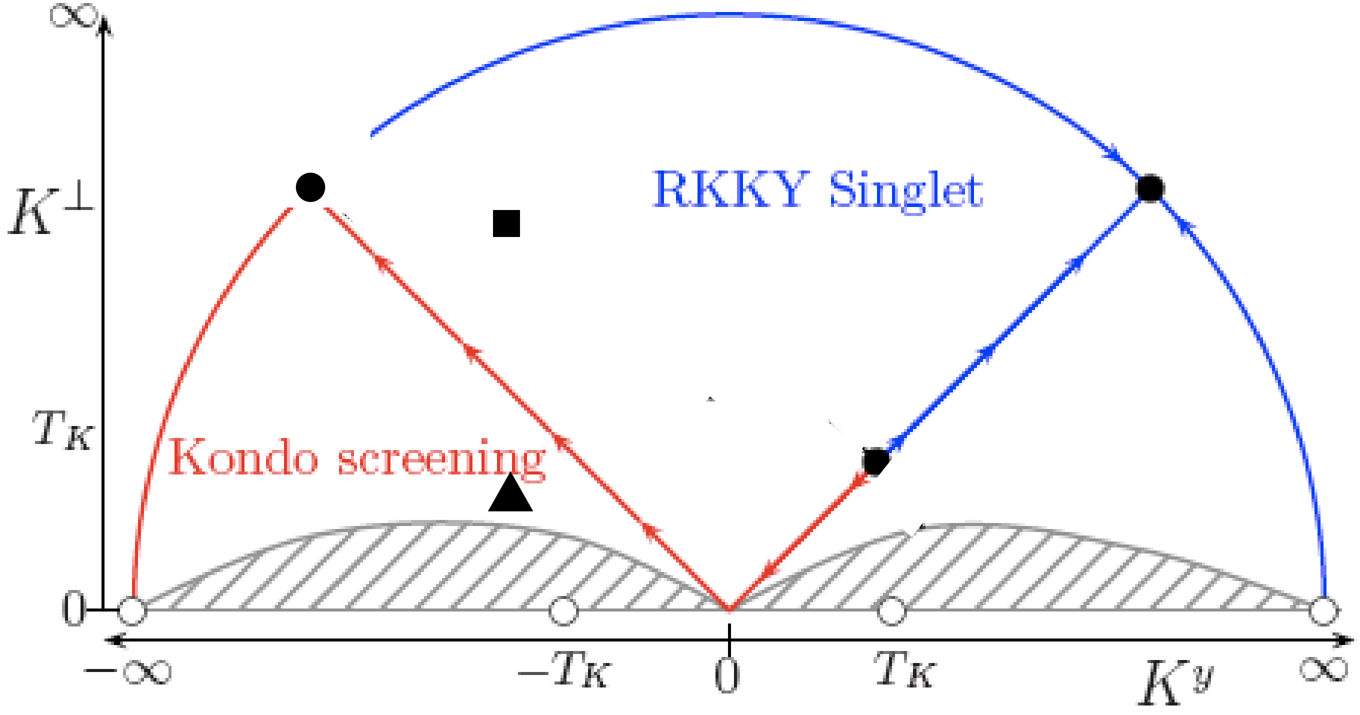

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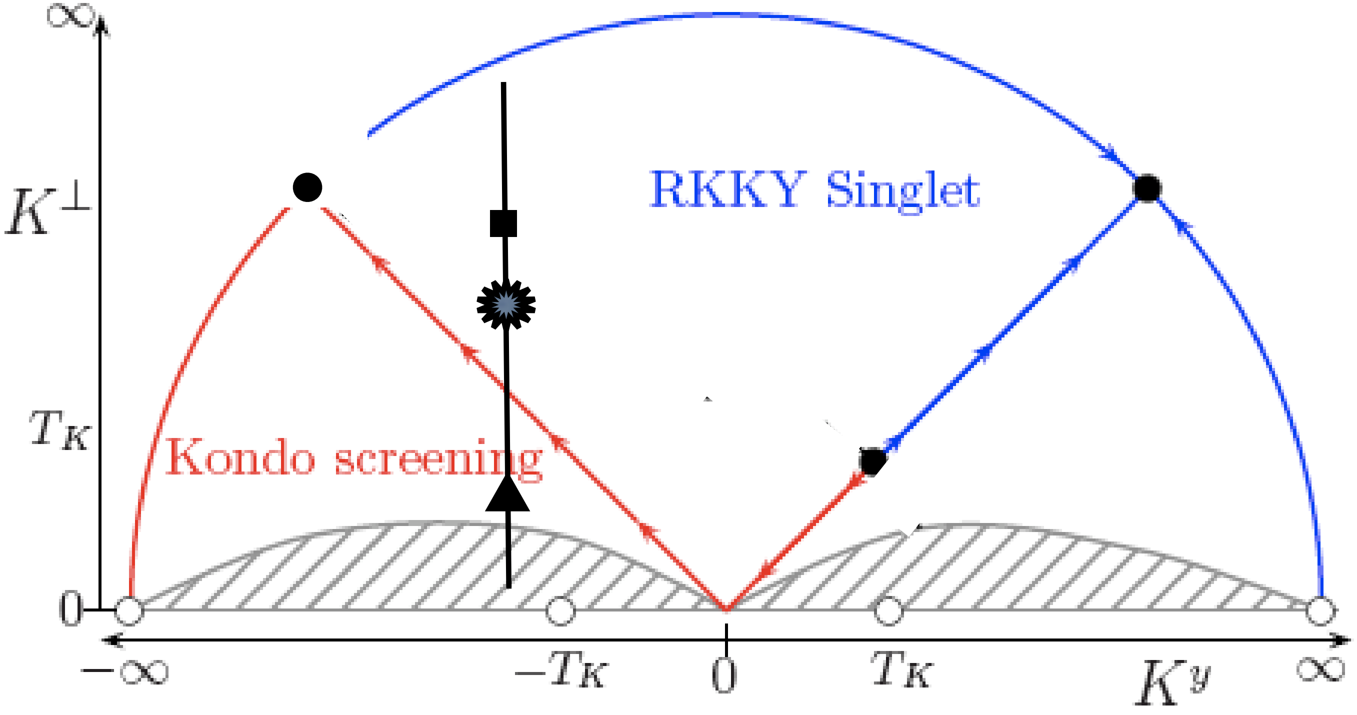

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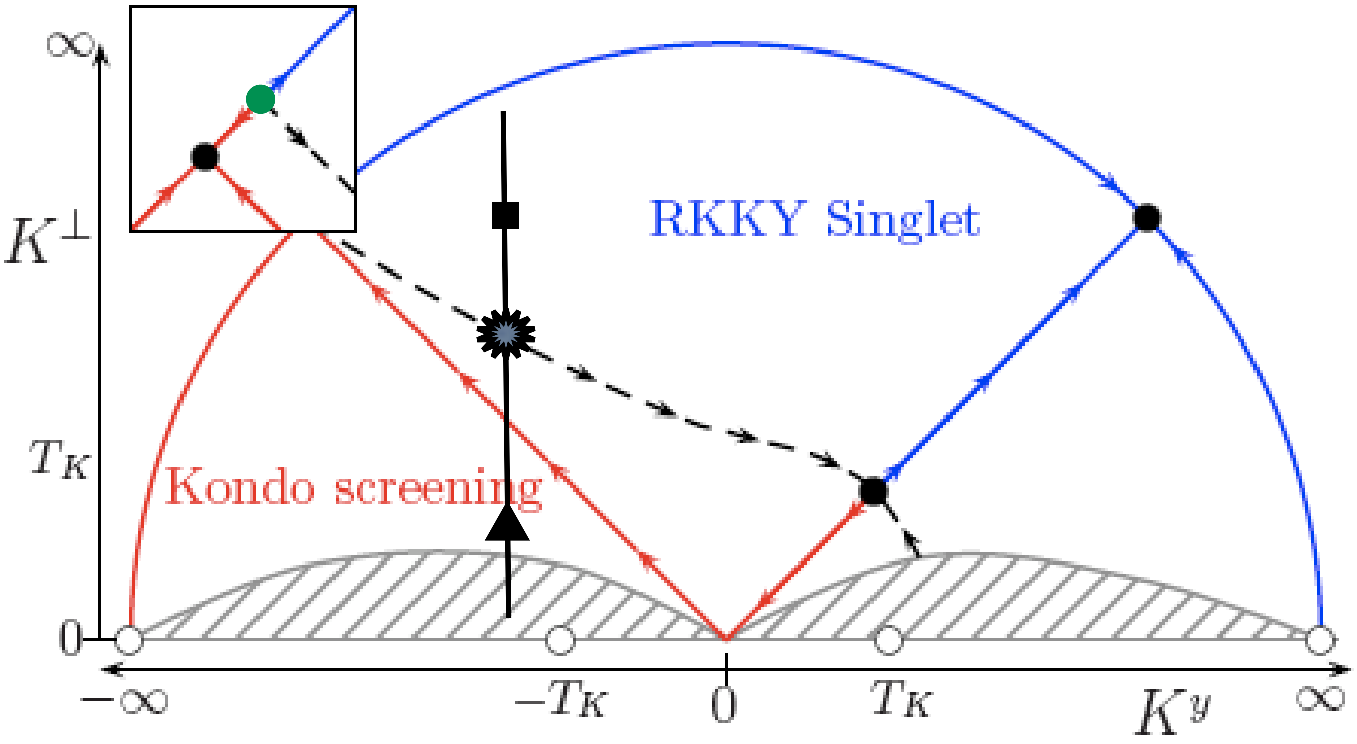

Summary

Two-impurity Kondo model, recent results & work in progress...

- *same* leading scaling correction to the critical boundary entanglement as for single-impurity Kondo model

- scaling corrections to the boundary entanglement with broken symmetries (parity, charge, spin anisotropy,...)? *work in progress*

- spin-orbit coupled electrons → RKKY interaction gets "twisted" with an Ising anisotropy

- SU(2) invariance recovered when |Rashba| = |Dresselhaus|

  good for RKKY-controlled two-qubit gates

- "fine-tuning" of $K^y, K^\perp \longrightarrow$ quantum critical behavior controlled by the isotropic TIKM fixed point

- possible new unstable fixed point for $(K^y, K^\perp) \rightarrow (K_0^y, \infty)$

$$H_{\mathrm{TIKM}}^{\mathrm{SO}} = H_{\mathrm{kin}} + J_1 \boldsymbol{S}_1 \cdot \boldsymbol{\sigma}_1 + J_2 \boldsymbol{S}_2' \cdot \boldsymbol{\sigma}_2 + K^{\perp} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2'$$

Critical behavior?

$$K^{\perp} = K^{y}, \ \theta \text{ arbitrary}$$

rotate also the spins of the electrons which couple to $\boldsymbol{S}_2'$

$$\psi_2 \rightarrow \psi_2' = e^{-i\theta\tau^{y}/2}\psi_2$$

↓

$$H_{\mathrm{TIKM}} = H_{\mathrm{kin}} + J_1 \boldsymbol{S}_1 \cdot \boldsymbol{\sigma}_1 + J_2 \boldsymbol{S'}_2 \cdot \boldsymbol{\sigma}_2' + K^{\perp} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2'$$

The model represented in a twisted spin basis = the ordinary TIKM

→ **same critical behavior for all** θ

$$H^{\mathrm{SO}}_{\mathrm{TIKM}} = H_{\mathrm{kin}} + J_1 \boldsymbol{S}_1 \cdot \boldsymbol{\sigma}_1 + J_2 \boldsymbol{S}_2 \cdot \boldsymbol{\sigma}_2 + K^{\perp} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + (K^y - K^{\perp}) S_1^y S_2^y$$

Critical behavior?

$K^{\perp} \neq K^y, \ \theta = 0$

$$SU(2) \longrightarrow U(1)$$

Kondo exchange anisotropies do not produce any RG-relevant or new correction-to-scaling operator

I. Affleck *et al.*, PRB **52**, 9528 (1995)

↓ ?

same critical behavior for all $K^{\perp} \neq K^y$

$$H_{\mathrm{TIKM}}^{\mathrm{SO}} = H_{\mathrm{kin}} + J_1 \boldsymbol{S}_1 \cdot \boldsymbol{\sigma}_1 + J_2 \boldsymbol{S}_2 \cdot \boldsymbol{\sigma}_2 + K^{\perp} \boldsymbol{S}_1 \cdot \boldsymbol{S}_2 + (K^y - K^{\perp}) S_1^y S_2^y$$

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same critical behavior for ~~all~~ $K^{\perp} \neq K^y$

$(K^{\perp}, K^y) \rightarrow (K^{\perp} + \delta, K^y - \delta)$