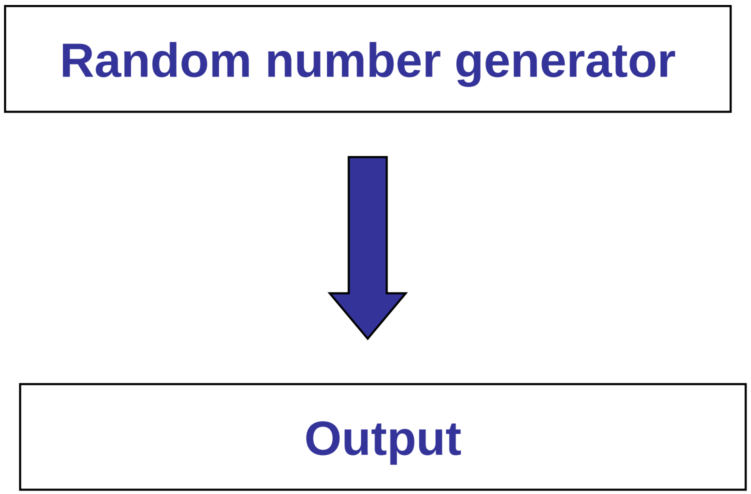
Random numbers

Content:

- Random number generators
- Uniform random numbers
- Non-uniform random numbers:
  1. Transformation method
  2. Rejection method

Random numbers



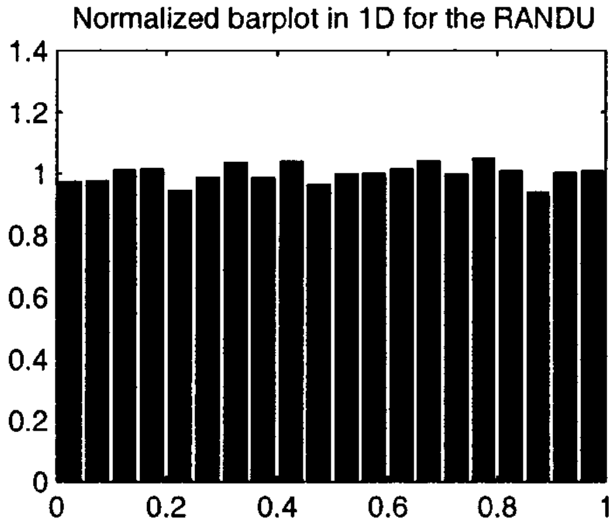
A perfect random number generator should produce numbers (”pseudorandom numbers”) that appear to be perfectly random, unless you happen to know both the algorithm and its internal state.

Is this a sequence of random numbers ?

0.242797434
0.696527659
0.540087878
0.256856381
0.985251605
0.123984143
0.801557302
0.773549854
0.052133180
0.202336564
0.670698583
0.431537956
0.858359396
0.446797788
0.330280125
0.017940610
0.527835488
0.330602377
0.434296846
0.227186769
0.327907294
0.137846455
0.785346388
0.317193448
0.070166930
0.295625895
0.584298729
0.309196233
0.660883903
0.475833952
0.341453462
0.808556079
0.402291059
0.305753976
0.807047486
0.047531273
0.858093679
0.980155766
0.478444427
0.215273425
0.100494504
0.011168224
0.704353213
0.064954675

Tests of random numbers

Do the random numbers have the correct distribution?

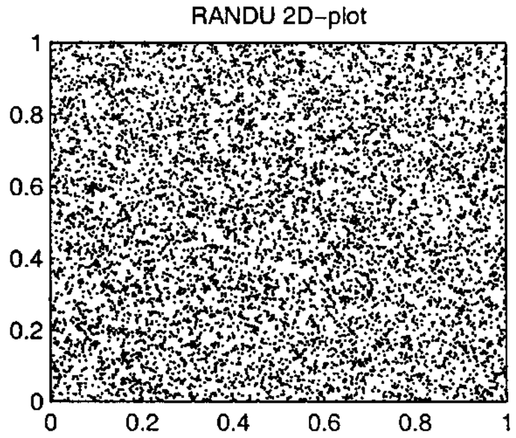


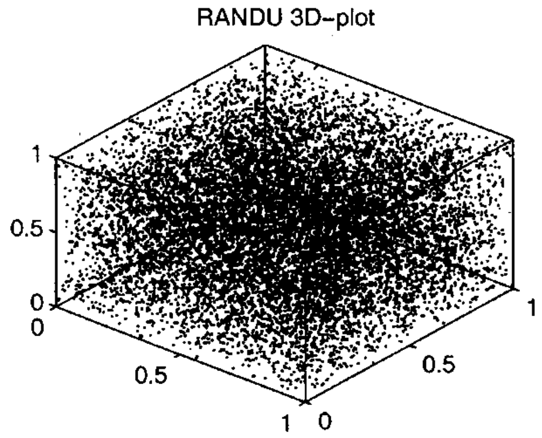
The infamous generator **RANDU** by IBM

”We can guarantee that each number is random individually, but we don’t guarantee that more than one of them is random.”

Tests of random numbers

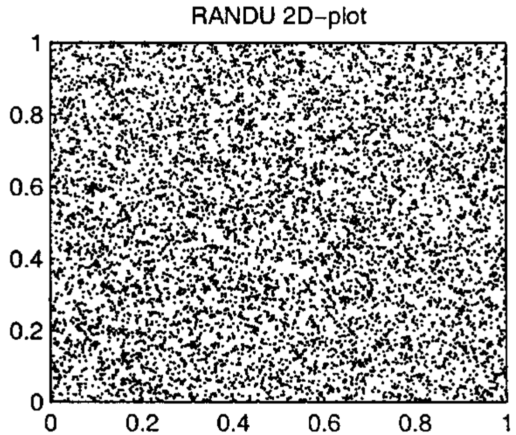
Are subsequent random numbers uncorrelated?

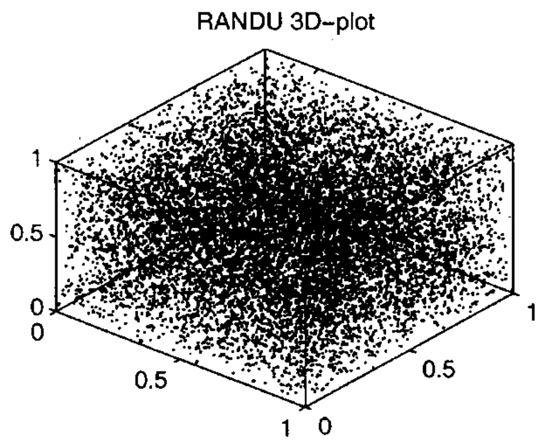


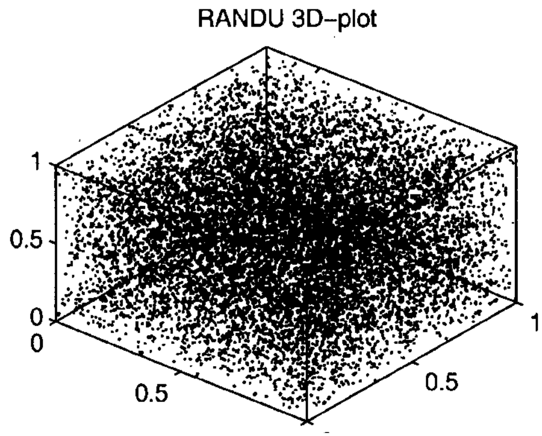


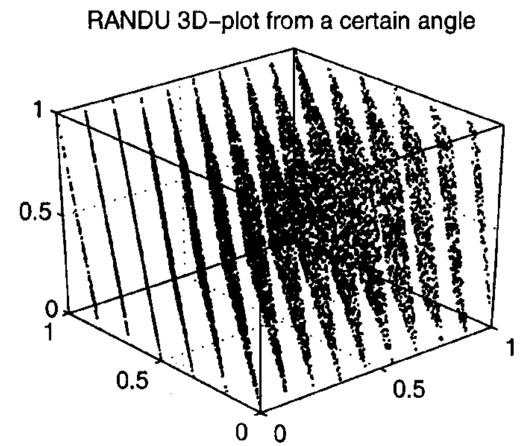
Tests of random numbers

Are subsequent random numbers uncorrelated?









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Non-uniform random numbers – Transformation method

ξ – uniform random number, η – non-uniform random number

Discrete case: $i = 1, \ldots, N$

Probability function

$$p_i \equiv p(i) \; ; \; \sum_{i=1}^{N} p_i = 1$$

Cumulative probability function

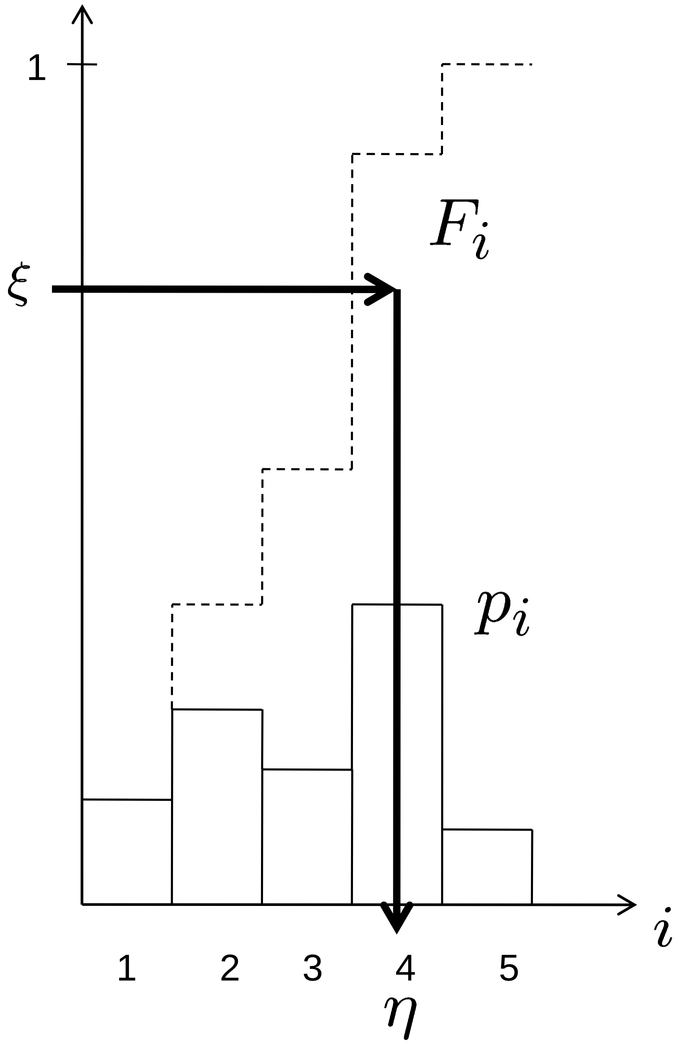
$$F_i \equiv \sum_{j=1}^{i} p_j$$

Transformation method

$$F_{i-1} \leq \xi \leq F_i$$

and

$$\eta = i$$



Non-uniform random numbers – Transformation method

ξ – uniform random number, η – non-uniform random number

Continuous case: x ; $a \leq x \leq b$

Probability function

$$p(x) \;;\; \int_a^b p(x)\, dx = 1$$

Cumulative probability function

$$F(x) \equiv \int_a^x p(x')dx'$$

Transformation method

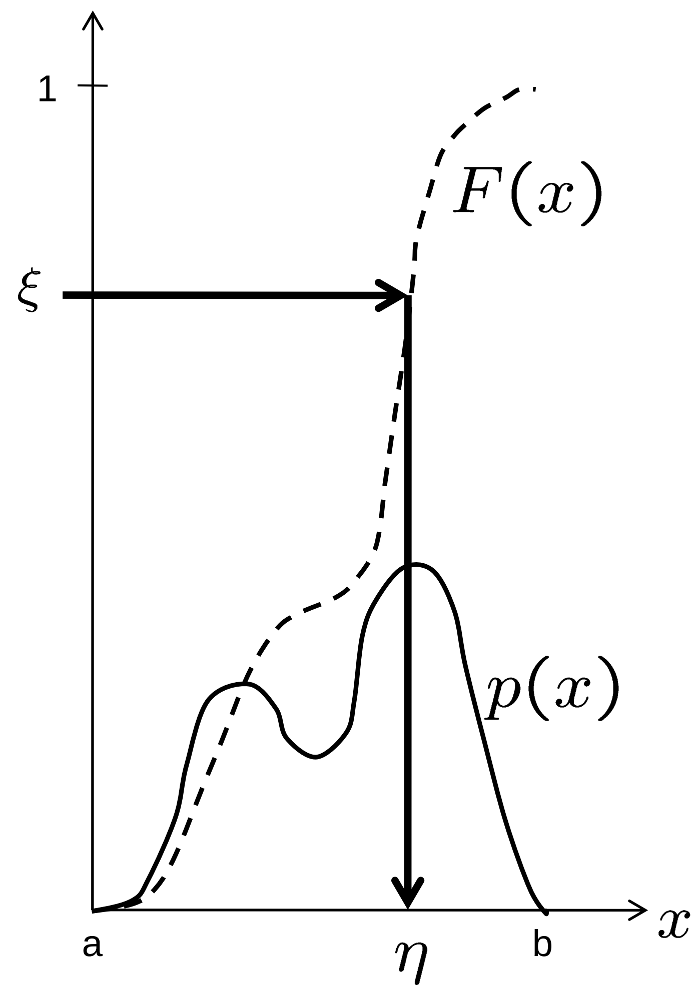
$$F(x - dx) \leq \xi < F(x)$$

i.e.

$$F(x) = \xi$$

and

$$\eta = F^{-1}(\xi)$$



Transformation method - examples

Uniform distribution

$$p(x) = \begin{cases} 1/(b-a) & \text{if} \quad a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

which implies that

$$\eta = a + (b-a)\xi$$

Exponential distribution

$$p(x) = \begin{cases} \lambda^{-1} \exp(-x/\lambda) & \text{if} \quad 0 \le x < \infty \\ 0 & \text{otherwise} \end{cases}$$

which implies that

$$\eta = -\lambda \ln(1-\xi)$$

or

$$\eta = -\lambda \ln(\xi)$$

Gaussian distribution

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-(x-\mu)^2/2\sigma^2\right]$$

$$-\infty < x < \infty$$

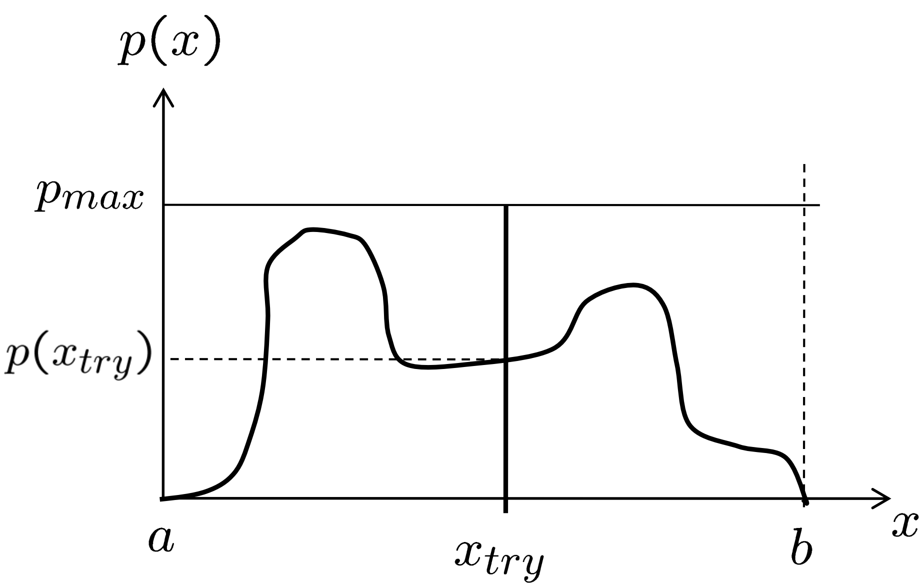
which implies that (Box-Müller method)

$$\begin{cases} \eta_1 = \mu + \sigma\sqrt{-2\ln \xi_1} \ \cos(2\pi\xi_2) \\ \eta_2 = \mu + \sigma\sqrt{-2\ln \xi_1} \ \sin(2\pi\xi_2) \end{cases}$$

Non-uniform random numbers – Rejection method

ξ – uniform random number, η – non-uniform random number

The *rejection method* is more general than the transformation method and can be used for both discrete and continuous random numbers.



Consider a probability distribution $p(x)$ on the interval [a,b]. Choose a value p_{max} such that

$$p_{max} \geq p(x), \quad a < x < b$$

1. Generate a uniform random number ξ_1 and determine a trial value

$$x_{try} = a + (b - a)\xi_1$$

2. Generate another random number ξ_2 and accept the trial value

$$\eta = x_{try} \quad \text{only if} \quad \xi_2 \leq \frac{p(x_{try})}{p_{max}}$$

Non-uniform random numbers – Rejection method

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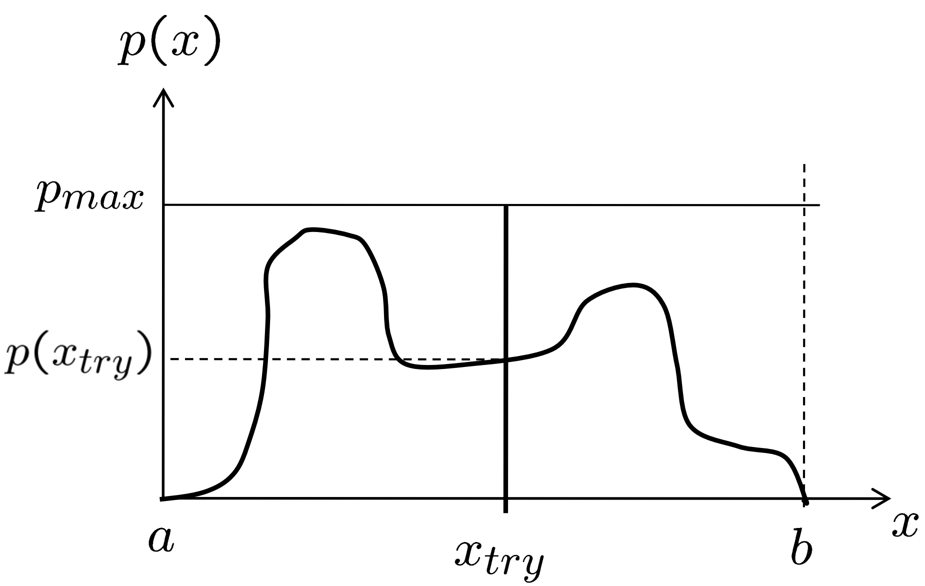
The *rejection method* is more general than the transformation method and can be used for both discrete and continuous random numbers.



The method can be made more efficient and also applicable on infinite intervals by instead of p_{max} introducing a comparison function

$$f(x) \geq p(x) \quad \forall x$$

and with $\int f(x)dx$ finite.