Strings, Membranes and Supergravity

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Abstract

We will start by reviewing some aspects of bosonic string theory and then move on to the fermionic supersymmetric string, i.e. the so called superstring. The focus will here be on the verification of the supersymmetry of the action. In chapter 3 we will consider eleven-dimensional supergravity and in particular solve the Bianchi identities. We will in chapter 4 finally examine the $p = 2$ brane in eleven-dimensional supergravity and the emphasis will here be on the $\varkappa$-symmetry of the action.

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Contents

1

Introduction

We are going to start by quickly reviewing some aspects of classical bosonic string theory in this chapter. For a more thorough treatment see e.g. [1] or [2]. The advantage of [1] is that it is considered as the standard reference and the advantage of [2] is that it uses the more modern framework of conformal field theory [3][4][5][6]. In chapter 2 we are going to look at fermionic strings and focus on the supersymmetry of the action. Chapter 3 deals with eleven-dimensional supergravity and our task there is to solve the Bianchi identities. In chapter 4 we are finally going to consider the $p = 2$ brane in eleven-dimensional supergravity and explicitly verify the $\varkappa$-symmetry of the action.

1.1 Bosonic String Theory

From general relativity [7] we know that the action for a point-particle is given by the length of the particle's path, its worldline, through curved spacetime. In analogy to this we take the action of the bosonic string to be given by the area which the string traces out as it propagates through spacetime. This area is called the worldsheet of the string. The action obtained in this way, the Nambu-Goto action, contains a squareroot, but it can be rewritten in an equivalent form without the root. We thus start from the Brink-di Vecchia-Howe action, also known as the Polyakov action [8]

$$A_M = -\frac{1}{4\pi\alpha'}\int_{t_1}^{t_2} dt \int_0^{\pi} d\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu (t,\sigma)\, \partial_\beta X^\nu (t,\sigma)\, \eta_{\mu\nu}$$

where α' is the Regge-slope parameter, $g^{\alpha\beta}$ is the worldsheet metric and $\eta_{\mu\nu}$ is the spacetime metric, which is assumed to be minkowskian since we consider an expansion around a flat spacetime (this explains the notation A_M). In this and the next chapter $\alpha, \beta, \ldots$ are worldsheet indices while $\mu, \nu, \ldots$ are spacetime indices. The scalar fields X^μ gives the mapping from the worldsheet into spacetime. The action has two important symmetries (well, all symmetries are important), reparametrization invariance given by

$$\sigma'^\alpha = \sigma'^\alpha\left(\sigma^\beta\right)$$

and local Weyl invariance

$$\begin{aligned} g'^{\alpha\beta} &= e^{\Lambda(\sigma)} g^{\alpha\beta} \\ X'^\mu &= X^\mu \end{aligned}$$

From the action we can get the equations of motion and by mode-expanding the solutions we can apply the procedure of canonical quantization to obtain a quantized theory. We can now use the symmetries above to fix the gauge and we choose conformal gauge, in which the action is conformally invariant after the gauge-fixing procedure. This allows us to use the framework of conformal field theory, which is exact, regarded as a two-dimensional field theory. The stress tensor, which generate conformal transformations, is now analytic and we can expand it as

$$T(z) = \sum_m L_m z^{-m-2}$$

where the coefficients satisfy the algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{D}{12} m\left(m^2-1\right)\delta_{m+n,0} \tag{1.1}$$

where D is the dimension of spacetime. This algebra is called the Virasoro algebra and since it is infinite dimensional our string theory will be severely constrained by the requirement of conformal symmetry. The second term in the right hand side of (1.1) is an anomaly term and in order to have conformal symmetry we must find a way to get this term to vanish, which we actually will below.

A more modern approach to the quantization of the string is to use the path integral. We then start with

$$W_M = \int \mathcal{D}X \mathcal{D}g_{\alpha\beta} e^{\frac{i}{\hbar} A_M\left[X, g_{\alpha\beta}\right]}$$

but we must now take care of the symmetries mentioned above since these cause the integral to diverge badly. In order to do this we can use the

Faddeev-Popov method from quantum field theory [9] and introduce ghost fields. We can again mode-expand these fields and quantize them, but we will get some problems with our vacuum. It turns out that the ordinary vacuum has zero norm and we must define several new vacua. This can however be done consistently and poses thus no big problem. After this procedure we see that we get additional terms from the ghosts in the Virasoro algebra and that the conformal anomaly vanishes for $D = 26$. This is the reason why one says that bosonic strings must live in 26 dimensions.

It is not a simple matter to define what should be considered as a physical state in string theory. One good way is to use another, more complex, symmetry of the action namely the BRST-symmetry, which involves an anticommuting parameter. For a good introductory treatment of BRST-quantization see [10]. This symmetry gives as usual rise to a conserved current, J_Q, and a conserved charge, Q. It turns out that

$$Q^2 = 0$$

just when we have $D = 26$. We then define a physical state by requiring that

$$Q\left|\Psi\right\rangle = 0$$

where

$$\left|\Psi\right\rangle \neq Q\left|\chi\right\rangle$$

, i.e. the state must be invariant under the transformation generated by the BRST charge and we exclude the trivially BRST invariant states. If $\left|\Psi\right\rangle = Q\left|\chi\right\rangle$ then the two states $\left|\Psi\right\rangle$ and $\left|\chi\right\rangle$ form a BRST doublet and, as follows from the hermiticity and nilpotency of Q, the state $\left|\Psi\right\rangle$ has zero norm. In order to only have states with non-vanishing norm we therefore exclude this type of states. We are now going to turn to the fermionic string and continue with a more explicit treatment.

2

The Fermionic String

In order to be able to describe fermionic particles we must include some anticommuting entity in the action and we will use anticommuting spinors. Since the spinors have more degrees of freedom than the old bosonic action we must require the total action to have some symmetry in order to reduce the fermionic degrees of freedom to equal the bosonic ones. This symmetry will of course be local supersymmetry and below we will explicitly verify this symmetry. After a thorough analysis [2] one arrives at the action

$$\begin{aligned} S &= -\frac{1}{8\pi}\int d^2\sigma\sqrt{g}\,\Big[g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu + 2i\bar{\psi}^\mu\rho^\alpha\partial_\alpha\psi_\mu \\ &\quad -i\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi^\mu\left(\partial_\beta X_\mu - \frac{i}{4}\bar{\chi}_\beta\psi_\mu\right)\Big] \end{aligned}$$

We will use the notation from [2] and we have collected some conventions and useful formulas concerning spinor algebra in appendix A. As in chapter 1 $\alpha, \beta, \ldots$ are worldsheet indices while $\mu, \nu, \ldots$ are spacetime indices. For a briefer treatment than the one found in [2] see [11].

2.1 The Symmetries of the Action

The action is invariant under five local symmetries

(i) **supersymmetry**

$$\delta_\epsilon X^\mu = i\bar{\epsilon}\psi^\mu$$

$$\begin{aligned}
\delta_\epsilon \psi^\mu &= \frac{1}{2}\rho^\alpha\left(\partial_\alpha X^\mu - \frac{i}{2}\bar{\chi}_\alpha\psi^\mu\right)\epsilon \\
\delta_\epsilon e_\alpha{}^a &= \frac{i}{2}\bar{\epsilon}\rho^a\chi_\alpha \\
\delta_\epsilon \chi_\alpha &= 2D_\alpha\epsilon
\end{aligned}$$

where $\epsilon(\sigma,\tau)$ is a Majorana spinor which parametrizes supersymmetry transformations and D_α is a covariant derivative with torsion

$$\begin{aligned}
D_\alpha\epsilon &= \partial_\alpha\epsilon - \frac{1}{2}\omega_\alpha\bar{\rho}\epsilon \\
\omega_\alpha &= -\frac{1}{2}\epsilon^{ab}\omega_{\alpha ab} = \omega_\alpha(e) + \frac{i}{4}\bar{\chi}_\alpha\bar{\rho}\rho^\beta\chi_\beta \\
\omega_\alpha(e) &= -\frac{1}{e}e_{\alpha a}\epsilon^{\beta\gamma}\partial_\beta e_\gamma{}^a
\end{aligned}$$

where $\omega_\alpha(e)$ is the spin connection without torsion. The variation of $e_\alpha{}^a$ implies that

$$\delta_\epsilon g_{\alpha\beta} = i\bar{\epsilon}\rho_{(\alpha}\chi_{\beta)}$$

which we will use in the next section.

(ii) **Weyl transformations**

$$\begin{aligned}
\delta_\Lambda X^\mu &= 0 \\
\delta_\Lambda \psi^\mu &= -\frac{1}{2}\Lambda\psi^\mu \\
\delta_\Lambda e_\alpha{}^a &= \Lambda e_\alpha{}^a \\
\delta_\Lambda \chi_\alpha &= \frac{1}{2}\Lambda\chi_\alpha
\end{aligned}$$

(iii) **super-Weyl transformations**

$$\begin{aligned}
\delta_\eta \chi_\alpha &= \rho_\alpha\eta \\
\delta_\eta(others) &= 0
\end{aligned}$$

with $\eta(\sigma,\tau)$ being a Majorana spinor parameter.

(iv) **two-dimensional Lorentz transformations**

$$\begin{aligned}
\delta_l X^\mu &= 0 \\
\delta_l \psi^\mu &= \frac{1}{2}l\bar{\rho}\psi^\mu \\
\delta_l e_\alpha{}^a &= l\epsilon^a{}_b e_\alpha{}^b \\
\delta_l \chi_\alpha &= \frac{1}{2}l\bar{\rho}\chi_\alpha
\end{aligned}$$

(v) **reparametrizations**

$$
\begin{aligned}
\delta_\xi X^\mu &= \xi^\beta \partial_\beta X^\mu \\
\delta_\xi \psi^\mu &= \xi^\beta \partial_\beta \psi^\mu \\
\delta_\xi e_\alpha{}^a &= \xi^\beta \partial_\beta e_\alpha{}^a + e_\beta{}^a \partial_\alpha \xi^\beta \\
\delta_\xi \chi_\alpha &= \xi^\beta \partial_\beta \chi_\alpha + \chi_\beta \partial_\alpha \xi^\beta
\end{aligned}
$$

All of these are easy to verify except for the supersymmetry, which we will focus on in the next section.

2.2 The Local Supersymmetry

We are now going to verify the local supersymmetry explicitly. We will make use of the familiar results [7]

$$
\begin{aligned}
\delta\sqrt{g} &= \frac{1}{2}\sqrt{g} g^{\alpha\beta} \delta g_{\alpha\beta} \\
\delta g^{\alpha\beta} &= -g^{\alpha\gamma} g^{\beta\delta} \delta g_{\gamma\delta}
\end{aligned}
$$

The variation of the action naturally gives four terms, where the first term comes from the variation of $\sqrt{g}$. After dropping the factor $\sqrt{g}$ common to all terms and suppressing the spacetime indices we get

$$
\begin{aligned}
I &= \frac{1}{2} i\bar{\epsilon}\rho^\xi \chi_\xi \left[g^{\alpha\beta}\partial_\alpha X \partial_\beta X + 2i\bar{\psi}\rho^\alpha \partial_\alpha \psi - i\bar{\chi}_\alpha \rho^\beta \rho^\alpha \psi \left(\partial_\beta X - \frac{i}{4}\bar{\chi}_\beta \psi \right) \right] \\
II &= -i\bar{\epsilon}\rho^{(\alpha}\chi^{\beta)} \partial_\alpha X \partial_\beta X + 2ig^{\alpha\beta}\partial_\alpha (\bar{\epsilon}\psi) \partial_\beta X \\
III &= -i\partial_\xi X \bar{\epsilon}\rho^\xi \rho^\alpha \nabla_\alpha \psi - \frac{1}{2}(\bar{\chi}_\xi \psi)\bar{\epsilon}\rho^\xi \rho^\alpha \nabla_\alpha \psi + 2\bar{\psi}\bar{\epsilon}\rho^{(\alpha}\chi^{\beta)} e_\beta{}^a \rho_a \nabla_\alpha \psi \\
&\quad -\bar{\psi} g^{\alpha\beta} \bar{\epsilon}\rho^a \chi_\beta \rho_a \nabla_\alpha \psi - i\nabla_\alpha \bar{\psi}\rho^\alpha \rho^\xi \left(\partial_\xi X - \frac{i}{2}\bar{\chi}_\xi \psi \right) \epsilon \\
&= -i\left(\bar{\epsilon}\rho^\xi \rho^\alpha \nabla_\alpha \psi\right) \partial_\xi X - \frac{1}{2}\left(\bar{\epsilon}\rho^\xi \rho^\alpha \nabla_\alpha \psi\right)\left(\bar{\chi}_\xi \psi\right) \\
&\quad +2\left(\bar{\epsilon}\rho^{(\alpha}\chi^{\beta)}\right)\left(\bar{\psi}\rho_\beta \nabla_\alpha \psi\right) - \left(\bar{\epsilon}\rho^\xi \chi^\alpha\right)\left(\bar{\psi}\rho_\xi \nabla_\alpha \psi\right) \\
&\quad -i\left(\bar{\epsilon}\rho^\xi \rho^\alpha \nabla_\alpha \psi\right) \partial_\xi X - \frac{1}{2}\left(\bar{\epsilon}\rho^\xi \rho^\alpha \nabla_\alpha \psi\right)\left(\bar{\chi}_\xi \psi\right) \\
&= -2i\left(\bar{\epsilon}\rho^\xi \rho^\alpha \nabla_\alpha \psi\right) \partial_\xi X - \left(\bar{\epsilon}\rho^\xi \rho^\alpha \nabla_\alpha \psi\right)\left(\bar{\chi}_\xi \psi\right) + \left(\bar{\epsilon}\rho^\alpha \chi^\beta\right)\left(\bar{\psi}\rho_\beta \nabla_\alpha \psi\right) \\
IV &= \left(\partial_\beta X - \frac{i}{4}\bar{\chi}_\beta \psi \right) \left[-2i\left(\bar{\psi}\rho^\alpha \rho^\beta D_\alpha \epsilon\right) - \bar{\chi}_\alpha \bar{\epsilon}\rho^{(\beta}\chi^{\xi)} \rho_\xi \rho^\alpha \psi \right. \\
&\quad \left. -\bar{\chi}_\alpha \rho^\beta \bar{\epsilon}\rho^{(\alpha}\chi^{\xi)}\rho_\xi \psi + \frac{1}{2}\bar{\chi}_\alpha \rho_a \rho^\alpha \psi \left(\bar{\epsilon}\rho^a \chi^\beta\right) + \frac{1}{2}\bar{\chi}_\alpha \rho^\beta \rho_a \psi \left(\bar{\epsilon}\rho^a \chi^\alpha\right) \right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{i}{2}\bar{\chi}_\alpha\rho^\beta\rho^\alpha\rho^\xi\left(\partial_\xi X-\frac{i}{2}\bar{\chi}_\xi\psi\right)\epsilon\left(\partial_\beta X-\frac{i}{4}\bar{\chi}_\beta\psi\right)\\
&-i\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi\left[i\partial_\beta\left(\bar{\epsilon}\psi\right)-\frac{i}{8}\bar{\chi}_\beta\rho^\xi\left(\partial_\xi X-\frac{i}{2}\bar{\chi}_\xi\psi\right)\epsilon-\frac{i}{2}\bar{\psi}D_\beta\epsilon\right]
\end{aligned}
$$

Remember here that in D_α there is a term $\sim\chi^2$. We now collect terms according to their order in the fields.

$\chi\psi$:

$$
\begin{aligned}
&\quad 2ig^{\alpha\beta}\partial_\alpha\left(\bar{\epsilon}\psi\right)\partial_\beta X-2i\left(\bar{\epsilon}\rho^\beta\rho^\alpha\nabla_\alpha\psi\right)\partial_\beta X-2i\left(\bar{\psi}\rho^\alpha\rho^\beta\nabla_\alpha\epsilon\right)\partial_\beta X\\
&= 2ig^{\alpha\beta}\nabla_\alpha\left(\bar{\epsilon}\psi\right)\partial_\beta X-2i\nabla_\alpha\left(\bar{\epsilon}\rho^\beta\rho^\alpha\psi\right)\partial_\beta X\\
&= -\frac{2i}{e}\partial_\alpha\left(\bar{\epsilon}\bar{\rho}\psi\right)\partial_\beta X\epsilon^{\beta\alpha}=[\text{P.I.}]\\
&\sim \left(\bar{\epsilon}\bar{\rho}\psi\right)\partial_\alpha\partial_\beta X\epsilon^{\beta\alpha}=0
\end{aligned}
$$

We have here used ordinary partial integration since the $\frac{1}{e}$ cancels the $\sqrt{g}$ we have dropped.

$\psi^2\chi$:

$$
\begin{aligned}
&\quad -\left(\bar{\epsilon}\rho^\xi\chi_\xi\right)\left(\bar{\psi}\rho^\alpha\partial_\alpha\psi\right)-\left(\bar{\epsilon}\rho^\xi\rho^\alpha\nabla_\alpha\psi\right)\left(\bar{\chi}_\xi\psi\right)+\left(\bar{\epsilon}\rho^\alpha\chi^\xi\right)\left(\bar{\psi}\rho_\xi\partial_\alpha\psi\right)\\
&\quad -\frac{1}{2}\left(\bar{\psi}\rho^\alpha\rho^\beta\nabla_\alpha\epsilon\right)\left(\bar{\chi}_\beta\psi\right)+\left(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi\right)\nabla_\beta\left(\bar{\epsilon}\psi\right)\\
&\quad -\frac{1}{2}\left(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi\right)\left(\bar{\psi}\nabla_\beta\epsilon\right)\\
&= -\left(\bar{\epsilon}\rho^\xi\chi_\xi\right)\left(\bar{\psi}\rho^\alpha\nabla_\alpha\psi\right)-\left(\bar{\epsilon}\rho^\xi\rho^\alpha\nabla_\alpha\psi\right)\left(\bar{\chi}_\xi\psi\right)+\left(\bar{\epsilon}\rho^\alpha\chi^\xi\right)\left(\bar{\psi}\rho_\xi\partial_\alpha\psi\right)\\
&\quad +\frac{1}{4}\left(\bar{\chi}_\beta\rho^\alpha\rho^\beta\nabla_\alpha\epsilon\right)\left(\bar{\psi}\psi\right)-\frac{1}{4}\left(\bar{\chi}_\beta\rho^\alpha\rho^\beta\nabla_\alpha\epsilon\right)\left(\bar{\psi}\psi\right)\\
&\quad +\left(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi\right)\left(\bar{\epsilon}\nabla_\beta\psi\right)\\
&= -\frac{1}{2}\left(\bar{\psi}\nabla_\alpha\psi\right)\left(\bar{\epsilon}\rho^\xi\rho^\alpha\chi_\xi-\bar{\epsilon}\rho^\xi\rho^\alpha\chi_\xi\right)+\left(\bar{\epsilon}\rho^\alpha\chi^\xi\right)\left(\bar{\psi}\rho_\xi\nabla_\alpha\psi\right)\\
&\quad -\frac{1}{2}\left(\bar{\psi}\rho^\sigma\nabla_\alpha\psi\right)\left(\bar{\epsilon}\rho_\sigma\rho^\xi\rho^\alpha\chi_\xi-\bar{\epsilon}\rho^\xi\rho^\alpha\rho_\sigma\chi_\xi\right)-\left(\bar{\epsilon}\rho^\xi\chi_\xi\right)\left(\bar{\psi}\rho^\alpha\nabla_\alpha\psi\right)\\
&\quad -\frac{1}{2}\left(\bar{\psi}\bar{\rho}\nabla_\alpha\psi\right)\left(\bar{\epsilon}\bar{\rho}\rho^\xi\rho^\alpha\chi_\xi-\bar{\epsilon}\rho^\xi\rho^\alpha\bar{\rho}\chi_\xi\right)\\
&= -\frac{1}{2}\left(\bar{\psi}\rho^\sigma\nabla_\alpha\psi\right)\left(\bar{\epsilon}\rho_\sigma\rho^\xi\rho^\alpha\chi_\xi-\bar{\epsilon}\rho_\sigma\rho^\xi\rho^\alpha\chi_\xi-2\left(\bar{\epsilon}\rho^\xi\delta^\alpha{}_\sigma\chi_\xi\right)\right.\\
&\quad \left.+2\left(\bar{\epsilon}\rho^\alpha\delta^\xi{}_\sigma\chi_\xi\right)\right)-\left(\bar{\epsilon}\rho^\xi\chi_\xi\right)\left(\bar{\psi}\rho^\alpha\nabla_\alpha\psi\right)+\left(\bar{\epsilon}\rho^\alpha\chi^\xi\right)\left(\bar{\psi}\rho_\xi\nabla_\alpha\psi\right)\\
&= 0
\end{aligned}
$$

$X^2\chi$:

$$\begin{aligned}
&\left(\frac{i}{2}\bar{\epsilon}\rho^{\xi}\chi_{\xi}g^{\alpha\beta} - i\bar{\epsilon}\rho^{\alpha}\chi^{\beta} - \frac{i}{2}\bar{\chi}_{\xi}\rho^{\beta}\rho^{\xi}\rho^{\alpha}\epsilon\right)\partial_{\alpha}X\partial_{\beta}X \\
= \;& \frac{i}{2}\bar{\epsilon}\left(\rho^{\xi}\chi_{\xi}g^{\alpha\beta} - \rho^{\xi}\rho^{\alpha}\rho^{\beta}\chi_{\xi}\right)\partial_{\alpha}X\partial_{\beta}X \\
= \;& -\frac{i}{2e}\bar{\epsilon}\rho^{\xi}\epsilon^{\alpha\beta}\bar{\rho}\chi_{\xi}\partial_{\alpha}X\partial_{\beta}X \\
= \;& 0
\end{aligned}$$

where the last step follows from the symmetries in α and β.

$\chi^3\psi^2$:

$$\begin{aligned}
&-\frac{i}{8}\left(\bar{\epsilon}\rho^{\xi}\chi_{\xi}\right)\left(\bar{\chi}_{\alpha}\rho^{\beta}\rho^{\alpha}\psi\right)\left(\bar{\chi}_{\beta}\psi\right) + \frac{i}{16}\left(\bar{\psi}\rho^{\alpha}\rho^{\beta}\bar{\rho}\epsilon\right)\left(\bar{\chi}_{\alpha}\bar{\rho}\rho^{\xi}\chi_{\xi}\right)\left(\bar{\chi}_{\beta}\psi\right) \\
&+\frac{i}{16}\left(\bar{\chi}_{\alpha}\rho^{\beta}\rho^{\alpha}\psi\right)\left(\bar{\chi}_{\beta}\rho^{\xi}\epsilon\right)\left(\bar{\chi}_{\xi}\psi\right) + \frac{i}{4}\left(\bar{\chi}_{\alpha}\rho_{\xi}\rho^{\alpha}\psi\right)\left(\bar{\epsilon}\rho^{(\beta}\chi^{\xi)}\right)\left(\bar{\chi}_{\beta}\psi\right) \\
&+\frac{i}{4}\left(\bar{\chi}_{\alpha}\rho^{\beta}\rho_{\xi}\psi\right)\left(\bar{\epsilon}\rho^{(\alpha}\chi^{\xi)}\right)\left(\bar{\chi}_{\beta}\psi\right) - \frac{i}{8}\left(\bar{\chi}_{\alpha}\rho_{\xi}\rho^{\alpha}\psi\right)\left(\bar{\epsilon}\rho^{\xi}\chi^{\beta}\right)\left(\bar{\chi}_{\beta}\psi\right) \\
&-\frac{i}{8}\left(\bar{\chi}_{\alpha}\rho^{\beta}\rho_{\xi}\psi\right)\left(\bar{\epsilon}\rho^{\xi}\chi^{\alpha}\right)\left(\bar{\chi}_{\beta}\psi\right) + \frac{i}{16}\left(\bar{\chi}_{\alpha}\rho^{\beta}\rho^{\alpha}\rho^{\xi}\epsilon\right)\left(\bar{\chi}_{\xi}\psi\right)\left(\bar{\chi}_{\beta}\psi\right) \\
&+\frac{i}{16}\left(\bar{\chi}_{\alpha}\rho^{\beta}\rho^{\alpha}\psi\right)\left(\bar{\psi}\bar{\rho}\epsilon\right)\left(\bar{\chi}_{\beta}\bar{\rho}\rho^{\xi}\chi_{\xi}\right) \\
= \;& \left[\text{drop } -\frac{i}{16}\right] = -\left(\bar{\epsilon}\rho^{\xi}\chi_{\xi}\right)\left(\bar{\chi}_{\alpha}\rho^{\beta}\rho^{\alpha}\chi_{\beta}\right) + \frac{3}{2}\left(\bar{\epsilon}\rho^{\beta}\chi^{\xi}\right)\left(\bar{\chi}_{\alpha}\rho_{\xi}\rho^{\alpha}\chi_{\beta}\right) \\
&-\frac{1}{2}\left(\bar{\epsilon}\rho^{\xi}\rho^{\alpha}\rho^{\beta}\bar{\chi}_{\alpha}\right)\left(\bar{\chi}_{\beta}\chi_{\xi}\right) - \left(\bar{\epsilon}\bar{\rho}\rho^{\beta}\rho^{\alpha}\bar{\chi}_{\beta}\right)\left(\bar{\chi}_{\alpha}\bar{\rho}\rho^{\xi}\chi_{\xi}\right) \\
= \;& -\left(\bar{\epsilon}\rho^{\xi}\chi_{\xi}\right)\left(\bar{\chi}_{\alpha}\bar{\rho}\chi_{\beta}\right)\frac{1}{e}\epsilon^{\beta\alpha} + \frac{1}{2}\left(\bar{\epsilon}\rho^{\beta}\chi_{\alpha}\right)\left(\bar{\chi}^{\alpha}\chi_{\beta}\right) \\
&+\frac{3}{2}\left(\bar{\epsilon}\rho^{\beta}\chi_{\xi}\right)\left(\bar{\chi}_{\alpha}\bar{\rho}\chi_{\beta}\right)\frac{1}{e}\epsilon^{\xi\alpha} - \frac{1}{2}\left(\bar{\epsilon}\rho^{\alpha}\rho^{\beta}\rho^{\xi}\chi_{\alpha}\right)\left(\bar{\chi}_{\beta}\chi_{\xi}\right) \\
&-\left(\bar{\epsilon}\bar{\rho}\rho^{\beta}\rho^{\alpha}\chi_{\beta}\right)\left(\bar{\chi}_{\alpha}\bar{\rho}\rho^{\xi}\chi_{\xi}\right) \\
= \;& -\frac{1}{2}\left(\bar{\epsilon}\rho^{\xi}\chi_{\xi}\right)\left(\bar{\chi}_{\alpha}\bar{\rho}\chi_{\beta}\right)\frac{1}{e}\epsilon^{\beta\alpha} + \frac{3}{2}\left(\bar{\epsilon}\rho^{\beta}\chi_{\xi}\right)\left(\bar{\chi}_{\alpha}\bar{\rho}\chi_{\beta}\right)\frac{1}{e}\epsilon^{\xi\alpha} \\
&+\frac{1}{2}\left(\bar{\epsilon}\bar{\rho}\rho^{\xi}\chi_{\alpha}\right)\left(\bar{\chi}_{\beta}\chi_{\xi}\right)\frac{1}{e}\epsilon^{\alpha\beta} - \left(\bar{\epsilon}\chi_{\beta}\right)\left(\bar{\chi}_{\alpha}\bar{\rho}\rho^{\xi}\chi_{\xi}\right)\frac{1}{e}\epsilon^{\beta\alpha} \\
= \;& \left[\text{drop } \frac{1}{e}\right] = \left(\bar{\epsilon}\rho^{\xi}\chi_{\xi}\right)\left(\bar{\chi}_{\alpha}\bar{\rho}\chi_{\beta}\right)\epsilon^{\alpha\beta} - \left(\bar{\epsilon}\rho^{\beta}\rho^{\sigma}\chi_{\beta}\right)\left(\bar{\chi}_{\alpha}\bar{\rho}\rho^{\sigma}\chi_{\xi}\right)\epsilon^{\xi\alpha} \\
&+\left(\bar{\epsilon}\chi^{\sigma}\right)\left(\bar{\chi}_{\alpha}\bar{\rho}\rho_{\sigma}\chi_{\beta}\right)\epsilon^{\beta\alpha} \\
= \;& \left(\bar{\epsilon}\bar{\rho}\chi_{\xi}\right)\left(\bar{\chi}_{\alpha}\rho^{\xi}\chi_{\beta}\right)\epsilon^{\alpha\beta} + \left(\bar{\epsilon}\rho_{\sigma}\chi_{\xi}\right)\left(\bar{\chi}_{\alpha}\bar{\rho}\rho^{\sigma}\rho^{\xi}\chi_{\beta}\right)\epsilon^{\alpha\beta}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2}\left(\bar{\epsilon}\rho^\xi\chi_\xi\right)\left(\bar{\chi}_\alpha\rho^\alpha\rho^\beta\chi_\beta\right)\\
&= 0
\end{aligned}$$

$\chi^2\psi X$: We start by dropping the common term $\partial_\beta X$ and get

$$\begin{aligned}
&\frac{1}{2}\left(\bar{\epsilon}\rho^\xi\chi_\xi\right)\left(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\psi\right)+\frac{1}{4}\left(\bar{\epsilon}\bar{\rho}\rho^\beta\rho^\alpha\psi\right)\left(\bar{\chi}_\alpha\bar{\rho}\rho^\xi\chi_\xi\right)\\
&-\frac{3}{8}\left(\bar{\epsilon}\rho^\beta\chi_\xi\right)\left(\bar{\chi}_\alpha\rho^\xi\rho^\alpha\psi\right)-\frac{1}{2}\left(\bar{\epsilon}\rho^\alpha\chi_\xi\right)\left(\bar{\chi}_\alpha\rho^\beta\rho^\xi\psi\right)\\
&-\frac{1}{8}\left(\bar{\chi}_\xi\psi\right)\left(\bar{\chi}_\alpha\rho^\xi\rho^\alpha\rho^\beta\epsilon\right)-\frac{1}{4}\left(\bar{\chi}_\xi\psi\right)\left(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\rho^\xi\epsilon\right)
\end{aligned}$$

By now Fierzing together ψ and ϵ we get three types of terms, namely (after dropping the common factor of $-\frac{1}{2}$)

$$\begin{aligned}
I &= \frac{1}{4}\left(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\rho^\xi\chi_\xi\right)-\frac{1}{2e}\epsilon^{\beta\alpha}\left(\bar{\chi}_\alpha\bar{\rho}\rho^\xi\chi_\xi\right)\\
&\quad -\frac{1}{2}\left(\bar{\chi}_\alpha\rho_\xi\rho^\alpha\rho^\beta\chi_\xi\right)-\frac{1}{2}\left(\bar{\chi}_\alpha\rho^\beta\rho^\xi\rho^\alpha\chi_\xi\right)\\
&= \frac{1}{2}\left(\bar{\chi}^\beta\rho^\xi\chi_\xi\right)-\frac{1}{2}\left(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\rho^\xi\chi_\xi\right)\\
&= \left(\frac{1}{2}-1+\frac{1}{2}\right)\left(\bar{\chi}^\beta\rho^\xi\chi_\xi\right)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
II &= \frac{1}{4}\left(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\bar{\rho}\rho^\xi\chi_\xi\right)-\frac{3}{8}\left(\bar{\chi}_\alpha\rho^\xi\rho^\alpha\bar{\rho}\rho^\beta\chi_\xi\right)\\
&\quad -\frac{1}{2}\left(\bar{\chi}_\alpha\rho^\beta\rho^\xi\bar{\rho}\rho^\alpha\chi_\xi\right)-\frac{1}{2}\left(\bar{\chi}^\beta\bar{\rho}\rho^\xi\chi_\xi\right)\\
&= \left(\frac{1}{2}-1+1-\frac{1}{2}\right)\left(\bar{\chi}^\beta\bar{\rho}\rho^\xi\chi_\xi\right)\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
III &= \frac{1}{2}\left(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\rho^\sigma\rho^\xi\chi_\xi\right)-\frac{3}{8}\left(\bar{\chi}_\alpha\rho^\xi\rho^\alpha\rho^\sigma\rho^\beta\chi_\xi\right)\\
&\quad -\frac{1}{2}\left(\bar{\chi}_\alpha\rho^\beta\rho^\xi\rho^\sigma\rho^\alpha\chi_\xi\right)+\frac{1}{8}\left(\bar{\chi}_\alpha\rho^\xi\rho^\alpha\rho^\beta\rho^\sigma\chi_\xi\right)\\
&\quad +\frac{1}{4}\left(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\rho^\xi\rho^\sigma\chi_\xi\right)\\
&= \frac{1}{2}\left(\bar{\chi}_\alpha\rho^\beta\rho^\alpha\rho^\sigma\rho^\xi\chi_\xi\right)-\frac{1}{2}\left(\bar{\chi}_\alpha\rho^\beta\rho^\xi\rho^\sigma\rho^\alpha\chi_\xi\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2e}\epsilon^{\alpha\xi}\left(\bar{\chi}_\alpha\bar{\rho}\chi_\xi\right)g^{\beta\sigma}+\frac{1}{2}\left(\bar{\chi}_\alpha\rho^\xi\rho^\sigma\chi_\xi\right)g^{\alpha\beta}-\frac{1}{2}\left(\bar{\chi}_\alpha\rho^\alpha\rho^\beta\chi_\xi\right)g^{\xi\sigma}\\
=\;&\frac{1}{2e}\epsilon^{\alpha\xi}\left(\bar{\chi}_\alpha\bar{\rho}\chi_\xi\right)g^{\beta\sigma}+\frac{1}{2e}\epsilon^{\beta\alpha}\left(\bar{\chi}_\alpha\bar{\rho}\chi^\sigma\right)+\frac{1}{2}\epsilon^{\xi\sigma}\left(\bar{\chi}^\beta\bar{\rho}\chi_\xi\right)\\
=\;&\frac{1}{2e}\left(\bar{\chi}_\alpha\bar{\rho}\chi_\xi\right)\left(\epsilon^{\alpha\xi}g^{\beta\sigma}+\epsilon^{\xi\beta}g^{\alpha\sigma}+\epsilon^{\xi\sigma}g^{\alpha\beta}\right)\\
\sim\;&\epsilon_{\alpha\xi}\left(\epsilon^{\alpha\xi}g^{\beta\sigma}+\epsilon^{\xi\beta}g^{\alpha\sigma}+\epsilon^{\xi\sigma}g^{\alpha\beta}\right)\\
=\;&\left(2g^{\beta\sigma}-g^{\beta\sigma}-g^{\beta\sigma}\right)\\
=\;&0
\end{aligned}
$$

One disadvantage of this action is that it can not be generalized to objects with dimension greater than one, i.e. membranes and so on. The problem arises when one tries to supersymmetrize the worldvolume cosmological constant which is included in the higher-dimensional actions but is absent in the string case. One has to introduce an Einstein-Hilbert term which makes the action inequivalent to the bosonic action in the limit that all fermions vanish. There is however another action, the Green-Schwarz action, that can be generalized and we will consider the two-dimensional case, i.e. the membrane, in chapter 4.

3

Eleven-Dimensional Supergravity

We are now going to study eleven-dimensional supergravity [12] and in the next chapter we are going to look at a membrane living in this theory.

3.1 Conventions and Definitions

Indices:

$$\left.\begin{array}{ll} m,n,p,\dots & \text{curved vector} \\ \mu,\nu,\rho,\dots & \text{curved "spinor"} \end{array}\right\} \begin{array}{c} M,N,P,\dots \text{ curved} \\ \text{super indices} \end{array}$$

$$\left.\begin{array}{ll} a,b,c,\dots & \text{flat vector} \\ \alpha,\beta,\gamma,\dots & \text{flat spinor} \end{array}\right\} A,B,C,\dots \text{ flat super indices}$$

Coordinates:

$$(X^m, \theta^\mu) \equiv Z^M$$

where X^m has 11 bosonic degrees of freedom and θ^μ has 32 real fermionic degrees of freedom since θ^μ is a Majorana spinor.

Signature:

$$(-,+,+,\cdots,+)$$

Γ-matrices:

$$\left\{\Gamma^a, \Gamma^b\right\} = 2\eta^{ab}$$

Forms:

$$(dX^m, d\theta^\mu) \equiv dZ^M$$

and

$$\begin{aligned} dX^m \wedge dX^n &= -dX^n \wedge dX^m \\ d\theta^\mu \wedge d\theta^\nu &= +d\theta^\nu \wedge d\theta^\mu \\ dX^m \wedge d\theta^\mu &= -d\theta^\mu \wedge dX^m \end{aligned}$$

which we will write as

$$dZ^M \wedge dZ^N = -(-)^{MN} dZ^N \wedge dZ^M$$

where in the exponent of $(-)$ $M = 0$ for a bosonic index and $M = 1$ for a fermionic index.

Vielbein:

$$E^A = dZ^M E_M{}^A (Z)$$

Spin connection:

$$\omega_A{}^B = dZ^M \omega_{MA}{}^B$$

where $\omega_A{}^B \in \widetilde{SO(1,10)}$ (i.e. the Lie algebra) and since this is no super algebra we have

$$\omega_a{}^\beta = \omega_\alpha{}^b = 0 \tag{3.1}$$

i.e. vectors and spinors don't mix.

Superforms:

$$B_{(p)} = \frac{1}{p!} dZ^{M_p} \wedge \ldots \wedge dZ^{M_1} B_{M_1 \cdots M_p}$$

Exterior derivative:

$$dB_{(p)} \equiv \left(dZ^M \partial_M\right) \wedge B_{(p)} = \left(dX^m \frac{\partial}{\partial X^m} + d\theta^\mu \frac{\partial}{\partial \theta^\mu}\right) \wedge B_{(p)}$$

and

$$d^2 = 0 \tag{3.2}$$

which just follows from the commutative or anti-commutative properties of the coordinate differentials and derivatives respectively. For a p-form B we then get

$$\begin{aligned} dB_{(p)} &= \frac{1}{p!} dZ^{M_p} \wedge \ldots \wedge dZ^{M_1} \wedge dB_{M_1 \cdots M_p} \\ &= \frac{1}{p!} dZ^{M_p} \wedge \ldots \wedge dZ^{M_1} \wedge dZ^M \partial_M B_{M_1 \cdots M_p} \\ &= \frac{1}{(p+1)!} dZ^{M_{p+1}} \wedge \ldots \wedge dZ^{M_1} H_{M_1 \cdots M_{p+1}} = H_{(p+1)} \end{aligned}$$

with

$$H_{M_1 \cdots M_{p+1}} = \partial_{M_1} B_{M_2 \cdots M_{p+1}} + \begin{array}{c} graded \\ cyclic \end{array} (p\ terms)$$

where we have used the symmetries of $B_{M_1 \cdots M_p}$.

Right action:

$$d\left(A_{(p)} \wedge B_{(q)}\right) = A_{(p)} \wedge dB_{(q)} + (-1)^q \, dA_{(p)} \wedge B_{(q)} \tag{3.3}$$

Covariant derivative:

$$D = d + \omega$$

and on a tensor we have

$$\begin{aligned} DT^A &= dT^A + T^B \wedge \omega_B{}^A \\ DT_A &= dT_A - \omega_A{}^B \wedge T_B \end{aligned}$$

Torsion:

$$T^A = DE^A = dE^A + E^B \wedge \omega_B{}^A \tag{3.4}$$

Curvature:

$$R_A{}^B = d\omega_A{}^B + \omega_A{}^C \wedge \omega_C{}^B$$

3.2 Bianchi identities

We are now going to obtain three Bianchi identities which the various fields must satisfy.

$$\begin{aligned} DR_A{}^B &= D\left(d\omega_A{}^B + \omega_A{}^C \wedge \omega_C{}^B\right) = [3.2] \\ &= d\left(\omega_A{}^C \wedge \omega_C{}^B\right) + d\omega_A{}^C \wedge \omega_C{}^B - \omega_A{}^C \wedge d\omega_C{}^B \\ &\quad +\omega_A{}^C \wedge \omega_C{}^D \wedge \omega_D{}^B - \omega_A{}^D \wedge \omega_D{}^C \wedge \omega_C{}^B \\ &= [3.3] = 0 \end{aligned}$$

and

$$\begin{aligned}
DT^A &= D\left(dE^A + E^B \wedge \omega_B{}^A\right) = [3.2] \\
&= dE^B \wedge \omega_B{}^A + E^B \wedge \omega_B{}^C \wedge \omega_C{}^A + d\left(E^B \wedge \omega_B{}^A\right) = [3.3] \\
&= dE^B \wedge \omega_B{}^A + E^B \wedge \omega_B{}^C \wedge \omega_C{}^A + E^B \wedge d\omega_B{}^A - dE^B \wedge \omega_B{}^A \\
&= E^B \wedge \omega_B{}^C \wedge \omega_C{}^A + E^B \wedge d\omega_B{}^A = E^B \wedge R_B{}^A
\end{aligned}$$

The last identity is

$$dH_{(4)} = 0$$

which arises from the fact that we want the 4-form field strength H to be described by a 3-form potential B according to $H = dB$. That this is actually possible is dimensional dependent and is thus one of the special features of 11-dimensional supergravity. It turns out that it is also possible to include a 7-form field strength, which is currently a popular subject. The Bianchi identities are thus

$$\begin{aligned}
DR_A{}^B &= 0 \\
DT^A &= E^B \wedge R_B{}^A \\
dH_{(4)} &= 0
\end{aligned}$$

We can now rewrite the last two relations in a more convenient form.

$$\begin{aligned}
DT^A &\equiv D\left(\frac{1}{2}E^D \wedge E^C T_{CD}{}^A\right) \\
&= \frac{1}{2}E^D \wedge E^C\left(DT_{CD}{}^A\right) + \frac{1}{2}E^D \wedge \left(DE^C\right) T_{CD}{}^A \\
&\quad -\frac{1}{2}\left(DE^D\right) \wedge E^C T_{CD}{}^A \\
&= \frac{1}{2}E^D \wedge E^C\left(DT_{CD}{}^A\right) + \frac{1}{2}E^D \wedge T^C T_{CD}{}^A - \frac{1}{2}T^D \wedge E^C T_{CD}{}^A \\
&= \frac{1}{2}E^D \wedge E^C\left(DT_{CD}{}^A\right) + \frac{1}{2}E^D \wedge T^C T_{CD}{}^A \\
&\quad + (-)^{CD}(-)^{CD}\frac{1}{2}E^C \wedge T^D T_{DC}{}^A \\
&= \left[D = E^A D_A\right] = \frac{1}{2}E^D \wedge E^C \wedge E^B\left(D_B T_{CD}{}^A + T_{BC}{}^E T_{ED}{}^A\right)
\end{aligned}$$

and

$$E^D R_D{}^A = \frac{1}{2}E^D \wedge E^C \wedge E^B R_{BCD}{}^A$$

gives

$$D_B T_{CD}{}^A + T_{BC}{}^E T_{ED}{}^A - R_{BCD}{}^A + \begin{matrix} \mathit{graded} \\ \mathit{cyclic} \end{matrix} \ (BCD) = 0 \qquad (3.5)$$

where the graded cyclic terms come from the implementation of the symmetries of the form part. Similarly

$$\begin{aligned}
dH_{(4)} &\equiv d\left(\frac{1}{4!}E^E \wedge E^D \wedge E^C \wedge E^B H_{BCDE}\right) = 0 \\
&\Rightarrow E^E \wedge E^D \wedge E^C \wedge E^B DH_{BCDE} + 4E^E \wedge E^D \wedge E^C \wedge T^F H_{FCDE} \\
&= \left[T^F \equiv \frac{1}{2}E^B \wedge E^A T_{AB}{}^F\right] \\
&= E^E \wedge E^D \wedge E^C \wedge E^B \wedge E^A D_A H_{BCDE} \\
&\quad +2E^E \wedge E^D \wedge E^C \wedge E^B \wedge E^A T_{AB}{}^F H_{FCDE} \\
&= 0
\end{aligned}$$

which gives

$$D_A H_{BCDE} + 2T_{AB}{}^F H_{FCDE} + \begin{matrix} graded \\ perm \end{matrix} \; (ABCDE) = 0$$

that can be rewritten in term of graded *cyclic* sums as

$$\begin{aligned}
0 &= \sum_{(ABCDE)} \left(D_A H_{BCDE} + T_{AB}{}^F H_{FCDE}\right) \\
&\quad - \sum_{(ADBEC)} (-)^{CD+BD+CE}\, T_{AD}{}^F H_{FBEC}
\end{aligned} \tag{3.6}$$

We have thus got rid of the forms and are left with two "ordinary" tensor equations.

3.3 Examining the Bianchi Identities

We now want to examine equation (3.5) and (3.6). The reason for not looking at the third Bianchi identity is that it turns out that (3.5) completely determines R and that the third Bianchi identity then is automatically satisfied.

We start by examining (3.5) and get eight different cases:

1 $(BCD, A) = (\beta\gamma\delta, a)$

$$D_{(\beta}T_{\gamma\delta)}{}^a + T_{(\beta\gamma}{}^E T_{|E|\delta)}{}^a - R_{(\beta\gamma\delta)}{}^a = 0 \tag{3.7}$$

Here $R_{(\beta\gamma\delta)}{}^a = 0$ since $R_{(\beta\gamma\delta)}{}^a$ doesn't belong to $\widetilde{SO(1,10)}$. If we now look at the dimensions of the torsion components we get

	dim $\left([length]^{-1} = [mass]\right)$
$T_{\gamma\delta}{}^a$	0
$T_{\gamma\delta}{}^\varepsilon$	1/2
$T_{e\delta}{}^a$	1/2

where the dimension is obtained by counting a fermionic index as 1/2 and a bosonic index as 1 where lower indices give a positive contribution while upper indices give a negative contribution. This procedure follows from the definition of $T_{AB}{}^{C}$. The only fields we want to have in the theory are

	dim $\left([length]^{-1} = [mass]\right)$
R_{abcd}	2
$S_{ab}{}^{\gamma} \sim \partial_{[a}\psi_{b]}$	3/2
H_{abcd}	1

which are the field strengths of

	degrees of freedom
$e_m{}^a$	44 bosonic
$\psi_m{}^\alpha$	128 fermionic
B_{mnp}	84 bosonic

Since there are no fields of dimension 1/2 and since there are no constants that have dimensions we must have

$$T_{\gamma\delta}{}^{\varepsilon} = T_{e\delta}{}^{a} = 0 \tag{3.8}$$

In order to have the supersymmetry algebra satisfied we must have $T_{\gamma\delta}{}^{a} \neq 0$ and since the index structure of $T_{\gamma\delta}{}^{a}$ matches that of the gamma matrices and since we have no field with dimension 0 we set

$$T_{\gamma\delta}{}^{a} = 2i\left(\Gamma^a\right)_{\gamma\delta}$$

We now see that (3.7) is satisfied.

2 $(BCD, A) = (\beta\gamma\delta, \alpha)$

$$\begin{aligned} 0 &= D_{(\beta}T_{\gamma\delta)}{}^{\alpha} + T_{(\beta\gamma}{}^{E}T_{|E|\delta)}{}^{\alpha} - R_{(\beta\gamma\delta)}{}^{\alpha} \\ &= 2i\left(\Gamma^e\right)_{(\beta\gamma}T_{|e|\delta)}{}^{\alpha} - R_{(\beta\gamma\delta)}{}^{\alpha} \end{aligned}$$

where the dimension of $T_{e\delta}{}^{\alpha}$ is 1 and therefore $T_{e\delta}{}^{\alpha} \sim H$.

3 $(BCD, A) = (\beta\gamma d, a)$

$$\begin{aligned} 0 &= 2D_{(\beta}T_{\gamma)d}{}^{a} + D_dT_{\beta\gamma}{}^{a} + T_{\beta\gamma}{}^{e}T_{ed}{}^{a} + 2T_{d(\beta}{}^{\varepsilon}T_{|\varepsilon|\gamma)}{}^{a} \\ &\quad - R_{\beta\gamma d}{}^{a} - 2R_{d(\beta\gamma)}{}^{a} \\ &= 2i\left(\Gamma^e\right)_{\beta\gamma}T_{ed}{}^{a} + 4iT_{d(\beta}{}^{\varepsilon}\left(\Gamma^a\right)_{|\varepsilon|\gamma)} - R_{\beta\gamma d}{}^{a} \end{aligned}$$

where we have used that

$$T_{\gamma d}{}^{a} = 0$$

since its dimension is 1/2. Now

$$R_{\beta\gamma(da)} = 0$$

gives

$$(\Gamma^e)_{\beta\gamma} T_{e(da)} + 2T_{(d(\beta}{}^{\varepsilon} \left(\Gamma_{a)}\right)_{|\varepsilon|\gamma)} = 0$$

4 $(BCD, A) = (b\gamma\delta, \alpha)$

$$\begin{aligned} 0 &= D_b T_{\gamma\delta}{}^{\alpha} + 2D_{(\gamma} T_{\delta)b}{}^{\alpha} + T_{\gamma\delta}{}^{e} T_{eb}{}^{\alpha} + 2T_{b(\gamma}{}^{\varepsilon} T_{|\varepsilon|\delta)}{}^{\alpha} \\ &\quad -2R_{b(\gamma\delta)}{}^{\alpha} - R_{\gamma\delta b}{}^{\alpha} \\ &= 2D_{(\gamma} T_{\delta)b}{}^{\alpha} + 2i\,(\Gamma^e)_{\gamma\delta} T_{eb}{}^{\alpha} - 2R_{b(\gamma\delta)}{}^{\alpha} \end{aligned}$$

5 $(BCD, A) = (\beta cd, a)$

$$\begin{aligned} 0 &= D_\beta T_{cd}{}^{a} + 2D_{[c} T_{d]\beta}{}^{a} + T_{cd}{}^{\alpha} T_{\alpha\beta}{}^{a} + 2T_{\beta[c}{}^{e} T_{|e|d]}{}^{a} \\ &\quad -2R_{\beta[cd]}{}^{a} - R_{cd\beta}{}^{a} \\ &= D_\beta T_{cd}{}^{a} + 2i\,(\Gamma^a)_{\alpha\beta} T_{cd}{}^{\alpha} - 2R_{\beta[cd]}{}^{a} \end{aligned}$$

6 $(BCD, A) = (bc\delta, \alpha)$

$$\begin{aligned} 0 &= D_\delta T_{bc}{}^{\alpha} + 2D_{[b} T_{c]\delta}{}^{\alpha} + T_{bc}{}^{e} T_{e\delta}{}^{\alpha} + 2T_{\delta[b}{}^{\varepsilon} T_{|\varepsilon|c]}{}^{\alpha} \\ &\quad -2R_{\delta[bc]}{}^{\alpha} - R_{bc\delta}{}^{\alpha} \\ &= D_\delta T_{bc}{}^{\alpha} + 2D_{[b} T_{c]\delta}{}^{\alpha} + T_{bc}{}^{e} T_{e\delta}{}^{\alpha} + 2T_{\delta[b}{}^{\varepsilon} T_{|\varepsilon|c]}{}^{\alpha} - R_{bc\delta}{}^{\alpha} \end{aligned}$$

7 $(BCD, A) = (bcd, a)$

$$0 = D_{[b} T_{cd]}{}^{a} + T_{[bc}{}^{e} T_{|e|d]}{}^{a} - R_{[bcd]}{}^{a}$$

8 $(BCD, A) = (bcd, \alpha)$

$$\begin{aligned} 0 &= D_{[b} T_{cd]}{}^{\alpha} + T_{[bc}{}^{\varepsilon} T_{|\varepsilon|d]}{}^{\alpha} + T_{[bc}{}^{e} T_{|e|d]}{}^{\alpha} - R_{[bcd]}{}^{\alpha} \\ &= D_{[b} T_{cd]}{}^{\alpha} + T_{[bc}{}^{\varepsilon} T_{|\varepsilon|d]}{}^{\alpha} + T_{[bc}{}^{e} T_{|e|d]}{}^{\alpha} \end{aligned}$$

We then continue by solving (3.6) and get seven more cases:

1 $(ABCDE) = (\alpha\beta\gamma\delta\varepsilon)$ Since $H_{(4)}$ has dimension -3 we have that

	dim $\left([length]^{-1} = [mass]\right)$
$H_{\alpha\beta\gamma\delta}$	-1
$H_{a\beta\gamma\delta}$	-1/2

and thus

$$H_{\alpha\beta\gamma\delta} = H_{a\beta\gamma\delta} = 0$$

Equation (3.6) then only gives

$$0 = 0$$

using the constraints on T.

2 $(ABCDE) = (a\beta\gamma\delta\varepsilon)$

$$T_{(\delta\varepsilon}{}^{f} H_{|fa|\beta\gamma)} = 0 \tag{3.9}$$

where $H_{fa\beta\gamma}$ has dimension 0 and thus $H_{fa\beta\gamma} \sim (\Gamma_{fa})_{\beta\gamma}$. From this dimensional argument follows that the only non-vanishing components of H is H_{abcd} and $H_{fa\beta\gamma}$. Equation (3.9) implies that the Γ matrix identity

$$(\Gamma_a)_{(\alpha\beta}\left(\Gamma^{ab}\right)_{\gamma\delta)} = 0 \tag{3.10}$$

must be satisfied in order to have a consistent theory. To check equation (3.10) we contract it with all symmetric combinations of Γ matrices. It turns out that it suffices to contract with $(\Gamma^c)^{\gamma\delta}$, $(\Gamma_{cd})^{\gamma\delta}$ and $\left(\Gamma^{(5)}\right)^{\gamma\delta}$ since the other symmetric combinations are accounted for by "mirroring" in $\Gamma^{(11)}$. We thus have three cases:

(i) $(\Gamma^c)^{\gamma\delta}$

$$\begin{aligned}
& (\Gamma_a)_{\alpha\beta}\left(\Gamma^{ab}\right)_{\gamma\delta}(\Gamma^c)^{\gamma\delta} + 4\,(\Gamma_a)_{\gamma(\alpha}(\Gamma^c)^{\gamma\delta}\left(\Gamma^{ab}\right)_{\beta)\delta} \\
& + (\Gamma_a)_{\gamma\delta}(\Gamma^c)^{\gamma\delta}\left(\Gamma^{ab}\right)_{\alpha\beta} \\
= \;& 4\left(\Gamma_a\Gamma^c\Gamma^{ab}\right)_{(\alpha\beta)} + \mathrm{tr}\mathbf{1}\delta_a{}^c\left(\Gamma^{ab}\right)_{\alpha\beta} \\
= \;& 32\left(\Gamma^{cb}\right)_{\alpha\beta} + 4\left(\Gamma_a\Gamma^c\Gamma^a\Gamma^b - \Gamma_a\Gamma^c\eta^{ab}\right)_{(\alpha\beta)} \\
= \;& 32\left(\Gamma^{cb}\right)_{\alpha\beta} + 4\left(-9\Gamma^c\Gamma^b - \Gamma^b\Gamma^c\right)_{(\alpha\beta)} \\
= \;& -8\eta^{bc}C_{(\alpha\beta)} = 0
\end{aligned}$$

since $C_{\alpha\beta}$ is antisymmetric and

$$\begin{aligned}
\Gamma^{ab}\Gamma^c &= \Gamma^{abc} + \Gamma^a\eta^{bc} - \Gamma^b\eta^{ac} = \Gamma^{abc} + 2\Gamma^{[a}\eta^{b]c} \\
\Gamma^a\Gamma^{bc} &= \Gamma^{abc} + \eta^{ab}\Gamma^c - \eta^{ac}\Gamma^b = \Gamma^{abc} + 2\eta^{a[b}\Gamma^{c]}
\end{aligned}$$

which gives

$$\mathrm{tr}\left[\Gamma^{ab}\Gamma^{c}\right]=0$$

since $\Gamma^{(k)}$, which denotes the totally antisymmetrized product of k Γ matrices, is traceless since (d is the dimension of our space)

$$\mathrm{tr}\left[\Gamma_a\Gamma^{(k)}\Gamma^a\right]=\mathrm{tr}\left[\Gamma^{(k)}\Gamma^a\Gamma_a\right]=d\,\mathrm{tr}\left[\Gamma^{(k)}\right]$$

and

$$\begin{aligned}
\mathrm{tr}\left[\Gamma_a\Gamma^{(k)}\Gamma^b\right] &= \mathrm{tr}\left[\Gamma_a\Gamma^{p_1\cdots p_k}\Gamma^b\right]\\
&= \begin{cases}(-1)^k\,\mathrm{tr}\left[\Gamma^{(k)}\Gamma_a\Gamma^b\right] & \text{if } a\notin p_i\\ (-1)^{(k-1)}\,\mathrm{tr}\left[\Gamma^{(k)}\Gamma_a\Gamma^b\right] & \text{if } a\in p_i\end{cases}\quad i=1,\dots k\\
&\Rightarrow\ \mathrm{tr}\left[\Gamma_a\Gamma^{(k)}\Gamma^a\right]=(-1)^k\,(d-k-k)\,\mathrm{tr}\left[\Gamma^{(k)}\right]
\end{aligned}$$

gives

$$\Big(d-(-1)^k\,(d-2k)\Big)\,\mathrm{tr}\left[\Gamma^{(k)}\right]=0$$

and thus

$$\mathrm{tr}\left[\Gamma^{(k)}\right]=0 \tag{3.11}$$

except for $\Gamma^{(d)}$ in odd dimensions which is proportional to ϵ with d vector indices.

(ii) $(\Gamma_{cd})^{\gamma\delta}$

$$\begin{aligned}
&(\Gamma_a)_{\alpha\beta}\left(\Gamma^{ab}\right)_{\gamma\delta}(\Gamma_{cd})^{\gamma\delta}+4\,(\Gamma_a)_{\gamma(\alpha}\,(\Gamma_{cd})^{\gamma\delta}\left(\Gamma^{ab}\right)_{\beta)\delta}\\
&+(\Gamma_a)_{\gamma\delta}\,(\Gamma_{cd})^{\gamma\delta}\left(\Gamma^{ab}\right)_{\alpha\beta}\\
=\ &-2\mathrm{tr}\mathbf{1}\delta^{ab}_{[cd]}\,(\Gamma_a)_{\alpha\beta}+4\left(\underbrace{\Gamma_a\Gamma_{cd}\Gamma^a}_{7\Gamma_{cd}}\,\Gamma^b-\Gamma^b\Gamma_{cd}\right)_{(\alpha\beta)}\\
=\ &-64\delta^{ab}_{[cd]}\,(\Gamma_a)_{\alpha\beta}+4\left(6\Gamma_{cd}{}^{b}+16\Gamma_{[c}\delta_{d]}{}^{b}\right)_{(\alpha\beta)}\\
=\ &0
\end{aligned}$$

where we used that $\left(\Gamma_{cd}{}^{b}\right)_{\alpha\beta}$ is antisymmetric and that

$$\Gamma^{ab}\Gamma_{cd}=\Gamma^{ab}{}_{cd}-4\delta^{[a}{}_{[c}\Gamma^{b]}{}_{d]}-2\delta^{ab}_{[cd]}$$

which gives

$$\mathrm{tr}\left[\Gamma^{ab}\Gamma_{cd}\right]=-2\mathrm{tr}\mathbf{1}\delta^{ab}_{[cd]}=-64\delta^{ab}_{[cd]}$$

by using (3.11).

(iii) $\left(\Gamma^{(5)}\right)^{\gamma\delta}$

$$
\begin{aligned}
& \left(\Gamma_a\right)_{\alpha\beta}\left(\Gamma^{ab}\right)_{\gamma\delta}\left(\Gamma^{(5)}\right)^{\gamma\delta}+4\left(\Gamma_a\right)_{\gamma(\alpha}\left(\Gamma^{(5)}\right)^{\gamma\delta}\left(\Gamma^{ab}\right)_{\beta)\delta} \\
& +\left(\Gamma_a\right)_{\gamma\delta}\left(\Gamma^{(5)}\right)^{\gamma\delta}\left(\Gamma^{ab}\right)_{\alpha\beta} \\
= & \; 4\left(\Gamma_a\Gamma^{(5)}\Gamma^{ab}\right)_{(\alpha\beta)}+\operatorname{tr}\left[\Gamma_a\Gamma^{(5)}\right]\left(\Gamma^{ab}\right)_{\alpha\beta} \\
= & \; 4\left(\Gamma_a\Gamma^{(5)}\Gamma^a\Gamma^b-\Gamma^b\Gamma^{(5)}\right)_{(\alpha\beta)} \\
= & \; -4\left(\Gamma^{(5)}\Gamma^b+\Gamma^b\Gamma^{(5)}\right)_{(\alpha\beta)} \\
= & \; -4\left(\Gamma^{(5)b}+\Gamma^{b(5)}\right)_{(\alpha\beta)} \\
= & \; 0
\end{aligned}
$$

since

$$
\begin{aligned}
\Gamma^{(k)}\Gamma^a &= \Gamma^{(k)a}+k\Gamma^{[c_1\cdots c_{k-1}}\eta^{c_k]a} \\
\Gamma^a\Gamma^{(k)} &= \Gamma^{a(k)}+k\eta^{a[c_1}\Gamma^{c_2\cdots c_k]}
\end{aligned}
$$

and $\left(\Gamma^{(4)}\right)_{\alpha\beta}$ is antisymmetric..

3 $(ABCDE)=(ab\gamma\delta\varepsilon)$ Gives only

$$0=0$$

4 $(ABCDE)=(a\beta c\delta\varepsilon)$ This one is also trivially satisfied

$$0=0$$

5 $(ABCDE)=(abc\delta\varepsilon)$

$$
\begin{aligned}
0 &= T_{\delta\varepsilon}{}^f H_{fabc}-T_{\varepsilon a}{}^\eta H_{\eta bc\delta}+T_{c\delta}{}^\eta H_{\eta\varepsilon ab}+T_{ab}{}^f H_{fc\delta\varepsilon}+T_{bc}{}^f H_{f\delta\varepsilon a} \\
&\quad +T_{\varepsilon c}{}^\eta H_{\eta a\delta b}-T_{ca}{}^f H_{f\delta b\varepsilon}-T_{a\delta}{}^\eta H_{\eta b\varepsilon c}-T_{\delta b}{}^\eta H_{\eta\varepsilon ca}+T_{b\varepsilon}{}^\eta H_{\eta ca\delta} \\
&= T_{\delta\varepsilon}{}^f H_{fabc}+3T_{[ab}{}^f H_{|f|c]\delta\varepsilon}-6T_{(\varepsilon[a}{}^\eta H_{|\eta|bc]\delta)} \\
&= 2i\left(\Gamma^f\right)_{\delta\varepsilon}H_{fabc}+3T_{[ab}{}^f H_{|f|c]\delta\varepsilon}-6T_{(\varepsilon[a}{}^\eta H_{|\eta|bc]\delta)} \qquad (3.12)
\end{aligned}
$$

6 $(ABCDE)=(abcd\varepsilon)$

$$
\begin{aligned}
0 &= D_\varepsilon H_{abcd} \\
&\quad +T_{ab}{}^\eta H_{cd\eta\varepsilon}-T_{bc}{}^\eta H_{da\eta\varepsilon}+T_{cd}{}^\eta H_{ab\eta\varepsilon}+T_{d\varepsilon}{}^f H_{fabc}+T_{\varepsilon a}{}^f H_{fbcd} \\
&\quad +T_{ad}{}^\eta H_{bc\eta\varepsilon}-T_{db}{}^\eta H_{ca\eta\varepsilon}-T_{ca}{}^\eta H_{db\eta\varepsilon}-T_{b\varepsilon}{}^f H_{fcad}-T_{\varepsilon c}{}^f H_{fadb} \\
&= D_\varepsilon H_{abcd}+6T_{[ab}{}^\eta H_{cd]\eta\varepsilon}
\end{aligned}
$$

7 $(ABCDE)=(abcde)$

$$
\begin{aligned}
0 &= 5D_{[a}H_{bcde]}+5T_{[ab}{}^f H_{|f|cde]}-5T_{[ad}{}^f H_{|f|bec]} \\
&= 5D_{[a}H_{bcde]}+10T_{[ab}{}^f H_{|f|cde]}
\end{aligned}
$$

3.4 The Constraints

From dimensional arguments we have found that the only non-vanishing components of H is H_{abcd} and $H_{ab\gamma\delta}$, where we have used the normalization $H_{ab\gamma\delta} = 2i\,(\Gamma_{ab})_{\gamma\delta}$. On the same grounds we have argued that the only non-vanishing components of T is $T_{ab}{}^{c}$, $T_{ab}{}^{\gamma}$, $T_{a\beta}{}^{\gamma}$ and $T_{\alpha\beta}{}^{c}$, where we used the normalization $T_{\alpha\beta}{}^{c} = 2i\,(\Gamma^{c})_{\alpha\beta}$. We can actually come a bit further since we want T to be expressed in the physical fields, i.e. H, S and R. We then see that we must have $T_{ab}{}^{c} = 0$ since there is no way of constructing something with this dimension and index structure out of the physical fields. In this case one can also argue that the spin connection is undetermined up to a tensor and we can therefore shift the spin connection in order to get $T_{ab}{}^{c} = 0$. We also see that the only possible form of $T_{a\beta}{}^{\gamma}$ is

$$T_{a\beta}{}^{\gamma} = xH_{abcd}\left(\Gamma^{bcd}\right)_{\beta}{}^{\gamma} + yH_{a_1a_2a_3a_4}\left(\Gamma_{a}{}^{a_1a_2a_3a_4}\right)_{\beta}{}^{\gamma} \tag{3.13}$$

in order to get the correct dimension and index structure. The coefficients x and y will be determined below. We can't however do much about the last component, $T_{ab}{}^{\gamma}$, but it will help to at least divide it into irreducible parts according to

$$T_{ab}{}^{\gamma} = G_{ab}{}^{\gamma} + 2G_{[a}{}^{\beta}\left(\Gamma_{b]}\right)_{\beta}{}^{\gamma} + G^{\beta}\left(\Gamma_{ab}\right)_{\beta}{}^{\gamma} \tag{3.14}$$

where

$$G_{ab}{}^{\gamma}\left(\Gamma^{b}\right)_{\gamma}{}^{\alpha} = G_{a}{}^{\beta}\left(\Gamma^{a}\right)_{\beta}{}^{\alpha} = 0$$

We are now ready to solve for x and y. From (3.12) we get

$$\begin{aligned}
2i\left(\Gamma^{f}\right)_{\delta\varepsilon}H_{fabc} &= 6T_{(\varepsilon[a}{}^{\eta}H_{|\eta|bc]\delta)} \\
&= -12i\Big(-xH_{a_1a_2a_3[a}\left(\Gamma^{a_1a_2a_3}\right)_{(\varepsilon}{}^{\eta} \\
&\quad +yH_{a_1a_2a_3a_4}\left(\Gamma_{[a}{}^{a_1a_2a_3a_4}\right)_{(\varepsilon}{}^{\eta}\Big)\left(\Gamma_{bc]}\right)_{\delta)\eta} \\
&= 12i\Big(xH_{a_1a_2a_3[a}\left(\Gamma^{a_1a_2a_3}\Gamma_{bc]}\right)_{(\varepsilon\delta)} \\
&\quad -yH_{a_1a_2a_3a_4}\left(\Gamma_{[a}{}^{a_1a_2a_3a_4}\Gamma_{bc]}\right)_{(\varepsilon\delta)}\Big) \\
&= 12i\Big(xH_{a_1a_2a_3[a}\left(\Gamma^{a_1a_2a_3}{}_{bc]} - 6\Gamma^{[a_1}\delta^{a_2a_3]}_{bc]}\right) \\
&\quad +10yH_{a_1a_2a_3a_4}\eta_{b_1[a}\left(\Gamma^{[b_1a_1a_2a_3}{}_{b}\delta^{a_4]}{}_{c]}\right)\Big)_{(\varepsilon\delta)}
\end{aligned} \tag{3.15}$$

where we have used

$$\Gamma^{abc}\Gamma_{de} = \Gamma^{abc}{}_{de} - 6\Gamma^{[ab}{}_{[d}\delta^{c]}{}_{e]} - 6\Gamma^{[a}\delta^{bc]}_{de}$$

and

$$\Gamma^{abcde}\Gamma_{fg} = \Gamma^{abcde}{}_{fg} - 10\Gamma^{[abcd}{}_{[f}\delta^{e]}{}_{g]} - 20\Gamma^{[abc}\delta^{de]}_{fg}$$

and that $\left(\Gamma^{(3)}\right)_{\alpha\beta}$ and $\left(\Gamma^{(7)}\right)_{\alpha\beta}$ are antisymmetric. We can now determine x and y by identifying terms in (3.15) , and starting with $\Gamma^{(1)}$ gives

$$\begin{aligned} \left(\Gamma^f\right)_{\delta\varepsilon} H_{fabc} &= -36xH_{a_1a_2a_3[a}\left(\Gamma^{[a_1}\delta^{a_2a_3]}_{bc]}\right)_{\delta\varepsilon} \\ &= -36x\left(\Gamma^f\right)_{\delta\varepsilon} H_{fabc} \end{aligned}$$

and thus

$$\boxed{x = -\tfrac{1}{36}}$$

Continuing with $\Gamma^{(5)}$ gives

$$\begin{aligned} 0 &= \frac{1}{36}H_{a_1a_2a_3[a}\Gamma^{a_1a_2a_3}{}_{bc]} - 10yH_{a_1a_2a_3a_4}\eta_{b_1[a}\left(\Gamma^{[b_1a_1a_2a_3}{}_b\delta^{a_4]}{}_{c]}\right) \\ &= \frac{1}{36}H_{a_1a_2a_3[a}\Gamma^{a_1a_2a_3}{}_{bc]} \\ &\quad -2y\left(H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{[b}\eta_{ac]} + 4H_{[c|a_1a_2a_3|}\Gamma^{a_1a_2a_3}{}_{ab]}\right) \\ &= \frac{1}{36}H_{a_1a_2a_3[a}\Gamma^{a_1a_2a_3}{}_{bc]} + 8yH_{a_1a_2a_3[a}\Gamma^{a_1a_2a_3}{}_{bc]} \end{aligned}$$

and thus

$$\boxed{y = -\tfrac{1}{288}}$$

3.5 Solving the Bianchi Identities

Before we start extracting information from the Bianchi identities it will be useful to collect what they have given us so far

$$\underline{DT^A = E^B \wedge R_B{}^A}$$

$(\beta\gamma\delta, a)$	$0 = 0$
$\left\{\begin{array}{l}(\beta\gamma\delta,\alpha)\\(\beta\gamma d, a)\end{array}\right.$	$R_{(\beta\gamma\delta)}{}^\alpha = 2i\left(\Gamma^e\right)_{(\beta\gamma}T_{\lvert e\rvert\delta)}{}^\alpha$ $R_{\beta\gamma d}{}^a = 4iT_{d(\beta}{}^\varepsilon\left(\Gamma^a\right)_{\lvert\varepsilon\rvert\gamma)}$
$\left\{\begin{array}{l}(b\gamma\delta,\alpha)\\(\beta cd, a)\end{array}\right.$	$R_{b(\gamma\delta)}{}^\alpha = D_{(\gamma}T_{\delta)b}{}^\alpha + i\left(\Gamma^e\right)_{\gamma\delta}T_{eb}{}^\alpha$ $R_{\beta[cd]}{}^a = i\left(\Gamma^a\right)_{\alpha\beta}T_{cd}{}^\alpha$
$\left\{\begin{array}{l}(bc\delta,\alpha)\\(bcd, a)\end{array}\right.$	$R_{bc\delta}{}^\alpha = D_\delta T_{bc}{}^\alpha + 2D_{[b}T_{c]\delta}{}^\alpha + 2T_{\delta[b}{}^\varepsilon T_{\lvert\varepsilon\rvert c]}{}^\alpha$ $R_{[bcd]}{}^a = 0$
(bcd,α)	$D_{[b}T_{cd]}{}^\alpha + T_{[bc}{}^\varepsilon T_{\lvert\varepsilon\rvert d]}{}^\alpha = 0$

(3.16)

and

$$\underline{dH_{(4)} = 0}$$

$(\alpha\beta\gamma\delta\varepsilon)$	$0 = 0$
$(a\beta\gamma\delta\varepsilon)$	$(\Gamma_a)_{(\alpha\beta}\left(\Gamma^{ab}\right)_{\gamma\delta)} = 0$
$(ab\gamma\delta\varepsilon)$	$0 = 0$
$(a\beta c\delta\varepsilon)$	$0 = 0$
$(abc\delta\varepsilon)$	$\left(\Gamma^f\right)_{\delta\varepsilon} H_{fabc} = 6T_{(\varepsilon[a}{}^{\eta}\left(\Gamma_{bc]}\right)_{\lvert\eta\rvert\delta)}$
$(abcd\varepsilon)$	$D_\varepsilon H_{abcd} + 12iT_{[ab}{}^{\eta}\left(\Gamma_{cd]}\right)_{\eta\varepsilon} = 0$
$(abcde)$	$D_{[a}H_{bcde]} = 0$

(3.17)

where we have collected the equations according to their dimension. We are now going to check the three equations of dimension 1 for consistency. We used one of the equations to determine x and y so we must check that the obtained values are consistent with the other two equations. We have

$$\begin{aligned} R_{\beta\gamma da} &= 4iT_{d(\beta}{}^{\varepsilon}\left(\Gamma_a\right)_{\lvert\varepsilon\rvert\gamma)} = \\ &= -4i\left(\frac{1}{36}H_{db_1b_2b_3}\Gamma^{b_1b_2b_3}\Gamma_a + \frac{1}{288}H_{a_1a_2a_3a_4}\Gamma_d{}^{a_1a_2a_3a_4}\Gamma_a\right)_{(\beta\gamma)} \\ &= -i\left(\frac{3}{9}H_{db_1b_2b_3}\Gamma^{[b_1b_2}\delta^{b_3]}{}_a + \frac{1}{72}H_{a_1a_2a_3a_4}\Gamma_d{}^{a_1a_2a_3a_4}{}_a\right)_{(\beta\gamma)} \\ &= -i\left(\frac{1}{3}H_{dab_1b_2}\Gamma^{b_1b_2} + \frac{1}{72}H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{da}\right)_{(\beta\gamma)} \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} R_{\beta\gamma\delta\alpha} &\equiv \frac{1}{4}R_{\beta\gamma da}\left(\Gamma^{da}\right)_{\delta\alpha} = [3.18] \\ &= -i\left(\frac{1}{12}H_{dab_1b_2}\Gamma^{b_1b_2} + \frac{1}{4\cdot 72}H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{da}\right)_{\beta\gamma}\left(\Gamma^{da}\right)_{\delta\alpha} \end{aligned}$$

From (3.16) we also get

$$\begin{aligned} R_{(\beta\gamma\delta)}{}^{\alpha} &= 2i\left(\Gamma^e\right)_{(\beta\gamma}T_{\lvert e\rvert\delta)}{}^{\alpha} \\ &= -2i\left(\Gamma^e\right)_{(\beta\gamma}\left(\frac{1}{36}H_{\lvert ebcd\rvert}\Gamma^{bcd} + \frac{1}{288}H_{\lvert a_1a_2a_3a_4}\Gamma_{e\rvert}{}^{a_1a_2a_3a_4}\right)_{\delta)}{}^{\alpha} \end{aligned}$$

which means that

$$\begin{aligned} &\left(24H_{cdb_1b_2}\Gamma^{b_1b_2} + H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{cd}\right)_{(\beta\gamma}\left(\Gamma^{cd}\right)_{\delta)}{}^{\alpha} \\ = &\left(\Gamma^e\right)_{(\beta\gamma}\left(16H_{\lvert ebcd\rvert}\Gamma^{bcd} + 2H_{\lvert a_1a_2a_3a_4}\Gamma_{e\rvert}{}^{a_1a_2a_3a_4}\right)_{\delta)}{}^{\alpha} \end{aligned} \tag{3.19}$$

must be satisfied for consistency. In order to check (3.19) we use the same method as used above, i.e. we contract with the symmetric Γ-matrices.

(i) $(\Gamma_a)^{\beta\gamma}$

$$
\begin{aligned}
& \left(24H_{cdb_1b_2}\mathrm{tr}\left[\Gamma^{b_1b_2}\Gamma_a\right] + H_{a_1a_2a_3a_4}\mathrm{tr}\left[\Gamma^{a_1a_2a_3a_4}{}_{cd}\Gamma_a\right]\right)\left(\Gamma^{cd}\right)_\delta{}^\alpha \\
& +2\left(24H_{cdb_1b_2}\Gamma^{b_1b_2} + H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{cd}\right)\left(\Gamma_a\Gamma^{cd}\right)_\delta{}^\alpha \\
& -\mathrm{tr}\left[\Gamma^e\Gamma_a\right]\left(16H_{ebcd}\Gamma^{bcd} + 2H_{a_1a_2a_3a_4}\Gamma_e{}^{a_1a_2a_3a_4}\right)_\delta{}^\alpha \\
& -2\left(\Gamma^e\Gamma_a\right)\left(16H_{ebcd}\Gamma^{bcd} + 2H_{a_1a_2a_3a_4}\Gamma_e{}^{a_1a_2a_3a_4}\right)_\delta{}^\alpha \\
= \; & [\text{drop } \delta \text{ and } \alpha] \\
= \; & \left(48H_{cdb_1b_2}\Gamma^{b_1b_2} + 2H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{cd}\right)\left(\Gamma^{cd}\Gamma_a + 4\delta_a{}^{[c}\Gamma^{d]}\right)_\delta{}^\alpha \\
& -32\left(16H_{abcd}\Gamma^{bcd} + 2H_{a_1a_2a_3a_4}\Gamma_a{}^{a_1a_2a_3a_4}\right) \\
& -2\left(-\Gamma_a\Gamma^e + 2\delta^e{}_a\right)\left(16H_{ebcd}\Gamma^{bcd} + 2H_{a_1a_2a_3a_4}\Gamma_e{}^{a_1a_2a_3a_4}\right) \\
= \; & 48H_{cdb_1b_2}\Gamma^{b_1b_2cd}\Gamma_a + 192H_{adb_1b_2}\Gamma^{b_1b_2d} - 84H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}\Gamma_a \\
& +8H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{ad}\Gamma^d \\
& -32\left(16H_{abcd}\Gamma^{bcd} + 2H_{a_1a_2a_3a_4}\Gamma_a{}^{a_1a_2a_3a_4}\right) \\
& +2\Gamma_a\left(16H_{ebcd}\Gamma^{ebcd} + 14H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}\right) \\
& -4\left(16H_{abcd}\Gamma^{bcd} + 2H_{a_1a_2a_3a_4}\Gamma_a{}^{a_1a_2a_3a_4}\right) \\
= \; & 48H_{cdb_1b_2}\left(\Gamma^{b_1b_2cd}{}_a + 4\Gamma^{b_1b_2c}\delta^d{}_a\right) + 192H_{adb_1b_2}\Gamma^{b_1b_2d} \\
& -84H_{a_1a_2a_3a_4}\left(\Gamma^{a_1a_2a_3a_4}{}_a + 4\Gamma^{a_1a_2a_3}\delta^{a_4}{}_a\right) + 48H_{bcde}\Gamma^{bcde}{}_a \\
& -32\left(16H_{abcd}\Gamma^{bcd} + 2H_{a_1a_2a_3a_4}\Gamma_a{}^{a_1a_2a_3a_4}\right) \\
& +32H_{ebcd}\left(\Gamma_a{}^{ebcd} + 4\delta_a{}^e\Gamma^{bcd}\right) \\
& +28H_{a_1a_2a_3a_4}\left(\Gamma_a{}^{a_1a_2a_3a_4} + 4\delta_a{}^{a_1}\Gamma^{a_2a_3a_4}\right) \\
& -4\left(16H_{abcd}\Gamma^{bcd} + 2H_{a_1a_2a_3a_4}\Gamma_a{}^{a_1a_2a_3a_4}\right) \\
= \; & H_{bcde}\Gamma_a{}^{bcde}\left(48 - 84 + 48 - 64 + 32 + 28 - 8\right) \\
& +H_{abcd}\Gamma^{bcd}\left(-192 + 192 + 336 - 512 + 128 + 112 - 64\right) \\
= \; & 0
\end{aligned}
$$

where we used that

$$\Gamma^{(k)}{}_c\Gamma^c = (11-k)\,\Gamma^{(k)}$$

$$\Gamma^{(k)}{}_{cd}\Gamma^{cd} = -(11-k)(10-k)\,\Gamma^{(k)}$$

and

$$\Gamma_a\Gamma^{bc} = \Gamma^{bc}\Gamma_a + 4\delta_a{}^{[b}\Gamma^{c]}$$

(ii) $(\Gamma_{ab})^{\beta\gamma}$

$$
\begin{aligned}
& \left(24H_{cdb_1b_2}\mathrm{tr}\left[\Gamma^{b_1b_2}\Gamma_{ab}\right] + H_{a_1a_2a_3a_4}\mathrm{tr}\left[\Gamma^{a_1a_2a_3a_4}{}_{cd}\Gamma_{ab}\right]\right)\left(\Gamma^{cd}\right)_\delta{}^\alpha \\
& 2\left(24H_{cdb_1b_2}\Gamma^{b_1b_2} + H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{cd}\right)\left(\Gamma_{ab}\Gamma^{cd}\right)_\delta{}^\alpha \\
& -\mathrm{tr}\left[\Gamma_{ab}\Gamma^f\right]\left(16H_{fcde}\Gamma^{cde} + 2H_{a_1a_2a_3a_4}\Gamma_f{}^{a_1a_2a_3a_4}\right)_\delta{}^\alpha \\
& -2\Gamma^f\Gamma_{ab}\left(16H_{fcde}\Gamma^{cde} + 2H_{a_1a_2a_3a_4}\Gamma_f{}^{a_1a_2a_3a_4}\right)_\delta{}^\alpha \\
= \; & [\text{drop } \delta \text{ and } \alpha] \\
= \; & -48\cdot 32H_{abcd}\Gamma^{cd} \\
& +\left(48H_{cdb_1b_2}\Gamma^{b_1b_2} + 2H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{cd}\right)\left(\Gamma^{cd}\Gamma_{ab} - 8\delta_{[a}{}^{[c}\Gamma_{b]}{}^{d]}\right) \\
& -\left(\Gamma_{ab}\Gamma^f + 4\delta^f{}_{[a}\Gamma_{b]}\right)\left(32H_{fcde}\Gamma^{cde} + 4H_{a_1a_2a_3a_4}\Gamma_f{}^{a_1a_2a_3a_4}\right) \\
= \; & -48\cdot 32H_{abcd}\Gamma^{cd} \\
& +48H_{cdb_1b_2}\Gamma^{b_1b_2cd}\Gamma_{ab} + 8\cdot 48H_{db_1b_2[a}\Gamma^{b_1b_2}\Gamma_{b]}{}^d \\
& -2\cdot 6\cdot 7H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}\Gamma_{ab} \\
& +16H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{[a|d|}\left(\Gamma^d\Gamma_{b]} - \delta^d{}_{b]}\right) \\
& -32H_{fcde}\Gamma_{ab}\Gamma^{fcde} + 4\cdot 32H_{cde[a}\left(\Gamma_{b]}{}^{cde} + 3\delta_{b]}{}^c\Gamma^{de}\right) \\
& -4\cdot 7H_{a_1a_2a_3a_4}\Gamma_{ab}\Gamma^{a_1a_2a_3a_4} + 16H_{a_1a_2a_3a_4}\Gamma_{[a}\Gamma_{b]}{}^{a_1a_2a_3a_4} \\
= \; & -48\cdot 32H_{abcd}\Gamma^{cd} - 36H_{cdef}\left(\Gamma^{cdef}{}_{ab} - 8\Gamma^{cde}{}_{[a}\delta^f{}_{b]} - 12\Gamma^{cd}\delta^{ef}_{[ab]}\right) \\
& +8\cdot 48H_{ecd[a}\left(-\Gamma^{cde}\Gamma_{b]} + \Gamma^{cd}\delta_{b]}{}^e\right) \\
& +16H_{a_1a_2a_3a_4}\left(6\Gamma^{a_1a_2a_3a_4}{}_{[a}\Gamma_{b]} - \Gamma^{a_1a_2a_3a_4}{}_{ab}\right) \\
& +128H_{cde[a}\Gamma_{b]}{}^{cde} - 384H_{abde}\Gamma^{de} \\
& -60H_{cdef}\left(\Gamma_{ab}{}^{cdef} - 8\delta_{[a}{}^{[c}\Gamma_{b]}{}^{def]} - 12\delta^{[cd}_{ab}\Gamma^{ef]}\right) \\
& +16H_{cdef}\left(\Gamma_{ab}{}^{cdef} - 4\delta^f{}_{[a}\Gamma^{cde}{}_{b]}\right) \\
= \; & -1536H_{abcd}\Gamma^{cd} - 36H_{cdef}\Gamma^{cdef}{}_{ab} - 288H_{cde[a}\Gamma^{cde}{}_{b]} \\
& +432H_{abcd}\Gamma^{cd} - 384\left(H_{cde[a}\Gamma^{cde}{}_{b]} - 3H_{abcd}\Gamma^{cd} + H_{abcd}\Gamma^{cd}\right) \\
& +16H_{cdef}\left(5\Gamma^{cdef}{}_{ab} - 24\Gamma^{cde}{}_{[a}\delta^f{}_{b]} + 6\Gamma^{cdef}\eta_{[ab]}\right) \\
& -128H_{cde[a}\Gamma^{cde}{}_{b]} - 384H_{abde}\Gamma^{de} \\
& -60H_{cdef}\Gamma_{ab}{}^{cdef} + 480H_{def[a}\Gamma^{def}{}_{b]} + 720H_{abef}\Gamma^{ef} \\
& +16H_{cdef}\Gamma^{cdef}{}_{ab} - 64H_{cde[a}\Gamma^{cde}{}_{b]} \\
= \; & +(-36+80-60+16)\,H_{cdef}\Gamma^{cdef}{}_{ab} \\
& +(-288-384+384-128+480-64)\,H_{cde[a}\Gamma^{cde}{}_{b]} \\
& +(432+768-1536-384+720)\,H_{abcd}\Gamma^{cd} \\
= \; & 0
\end{aligned}
$$

where we used that

$$\Gamma_{ab}\Gamma^{cd} = \Gamma^{cd}\Gamma_{ab} - 8\delta_{[a}{}^{[c}\Gamma_{b]}{}^{d]}$$

and

$$\begin{aligned}
\Gamma_{ab}\Gamma^{cdef} &= \Gamma_{ab}{}^{cdef} - 8\delta_{[a}{}^{[c}\Gamma_{b]}{}^{def]} - 12\delta^{[cd}_{ab}\Gamma^{ef]} \\
\Gamma^{abcd}\Gamma_{ef} &= \Gamma^{abcd}{}_{ef} - 8\Gamma^{[abc}{}_{[e}\delta^{d]}{}_{f]} - 12\Gamma^{[ab}\delta^{cd]}_{ef}
\end{aligned}$$

(iii) $\left(\Gamma_{(5)}\right)^{\beta\gamma}$ We first note that we use the definition

$$\mathrm{tr}\left[\Gamma^{a_1\cdots a_{11}}\right] \equiv 32\epsilon^{a_1\cdots a_{11}}$$

As above we now get

$$\begin{aligned}
& \left(24H_{cdb_1b_2}\mathrm{tr}\left[\Gamma^{b_1b_2}\Gamma_{(5)}\right] + H_{a_1a_2a_3a_4}\mathrm{tr}\left[\Gamma^{a_1a_2a_3a_4}{}_{cd}\Gamma_{(5)}\right]\right)\left(\Gamma^{cd}\right)_\delta{}^\alpha \\
& +2\left(24H_{cdb_1b_2}\Gamma^{b_1b_2} + H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{cd}\right)\left(\Gamma_{(5)}\Gamma^{cd}\right)_\delta{}^\alpha \\
& -\mathrm{tr}\left[\Gamma^e\Gamma_{(5)}\right]\left(16H_{ebcd}\Gamma^{bcd} + 2H_{a_1a_2a_3a_4}\Gamma_e{}^{a_1a_2a_3a_4}\right)_\delta{}^\alpha \\
& -2\left(\Gamma^e\Gamma_{(5)}\right)\left(16H_{ebcd}\Gamma^{bcd} + 2H_{a_1a_2a_3a_4}\Gamma_e{}^{a_1a_2a_3a_4}\right)_\delta{}^\alpha \\
= & \ [\text{drop } \delta \text{ and } \alpha] \\
& 32H_{abcd}\epsilon^{abcd}{}_{ef(5)}\Gamma^{ef} \\
& \left(48H_{cdb_1b_2}\Gamma^{b_1b_2} + 2H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{cd}\right)\left(\Gamma_{(5)}\Gamma^{cd}\right) \\
& -\left(\Gamma^e\Gamma_{(5)}\right)\left(32H_{ebcd}\Gamma^{bcd} + 4H_{a_1a_2a_3a_4}\Gamma_e{}^{a_1a_2a_3a_4}\right)
\end{aligned}$$

We now look at the five terms separately and use that

$$\begin{aligned}
\Gamma_{(5)}\Gamma^{ab} &= \Gamma_{(5)}{}^{ab} - 10\Gamma_{[c_1\cdots c_4}{}^{[a}\delta_{c_5]}{}^{b]} - 20\Gamma_{[c_1c_2c_3}\delta^{ab}_{c_4c_5]} \\
\Gamma^{ab}\Gamma_{(5)} &= \Gamma^{ab}{}_{(5)} - 10\delta^{[a}{}_{[c_1}\Gamma^{b]}{}_{c_2\cdots c_5]} - 20\delta^{ab}_{[c_1c_2}\Gamma_{c_3c_4c_5]}
\end{aligned}$$

and

$$\begin{aligned}
\Gamma_{(5)}\Gamma^a &= \Gamma_{(5)}{}^a + 5\Gamma_{[c_1\cdots c_4}\delta_{c_5]}{}^a \\
\Gamma^a\Gamma_{(5)} &= \Gamma^a{}_{(5)} + 5\delta^a{}_{[c_1}\Gamma_{c_2\cdots c_5]}
\end{aligned}$$

which gives

$$\Gamma_{(5)}\Gamma^{ab} = \Gamma^{ab}\Gamma_{(5)} - 20\Gamma_{[c_1\cdots c_4}{}^{[a}\delta_{c_5]}{}^{b]}$$

and

$$\Gamma^a\Gamma_{(5)} = -\Gamma_{(5)}\Gamma^a + 10\Gamma_{[c_1\cdots c_4}\delta_{c_5]}{}^a$$

We will also need

$$\begin{aligned}
\Gamma^{a_1\cdots a_4}\Gamma_{b_1\cdots b_5} = & \ \Gamma^{a_1\cdots a_4}{}_{b_1\cdots b_5} - 20\delta^{[a_1}{}_{[b_1}\Gamma^{a_2a_3a_4]}{}_{b_2\cdots b_5]} \\
& -120\delta^{[a_1a_2}_{[b_1b_2}\Gamma^{a_3a_4]}{}_{b_3b_4b_5]} + 240\delta^{[a_1a_2a_3}_{[b_1b_2b_3}\Gamma^{a_4]}{}_{b_4b_5]} \\
& +120\delta^{a_1a_2a_3a_4}_{[b_1b_2b_3b_4}\Gamma_{b_5]}
\end{aligned}$$

1.

$$\begin{aligned}
& 32 H_{abcd}\epsilon^{abcd}{}_{ef(5)}\Gamma^{ef} \\
= \;& 32 H_{abcd}\epsilon^{abcd}{}_{(5)ef}\Gamma^{ef} \\
= \;& 32 H_{abcd}\Gamma^{abcd}{}_{(5)ef}\Gamma^{ef} \\
= \;& -32\cdot 2 H_{abcd}\Gamma^{abcd}{}_{(5)}
\end{aligned}$$

2.

$$\begin{aligned}
& 48 H_{cdb_1b_2}\Gamma^{b_1b_2}\Gamma_{(5)}\Gamma^{cd} \\
= \;& 48 H_{cdb_1b_2}\Gamma^{b_1b_2}\left(\Gamma^{cd}\Gamma_{(5)} - 20\Gamma_{[c_1\cdots c_4}{}^{[c}\delta_{c_5]}{}^{d]}\right) \\
= \;& 48 H_{abcd}\Gamma^{abcd}\Gamma_{(5)} - 48\cdot 20 H_{abc[c_1}\Gamma^{ab}\Gamma^{c}{}_{c_2\cdots c_5]} \\
= \;& 48 H_{abcd}\Big(\Gamma^{abcd}{}_{b_1\cdots b_5} - 20\delta^{[a}{}_{[b_1}\Gamma^{bcd]}{}_{b_2\cdots b_5]} \\
& -120\delta^{[ab}_{[b_1b_2}\Gamma^{cd]}{}_{b_3b_4b_5]} + 240\delta^{[abc}_{[b_1b_2b_3}\Gamma^{d]}{}_{b_4b_5]} + 120\delta^{abcd}_{[b_1b_2b_3b_4}\Gamma_{b_5]}\Big) \\
& -48\cdot 20 H_{abc[c_1}\eta^{c_6c}\Big(\Gamma^{ab}{}_{c_2\cdots c_5]c_6} - 8\delta^{[a}{}_{c_2}\Gamma^{b]}{}_{c_3\cdots c_5]c_6} - 12\delta^{ab}_{c_2c_3}\Gamma_{c_4c_5]c_6}\Big) \\
= \;& 48 H_{abcd}\Gamma^{abcd}{}_{(5)} + 48\cdot 20 H_{abc[b_1}\Gamma^{abc}{}_{b_2\cdots b_5]} \\
& -48\cdot 120 H_{ab[b_1b_2}\Gamma^{ab}{}_{b_3b_4b_5]} - 48\cdot 240 H_{a[b_1b_2b_3}\Gamma^{a}{}_{b_4b_5]} \\
& +48\cdot 120 H_{[b_1b_2b_3b_4}\Gamma_{b_5]} \\
& -48\cdot 20 H_{abc[c_1}\left(\Gamma^{abc}{}_{c_2\cdots c_5]} + 8\delta^{a}{}_{c_2}\Gamma^{bc}{}_{c_3\cdots c_5]} - 12\delta^{ab}_{c_2c_3}\Gamma^{c}{}_{c_4c_5]}\right) \\
= \;& 48 H_{abcd}\Gamma^{abcd}{}_{(5)} + 48\cdot 20 H_{abc[b_1}\Gamma^{abc}{}_{b_2\cdots b_5]} \\
& -48\cdot 120 H_{ab[b_1b_2}\Gamma^{ab}{}_{b_3b_4b_5]} - 48\cdot 240 H_{a[b_1b_2b_3}\Gamma^{a}{}_{b_4b_5]} \\
& +48\cdot 120 H_{[b_1b_2b_3b_4}\Gamma_{b_5]} \\
& -48\cdot 20\left(H_{abc[c_1}\Gamma^{abc}{}_{c_2\cdots c_5]} - 8 H_{ab[c_1c_2}\Gamma^{ab}{}_{c_3\cdots c_5]} - 12 H_{a[c_1c_2c_3}\Gamma^{a}{}_{c_4c_5]}\right) \\
= \;& 48 H_{abcd}\Gamma^{abcd}{}_{(5)} + 48\cdot 40 H_{ab[b_1b_2}\Gamma^{ab}{}_{b_3b_4b_5]} + 0 H_{a[b_1b_2b_3}\Gamma^{a}{}_{b_4b_5]} \\
& +48\cdot 120 H_{[b_1b_2b_3b_4}\Gamma_{b_5]}
\end{aligned}$$

3.

$$\begin{aligned}
& 2 H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{cd}\Gamma_{(5)}\Gamma^{cd} \\
= \;& 2 H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{cd}\left(\Gamma^{cd}\Gamma_{(5)} - 20\Gamma_{[c_1\cdots c_4}{}^{[c}\delta_{c_5]}{}^{d]}\right) \\
= \;& -2\cdot 6\cdot 7 H_{abcd}\Gamma^{abcd}\Gamma_{(5)} + 2\cdot 20 H_{a_1a_2a_3a_4}\Gamma^{a_1a_2a_3a_4}{}_{[c_1|c|}\Gamma^{c}{}_{c_2\cdots c_5]} \\
= \;& -84 H_{abcd}\Gamma^{abcd}\Gamma_{(5)} \\
& +2\cdot 20 H_{abcd}\Gamma^{abcd}{}_{[c_1|e|}\left(\Gamma^{e}\Gamma_{c_2\cdots c_5]} - 4\delta^{e}{}_{c_2}\Gamma_{c_3\cdots c_5]}\right) \\
= \;& -84 H_{abcd}\Gamma^{abcd}\Gamma_{(5)} \\
& +2\cdot 20 H_{abcd}\left(6\Gamma^{abcd}{}_{[c_1}\Gamma_{c_2\cdots c_5]} - 4\Gamma^{abcd}{}_{[c_1c_2}\Gamma_{c_3\cdots c_5]}\right)
\end{aligned}$$

The first term here have we already calculated in 2 above and the two last terms becomes

$$
\begin{aligned}
& H_{abcd}\Gamma^{abcd}{}_{[c_1}\Gamma_{c_2\cdots c_5]} \\
= \; & H_{abcd}\eta_{e[c_1}\Gamma^{abcde}\Gamma_{c_2\cdots c_5]} \\
= \; & H_{abcd}\eta_{e[c_1}\Big(\Gamma^{abcde}{}_{c_2\cdots c_5]} + 20\delta^{[a}{}_{c_2}\Gamma^{bcde]}{}_{c_3c_4c_5]} \\
& -120\delta^{[ab}_{c_2c_3}\Gamma^{cde]}{}_{c_4c_5]} - 240\delta^{[abc}_{c_2c_3c_4}\Gamma^{de]}{}_{c_5]} + 120\delta^{[abcd}_{c_2c_3c_4c_5]}\Gamma^{e]}\Big) \\
= \; & H_{abcd}\Gamma^{abcd}{}_{(5)} + 16H_{abc[c_1}\Gamma^{abc}{}_{c_2\cdots c_5]} \\
& -72H_{ab[c_1c_2}\Gamma^{ab}{}_{c_3c_4c_5]} - 96H_{a[c_1c_2c_3}\Gamma^{a}{}_{c_4c_5]} + 24H_{[c_1c_2c_3c_4}\Gamma_{c_5]}
\end{aligned}
$$

and

$$
\begin{aligned}
& H_{abcd}\Gamma^{abcd}{}_{[c_1c_2}\Gamma_{c_3\cdots c_5]} \\
= \; & H_{abcd}\eta_{e[c_1}\eta_{|f|c_2}\Gamma^{abcdef}\Gamma_{c_3\cdots c_5]} \\
= \; & H_{abcd}\eta_{e[c_1}\eta_{|f|c_2}\Big(\Gamma^{abcdef}{}_{c_3c_4c_5]} - 18\delta^{[a}{}_{c_3}\Gamma^{bcdef]}{}_{c_4c_5]} \\
& -90\delta^{[ab}_{c_3c_4}\Gamma^{cdef]}{}_{c_5]} + 120\delta^{[abc}_{c_3c_4c_5]}\Gamma^{def]}\Big) \\
= \; & H_{abcd}\Gamma^{abcd}{}_{(5)} + 12H_{abc[c_1}\Gamma^{abc}{}_{c_2c_3c_4c_5]} \\
& -36H_{ab[c_1c_2}\Gamma^{ab}{}_{c_3c_4c_5]} - 24H_{a[c_1c_2c_3}\Gamma^{a}{}_{c_4c_5]}
\end{aligned}
$$

where we used that

$$
\begin{aligned}
\Gamma^{a_1\cdots a_5}\Gamma_{b_1\cdots b_4} = \; & \Gamma^{a_1\cdots a_5}{}_{b_1\cdots b_4} + 20\delta^{[a_1}{}_{[b_1}\Gamma^{a_2\cdots a_5]}{}_{b_2b_3b_4]} \\
& -120\delta^{[a_1a_2}_{[b_1b_2}\Gamma^{a_3a_4a_5]}{}_{b_3b_4]} - 240\delta^{[a_1a_2a_3}_{[b_1b_2b_3}\Gamma^{a_4a_5]}{}_{b_4]} \\
& +120\delta^{[a_1a_2a_3a_4}_{b_1b_2b_3b_4}\Gamma^{a_5]}
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma^{a_1\cdots a_6}\Gamma_{b_1b_2b_3} = \; & \Gamma^{a_1\cdots a_6}{}_{b_1b_2b_3} - 18\delta^{[a_1}{}_{[b_1}\Gamma^{a_2\cdots a_6]}{}_{b_2b_3]} \\
& -90\delta^{[a_1a_2}_{[b_1b_2}\Gamma^{a_3\cdots a_6]}{}_{b_3]} + 120\delta^{[a_1a_2a_3}_{b_1b_2b_3}\Gamma^{a_4a_5a_6]}
\end{aligned}
$$

Collecting all terms we get

$$
\begin{aligned}
& +H_{abcd}\Gamma^{abcd}{}_{(5)}(-84+240-160) \\
& +H_{abc[c_1}\Gamma^{abc}{}_{c_2c_3c_4c_5]}(-84\cdot 20 + 240\cdot 16 - 160\cdot 12) \\
& +H_{ab[c_1c_2}\Gamma^{ab}{}_{c_3c_4c_5]}(+84\cdot 120 - 240\cdot 72 + 160\cdot 36) \\
& +H_{a[c_1c_2c_3}\Gamma^{a}{}_{c_4c_5]}(+84\cdot 240 - 240\cdot 96 + 160\cdot 24) \\
& +H_{[c_1c_2c_3c_4}\Gamma_{c_5]}(-84\cdot 120 + 240\cdot 24) \\
= \; & -4H_{abcd}\Gamma^{abcd}{}_{(5)} + 240H_{abc[c_1}\Gamma^{abc}{}_{c_2c_3c_4c_5]} \\
& -1440H_{ab[c_1c_2}\Gamma^{ab}{}_{c_3c_4c_5]} + 960H_{a[c_1c_2c_3}\Gamma^{a}{}_{c_4c_5]} - 4320H_{[c_1c_2c_3c_4}\Gamma_{c_5]}
\end{aligned}
$$

4.

$$
\begin{aligned}
& -32 H_{ebcd} \Gamma^e \Gamma_{(5)} \Gamma^{bcd} \\
= \; & -32 H_{ebcd} \left(-\Gamma_{(5)} \Gamma^e + 10 \Gamma_{[c_1 \cdots c_4} \delta_{c_5]}{}^e \right) \Gamma^{bcd} \\
= \; & +32 H_{abcd} \Gamma_{(5)} \Gamma^{abcd} + 10 \cdot 32 H_{abc[c_1} \Gamma_{c_2 \cdots c_5]} \Gamma^{abc}
\end{aligned}
$$

We look at the two terms separately and get

$$
\begin{aligned}
& H_{abcd} \Gamma_{(5)} \Gamma^{abcd} \\
= \; & H_{abcd} \left(\Gamma^{abcd}{}_{(5)} + 20 \delta_{[b_1}{}^{[a} \Gamma_{b_2 \cdots b_5]}{}^{bcd]} \right. \\
& \left. -120 \delta^{[ab}_{[b_1 b_2} \Gamma_{b_3 b_4 b_5]}{}^{cd]} - 240 \delta^{[abc}_{[b_1 b_2 b_3} \Gamma_{b_4 b_5]}{}^{d]} + 120 \delta^{abcd}_{[b_1 b_2 b_3 b_4} \Gamma_{b_5]} \right) \\
= \; & H_{abcd} \Gamma^{abcd}{}_{(5)} - 20 H_{abc[b_1} \Gamma^{abc}{}_{b_2 \cdots b_5]} - 120 H_{ab[b_1 b_2} \Gamma_{b_3 b_4 b_5]}{}^{ab} \\
& +240 H_{a[b_1 b_2 b_3} \Gamma^a{}_{b_4 b_5]} + 120 H_{[b_1 b_2 b_3 b_4} \Gamma_{b_5]}
\end{aligned}
$$

and

$$
\begin{aligned}
& H_{abc[c_1} \Gamma_{c_2 \cdots c_5]} \Gamma^{abc} \\
= \; & H_{abc[c_1} \left(\Gamma_{c_2 \cdots c_5]}{}^{abc} - 12 \delta_{c_2}{}^{[a} \Gamma_{c_3 c_4 c_5]}{}^{bc]} \right. \\
& \left. -36 \delta^{[ab}_{c_2 c_3} \Gamma_{c_4 c_5]}{}^{c]} + 24 \delta^{abc}_{c_2 c_3 c_4} \Gamma_{c_5]} \right) \\
= \; & H_{abc[c_1} \Gamma^{abc}{}_{c_2 \cdots c_5]} + 12 H_{ab[c_1 c_2} \Gamma^{ab}{}_{c_3 c_4 c_5]} \\
& -36 H_{a[c_1 c_2 c_3} \Gamma^a{}_{c_4 c_5]} - 24 H_{[c_1 c_2 c_3 c_4} \Gamma_{c_5]}
\end{aligned}
$$

where we used that

$$
\begin{aligned}
\Gamma_{a_1 \cdots a_4} \Gamma^{b_1 b_2 b_3} = \; & \Gamma_{a_1 \cdots a_4}{}^{b_1 b_2 b_3} - 12 \delta_{[a_1}{}^{[b_1} \Gamma_{a_2 a_3 a_4]}{}^{b_2 b_3]} \\
& -36 \delta^{[b_1 b_2}_{[a_1 a_2} \Gamma_{a_3 a_4]}{}^{b_3]} + 24 \delta^{b_1 b_2 b_3}_{[a_1 a_2 a_3} \Gamma_{a_4]}
\end{aligned}
$$

Collecting terms we get

$$
\begin{aligned}
& +H_{abcd} \Gamma^{abcd}{}_{(5)} (32) \\
& +H_{abc[c_1} \Gamma^{abc}{}_{c_2 \cdots c_5]} (-20 \cdot 32 + 320) \\
& +H_{ab[c_1 c_2} \Gamma^{ab}{}_{c_3 c_4 c_5]} (-32 \cdot 120 + 320 \cdot 12) \\
& +H_{a[c_1 c_2 c_3} \Gamma^a{}_{c_4 c_5]} (32 \cdot 240 - 320 \cdot 36) \\
& +H_{[c_1 c_2 c_3 c_4} \Gamma_{c_5]} (32 \cdot 120 - 320 \cdot 24) \\
= \; & 32 H_{abcd} \Gamma^{abcd}{}_{(5)} - 320 H_{abc[c_1} \Gamma^{abc}{}_{c_2 \cdots c_5]} + 0 H_{ab[c_1 c_2} \Gamma^{ab}{}_{c_3 c_4 c_5]} \\
& -3840 H_{a[c_1 c_2 c_3} \Gamma^a{}_{c_4 c_5]} - 3840 H_{[c_1 c_2 c_3 c_4} \Gamma_{c_5]}
\end{aligned}
$$

5.

$$
\begin{aligned}
& -4H_{a_1a_2a_3a_4}\Gamma^e\Gamma_{(5)}\Gamma_e{}^{a_1a_2a_3a_4} \\
= \;& -4H_{abcd}\left(-\Gamma_{(5)}\Gamma^e + 10\Gamma_{[c_1\cdots c_4}\delta_{c_5]}{}^e\right)\Gamma_e{}^{abcd} \\
= \;& +4H_{abcd}\Gamma_{(5)}\Gamma^e\Gamma_e{}^{abcd} - 4\cdot 10 H_{abcd}\Gamma_{[c_1\cdots c_4}\Gamma_{c_5]}{}^{abcd} \\
= \;& 4\cdot 7H_{abcd}\Gamma_{(5)}\Gamma^{abcd} - 4\cdot 10H_{abcd}\eta_{e[c_5}\Gamma_{c_1\cdots c_4]}\Gamma^{abcde}
\end{aligned}
$$

The first term was calculated in 4 and the second term becomes

$$
\begin{aligned}
& H_{abcd}\eta_{e[c_5}\Gamma_{c_1\cdots c_4]}\Gamma^{abcde} \\
= \;& H_{abcd}\eta_{e[c_5}\left(\Gamma_{c_1\cdots c_4]}{}^{abcde} - 20\delta_{c_1}{}^{[a}\Gamma_{c_2c_3c_4]}{}^{bcde]}\right. \\
& \left. -120\delta^{[ab}_{c_1c_2}\Gamma_{c_3c_4]}{}^{cde]} + 240\delta^{[abc}_{c_1c_2c_3}\Gamma_{c_4]}{}^{de]} + 120\delta^{[abcd}_{c_1c_2c_3c_4]}\Gamma^{e]}\right) \\
= \;& H_{abcd}\Gamma^{abcd}{}_{(5)} - 16H_{abc[c_1}\Gamma^{abc}{}_{c_2\cdots c_5]} - 72H_{ab[c_1c_2}\Gamma^{ab}{}_{c_3c_4c_5]} \\
& +96H_{a[c_1c_2c_3}\Gamma^a{}_{c_4c_5]} + 24H_{[c_1c_2c_3c_4}\Gamma_{c_5]}
\end{aligned}
$$

where we used that

$$
\begin{aligned}
\Gamma_{a_1\cdots a_4}\Gamma^{b_1\cdots b_5} \;=\;& \Gamma_{a_1\cdots a_4}{}^{b_1\cdots b_5} - 20\delta_{[a_1}{}^{[b_1}\Gamma_{a_2a_3a_4]}{}^{b_2\cdots b_5]} \\
& -120\delta^{[b_1b_2}_{[a_1a_2}\Gamma_{a_3a_4]}{}^{b_3b_4b_5]} + 240\delta^{[b_1b_2b_3}_{[a_1a_2a_3}\Gamma_{a_4]}{}^{b_4b_5]} \\
& +120\delta^{[b_1b_2b_3b_4}_{a_1a_2a_3a_4}\Gamma^{b_5]}
\end{aligned}
$$

Collecting the terms gives us

$$
\begin{aligned}
& +H_{abcd}\Gamma^{abcd}{}_{(5)}\,(28-40) \\
& +H_{abc[c_1}\Gamma^{abc}{}_{c_2\cdots c_5]}\,(-28\cdot 20 + 40\cdot 16) \\
& +H_{ab[c_1c_2}\Gamma^{ab}{}_{c_3c_4c_5]}\,(-28\cdot 120 + 40\cdot 72) \\
& +H_{a[c_1c_2c_3}\Gamma^a{}_{c_4c_5]}\,(28\cdot 240 - 40\cdot 96) \\
& +H_{[c_1c_2c_3c_4}\Gamma_{c_5]}\,(28\cdot 120 - 40\cdot 24) \\
= \;& -12H_{abcd}\Gamma^{abcd}{}_{(5)} + 80H_{abc[c_1}\Gamma^{abc}{}_{c_2\cdots c_5]} - 480H_{ab[c_1c_2}\Gamma^{ab}{}_{c_3c_4c_5]} \\
& +2880H_{a[c_1c_2c_3}\Gamma^a{}_{c_4c_5]} + 2400H_{[c_1c_2c_3c_4}\Gamma_{c_5]}
\end{aligned}
$$

We now see that collecting the results from all the five terms gives us

$$
\begin{aligned}
& +H_{abcd}\Gamma^{abcd}{}_{(5)}\,(-64+48-4+32-12) \\
& +H_{abc[c_1}\Gamma^{abc}{}_{c_2\cdots c_5]}\,(240-320+80) \\
& +H_{ab[c_1c_2}\Gamma^{ab}{}_{c_3c_4c_5]}\,(40-30+0-10)\cdot 48 \\
& +H_{a[c_1c_2c_3}\Gamma^a{}_{c_4c_5]}\,(0+20-80+60)\cdot 48 \\
& +H_{[c_1c_2c_3c_4}\Gamma_{c_5]}\,(120-90-80+50)\cdot 48 \\
= \;& 0
\end{aligned}
$$

We are now ready to proceed to the equations of dimension 3/2. From

$$R_{\beta[cd]a} = \frac{1}{2}\left(R_{\beta cda} - R_{\beta dca}\right) = \frac{1}{2}\left(R_{\beta cda} + R_{\beta dac}\right)$$

we get

$$R_{\beta[cd]a} - R_{\beta[da]c} + R_{\beta[ac]d} = R_{\beta cda}$$

and analogously

$$R_{b(\gamma\delta)\alpha} = \frac{1}{2}\left(R_{b\gamma\delta\alpha} + R_{b\delta\gamma\alpha}\right) = \frac{1}{2}\left(R_{b\gamma\delta\alpha} + R_{b\delta\alpha\gamma}\right)$$

gives

$$R_{b(\gamma\delta)\alpha} - R_{b(\delta\alpha)\gamma} + R_{b(\alpha\gamma)\delta} = R_{b\gamma\delta\alpha}$$

where we used that

$$R_{b\gamma\delta\alpha} \equiv \frac{1}{4} R_{b\gamma da}\left(\Gamma^{da}\right)_{\delta\alpha}$$

We can now check the two equations with dimension 3/2 in (3.16)

$$R_{b(\gamma\delta)}{}^{\alpha} = D_{(\gamma}T_{\delta)b}{}^{\alpha} + i\left(\Gamma^{e}\right)_{\gamma\delta} T_{eb}{}^{\alpha}$$

and

$$R_{\beta[cd]}{}^{a} = i\left(\Gamma^{a}\right)_{\alpha\beta} T_{cd}{}^{\alpha}$$

With the formulas obtained above this gives us

$$\begin{gathered} D_{(\beta}T_{\delta)c\alpha} + i\left(\Gamma^{e}\right)_{\beta\delta} T_{ec\alpha} - D_{(\delta}T_{\alpha)c\beta} - i\left(\Gamma^{e}\right)_{\delta\alpha} T_{ec\beta} \\ + D_{(\alpha}T_{\beta)c\delta} + i\left(\Gamma^{e}\right)_{\alpha\beta} T_{ec\delta} \\ = -\tfrac{i}{4}\left(\Gamma^{da}\right)_{\delta\alpha}\left(T_{cd}\Gamma_{a} - T_{da}\Gamma_{c} + T_{ac}\Gamma_{d}\right)_{\beta} \end{gathered}$$

By using the last equation of dimension 3/2, i.e. the one in (3.17)

$$D_{\varepsilon} H_{abcd} = -12 i T_{[ab}{}^{\eta}\left(\Gamma_{cd]}\right)_{\eta\varepsilon}$$

we get

$$\begin{aligned} D_{(\beta}T_{\delta)a\alpha} &= -D_{(\beta}T_{|a|\delta)\alpha} = \frac{1}{36} D_{(\beta} H_{abcd}\left(\Gamma^{bcd}\right)_{\delta)\alpha} + \frac{1}{288} D_{(\beta} H_{bcde}\left(\Gamma_{a}{}^{bcde}\right)_{\delta)\alpha} \\ &= -\frac{12i}{36}\left(T_{[ab}\Gamma_{cd]}\right)_{(\beta}\left(\Gamma^{bcd}\right)_{\delta)\alpha} - \frac{12i}{288}\left(T_{[bc}\Gamma_{de]}\right)_{(\beta}\left(\Gamma_{a}{}^{bcde}\right)_{\delta)\alpha} \end{aligned}$$

and thus

$$\begin{aligned} & \frac{1}{4}\left(\Gamma^{dc}\right)_{\delta\alpha}\left(T_{ad}\Gamma_{c} - T_{dc}\Gamma_{a} + T_{ca}\Gamma_{d}\right)_{\beta} \\ & + \left(\Gamma^{e}\right)_{\beta\delta} T_{ea\alpha} - \left(\Gamma^{e}\right)_{\delta\alpha} T_{ea\beta} + \left(\Gamma^{e}\right)_{\alpha\beta} T_{ea\delta} \\ = \; & + \frac{1}{3}\left(T_{[ab}\Gamma_{cd]}\right)_{(\beta}\left(\Gamma^{bcd}\right)_{\delta)\alpha} + \frac{1}{24}\left(T_{[bc}\Gamma_{de]}\right)_{(\beta}\left(\Gamma_{a}{}^{bcde}\right)_{\delta)\alpha} \\ & - \frac{1}{3}\left(T_{[ab}\Gamma_{cd]}\right)_{(\delta}\left(\Gamma^{bcd}\right)_{\alpha)\beta} - \frac{1}{24}\left(T_{[bc}\Gamma_{de]}\right)_{(\delta}\left(\Gamma_{a}{}^{bcde}\right)_{\alpha)\beta} \\ & + \frac{1}{3}\left(T_{[ab}\Gamma_{cd]}\right)_{(\alpha}\left(\Gamma^{bcd}\right)_{\beta)\delta} + \frac{1}{24}\left(T_{[bc}\Gamma_{de]}\right)_{(\alpha}\left(\Gamma_{a}{}^{bcde}\right)_{\beta)\delta} \end{aligned}$$

By observing that this expression is symmetric in δ and α we contract it with the symmetric Γ-matrices.

(i) $(\Gamma_f)^{\delta\alpha}$

$$\begin{aligned}
& 2\left(T_{ea}\Gamma_f\Gamma^e\right)_\beta - 32T_{fa\beta} \\
= & -\frac{1}{6}\left(T_{[ab}\Gamma_{cd]}\Gamma_f\Gamma^{bcd}\right)_\beta - \frac{1}{3}\left(T_{[ab}\Gamma_{cd]}\Gamma_f\Gamma^{bcd}\right)_\beta - \frac{1}{6}\left(T_{[ab}\Gamma_{cd]}\Gamma_f\Gamma^{bcd}\right)_\beta \\
& \frac{1}{48}\left(T_{[bc}\Gamma_{de]}\Gamma_f\Gamma_a{}^{bcde}\right)_\beta - \frac{1}{24}\left(T_{[bc}\Gamma_{de]}\Gamma_f\Gamma_a{}^{bcde}\right)_\beta \\
& +\frac{1}{48}\left(T_{[bc}\Gamma_{de]}\Gamma_f\Gamma_a{}^{bcde}\right)_\beta \qquad (3.20) \\
= & -\frac{2}{3}\left(T_{[ab}\Gamma_{cd]}\Gamma_f\Gamma^{bcd}\right)_\beta
\end{aligned}$$

By using the decomposition into irreducible parts according to (3.14) we get

$$\begin{aligned}
T_{ab}\Gamma^b &= G_a\Gamma_b\Gamma^b - G_b\Gamma_a\Gamma^b + G\Gamma_{ab}\Gamma^b \\
&= 11G_a - G_b\delta^b{}_a + 10G\Gamma_a \\
&= 9G_a + 10G\Gamma_a
\end{aligned}$$

$$T_{ab}\Gamma^{ab} = -T_{ab}\Gamma^b\Gamma^a = -9G_a\Gamma^a - 10G\Gamma_a\Gamma^a = -110G$$

and

$$\begin{aligned}
T_{ab}\Gamma^{abc} &= T_{ab}\left(\Gamma^{ab}\Gamma^c - 2\Gamma^{[a}\eta^{b]c}\right) = -110G\Gamma^c + T^c{}_b\Gamma^b \\
&= -110G\Gamma^c + 18G^c + 20G\Gamma^c = 18G^c - 90G\Gamma^c
\end{aligned}$$

Since $G_{ab}{}^\gamma$ is analogous to the Weyl tensor in GR we want it to drop out and we also want to obtain conditions on G and G_a. By multiplying (3.20) by η^{af} we get

$$\begin{aligned}
& 2T_{ea}\Gamma^a\Gamma^e - 32T_{fa}\eta^{af} + \frac{2}{3}T_{[ab}\Gamma_{cd]}\Gamma^a\Gamma^{bcd} \\
= & \ 2\cdot 110G + \frac{1}{3}T_{ab}\Gamma_{cd}\Gamma^a\Gamma^{bcd} + \frac{1}{3}T_{cd}\Gamma_{ab}\Gamma^a\Gamma^{bcd} \\
= & \ 220G + \frac{1}{3}T_{ab}\left(\Gamma^a\Gamma_{cd} - 4\delta^a{}_{[c}\Gamma_{d]}\right)\Gamma^{bcd} - \frac{10\cdot 9}{3}T_{cd}\Gamma^{cd} \\
= & \ 220G - \frac{1}{3}\left(9G_b + 10G\Gamma_b\right)\Gamma_{cd}\Gamma^{cdb} - \frac{4\cdot 9}{3}T_{cb}\Gamma^{bc} + 30\cdot 110G \\
= & \ 32\cdot 110G + \frac{10\cdot 10\cdot 9\cdot 11}{3}G - 12\cdot 110G \\
= & \ 0
\end{aligned}$$

and thus

$$\boxed{G=0}$$

If we instead multiply (3.20) by Γ^f we get

$$\begin{aligned}
& 2T_{ea}\Gamma_f\Gamma^e\Gamma^f - 32T_{fa}\Gamma^f + \frac{2}{3}T_{[ab}\Gamma_{cd]}\Gamma_f\Gamma^{bcd}\Gamma^f \\
= \; & +2\cdot 11T_{ae}\Gamma^e - 4T_{ae}\Gamma^e + 32\cdot 9G_a - \frac{2\cdot 5}{3}T_{[ab}\Gamma_{cd]}\Gamma^{bcd} \\
= \; & 48\cdot 9G_a - \frac{5}{3}T_{ab}\Gamma_{cd}\Gamma^{bcd} - \frac{5}{3}T_{cd}\left(\Gamma_a\Gamma_b - \eta_{ab}\right)\Gamma^{bcd} \\
= \; & 48\cdot 9G_a + \frac{5\cdot 10\cdot 9}{3}9G_a - \frac{5\cdot 9}{3}T_{cd}\left(\Gamma^{cd}\Gamma_a + 4\delta_a{}^{[c}\Gamma^{d]}\right) + \frac{5}{3}T_{cd}\Gamma^{cd}{}_a \\
= \; & 198\cdot 48G_a - 15\cdot 4\cdot 9G_a + \frac{5}{3}\cdot 18G_a \\
= \; & 0
\end{aligned}$$

and thus

$$\boxed{G_a=0}$$

From the formulas above we thus see that

$$T_{ab}\Gamma^a = T_{ab}\Gamma^{ab} = 0$$

$$T_{ab}\Gamma^{abc} = 0$$

The last relation above is the equation of motion for the spin 3/2 field which is seen by varying the Lagrangian

$$\bar{\psi}_a\Gamma^{abc}\partial_b\psi_c$$

which gives

$$\partial_b\psi_c\Gamma^{abc} = 0$$

and from the index structure we have

$$T_{ab}{}^{\gamma} \sim \partial_{[a}\psi_{b]}$$

We can now see that $G_{ab}{}^{\gamma}$ actually drops out by using $G = G_a = 0$ in (3.20) which gives us

$$\begin{aligned}
& 2T_{ea}\Gamma_f\Gamma^e - 32T_{fa} + \frac{2}{3}T_{[ab}\Gamma_{cd]}\Gamma_f\Gamma^{bcd} \\
= \; & 2T_{ae}\Gamma^e\Gamma_f - 4T_{af} - 32T_{fa} + \frac{1}{3}T_{ab}\Gamma_{cd}\Gamma_f\Gamma^{bcd} + \frac{1}{3}T_{cd}\Gamma_{ab}\Gamma_f\Gamma^{bcd} \\
= \; & 28T_{af} + \frac{1}{3}T_{ab}\left(\Gamma_f\Gamma_{cd} - 4\eta_{f[c}\Gamma_{d]}\right)\Gamma^{cdb}
\end{aligned}$$

$$
\begin{aligned}
& +\frac{1}{3}T_{cd}\left(\Gamma_f\Gamma_{ab} - 4\eta_{f[a}\Gamma_{b]}\right)\Gamma^{bcd} \\
= \;& 28T_{af} - \frac{9\cdot 10}{3}T_{ab}\Gamma_f\Gamma^b - \frac{4}{3}T_{ab}\Gamma_d\Gamma^{db}{}_f + \frac{1}{3}T_{cd}\Gamma_f\left(\Gamma_a\Gamma_b - \eta_{ab}\right)\Gamma^{bcd} \\
& -\frac{2}{3}T_{cd}\left(\eta_{fa}\Gamma_b\Gamma^{bcd} - \eta_{fb}\Gamma_a\Gamma^{bcd}\right) \\
= \;& (28-60)\,T_{af} - \frac{4\cdot 9}{3}T_{ab}\Gamma^b{}_f + \frac{9}{3}T_{cd}\Gamma_f\Gamma_a\Gamma^{cd} - \frac{1}{3}T_{cd}\Gamma_f\Gamma^{cd}{}_a \\
& +\frac{2}{3}T_{cd}\Gamma_a\Gamma^{cd}{}_f \\
= \;& (28-60+12)\,T_{af} + 3T_{cd}\Gamma_f\left(\Gamma^{cd}\Gamma_a + 4\delta_a{}^{[c}\Gamma^{d]}\right) \\
& -\frac{1}{3}T_{cd}\eta_{ea}\left(\Gamma^{cde}\Gamma_f + 6\delta_f{}^{[c}\Gamma^{de]}\right) + \frac{2}{3}T_{cd}\eta_{ef}\left(\Gamma^{cde}\Gamma_a + 6\delta_a{}^{[c}\Gamma^{de]}\right) \\
= \;& (28-60+12)\,T_{af} + 3\cdot 4T_{cd}\delta_f{}^{[c}\Gamma^{d]}\Gamma_a + 12\cdot 2T_{af} - \frac{2}{3}T_{fd}\Gamma^d{}_a \\
& +\frac{4}{3}T_{ad}\Gamma^d{}_f \\
= \;& \left(28-60+12+24-\frac{4}{3}-\frac{8}{3}\right)T_{af} = 0
\end{aligned}
$$

Contracting with $(\Gamma_{fg})^{\delta\alpha}$ and $\left(\Gamma_{(5)}\right)^{\delta\alpha}$ gives nothing new, which we leave to the reader to verify. We now proceed to the equations with dimension 2. From (3.16) we get

$$R_{bc\delta\alpha} = D_\delta T_{bc\alpha} + 2D_{[b}T_{c]\delta\alpha} + 2T_{\delta[b}{}^{\varepsilon}T_{|\varepsilon|c]\alpha}$$

and contracting with $(\Gamma_{da})^{\delta\alpha}$ gives us

$$\frac{1}{4}R_{bcef}\left(\Gamma^{ef}\right)_{\delta\alpha}(\Gamma_{da})^{\delta\alpha} = \frac{1}{4}R_{bcef}\mathrm{tr}\left[-2\delta^{[ef]}_{da}\right] = -16R_{bcda}$$

and

$$
\begin{aligned}
& D_\delta T_{bc\alpha}(\Gamma_{da})^{\delta\alpha} + 2D_{[b}T_{c]\delta\alpha}(\Gamma_{da})^{\delta\alpha} + 2T_{\delta[b}{}^{\varepsilon}T_{|\varepsilon|c]\alpha}(\Gamma_{da})^{\delta\alpha} \\
= \;& [3.13] = D_\delta T_{bc\alpha}(\Gamma_{da})^{\delta\alpha} + 2\mathrm{tr}\left[T_{[b}T_{c]}\Gamma_{da}\right]
\end{aligned}
$$

so we have

$$R_{abcd} = -\frac{1}{16}D_\delta T_{ab\alpha}(\Gamma_{cd})^{\delta\alpha} - \frac{1}{8}\mathrm{tr}\left[T_{[a}T_{b]}\Gamma_{cd}\right]$$

Multiplying by η^{ac} then leads to

$$-16R_{bd} = D_\delta T_{bc}\Gamma_d{}^c + 2\mathrm{tr}\left[T_{[b}T_{c]}\Gamma_d{}^c\right]$$

and symmetrizing in b and d gives

$$R_{bd} = -\frac{1}{16}\left(\mathrm{tr}\left[T_{(b}T_{|c|}\Gamma_{d)}{}^{c} - T_{c}T_{(b}\Gamma_{d)}{}^{c}\right]\right)$$

where we have used that R_{bd} is symmetric since we have from the Bianchi identities (3.16) that

$$R_{[bcd]}{}^{a} = 0$$

and thus

$$\begin{aligned} 0 &= R_{[bcd]}{}^{a}\delta_{a}{}^{c} = \frac{1}{3}R_{dba}{}^{a} + \frac{1}{3}\left(R_{bd} - R_{db}\right) \\ &\Rightarrow R_{bd} = R_{db} \end{aligned}$$

We continue our work on R_{bd} and get

$$-\frac{1}{16}\mathrm{tr}\left[T_{(b}T_{|c|}\Gamma_{d)}{}^{c}\right] = \frac{1}{16}\mathrm{tr}\left[\Gamma_{(d}T_{b)}T_{c}\Gamma^{c}\right] - \frac{1}{16}\mathrm{tr}\left[T_{(b}T_{d)}\right]$$

and

$$\frac{1}{16}\mathrm{tr}\left[T_{c}T_{(b}\Gamma_{d)}{}^{c}\right] = \frac{1}{16}\mathrm{tr}\left[\Gamma^{c}T_{c}T_{(b}\Gamma_{d)}\right] - \frac{1}{16}\mathrm{tr}\left[T_{(d}T_{b)}\right]$$

By using our ansatz (3.13) we now get

$$\begin{aligned} \mathrm{tr}\left[T_{(b}T_{d)}\right] &= \frac{32\cdot 24}{288^2}\eta_{bd}H^2 - 32\cdot\left(\frac{96}{288^2} + \frac{6}{36^2}\right)H_{a(3)}H_{b}{}^{(3)} \\ &= \frac{8}{3\cdot 288}\eta_{bd}H^2 - \frac{5}{27}H_{a(3)}H_{b}{}^{(3)} \end{aligned}$$

$$\begin{aligned} \mathrm{tr}\left[\Gamma^{c}T_{c}T_{(b}\Gamma_{d)}\right] &= \mathrm{tr}\Bigg[\Gamma^{a}\left(-\frac{1}{72}H_{a(3)}\Gamma^{(3)} - \frac{1}{288}\Gamma_{a}\Gamma^{(4)}H_{(4)}\right) \\ &\qquad \left(-\frac{1}{24}H_{(b|(\tilde{3})|}\Gamma^{(\tilde{3})} - \frac{1}{288}\Gamma^{(\tilde{4})}\Gamma_{(b}H_{(\tilde{4})}\right)\Gamma_{d)}\Bigg] \\ &= \mathrm{tr}\Bigg[\left(-\frac{1}{72}H_{(4)}\Gamma^{(4)} - \frac{11}{288}\Gamma^{(4)}H_{(4)}\right) \\ &\qquad \left(-\frac{1}{24}\Gamma^{(\tilde{3})}{}_{(d}H_{b)(\tilde{3})} - \frac{1}{288}\eta_{bd}\Gamma^{(\tilde{4})}H_{(\tilde{4})}\right)\Bigg] \\ &= \frac{5\cdot 32\cdot 24}{96}\left(-\frac{1}{24}H_{a(3)}H_{b}{}^{(3)} + \frac{1}{288}\eta_{bd}H^2\right) \\ &= \frac{5}{36}\eta_{bd}H^2 - \frac{5}{3}H_{a(3)}H_{b}{}^{(3)} \end{aligned}$$

and

$$\begin{aligned}
\mathrm{tr}\left[\Gamma_{(d}T_{b)}T_c\Gamma^c\right] &= \mathrm{tr}\left[\Gamma_{(d}\left(-\frac{1}{72}H_{b)(3)}\Gamma^{(3)} - \frac{1}{288}\Gamma_{b)}\Gamma^{(4)}H_{(4)}\right)\right.\\
&\qquad\left.\left(-\frac{1}{24}H_{a(3)}\Gamma^{(3)} - \frac{1}{288}\Gamma^{(4)}\Gamma_a H_{(4)}\right)\Gamma^a\right]\\
&= \mathrm{tr}\left[\left(-\frac{1}{72}\Gamma_{(d}{}^{(3)}H_{b)(3)} - \frac{1}{288}\eta_{bd}\Gamma^{(4)}H_{(4)}\right)\right.\\
&\qquad\left.\left(+\frac{1}{24}H_{a(3)}\Gamma^{a(3)} - \frac{11}{288}\Gamma^{(4)}H_{(4)}\right)\right]\\
&= 32\cdot 24\left(-\frac{1}{72\cdot 288}H_d{}^{(3)}H_{b(3)} - \frac{1}{288^2}\eta_{bd}H^2\right)\\
&= -\frac{1}{108}\eta_{bd}H^2 - \frac{1}{27}H_d{}^{(3)}H_{b(3)}
\end{aligned}$$

where we used that

$$\begin{aligned}
T_a &= -\frac{1}{36}H_{a(3)}\Gamma^{(3)} - \frac{1}{288}\Gamma_a{}^{(4)}H_{(4)}\\
&= -\frac{1}{72}H_{a(3)}\Gamma^{(3)} - \frac{1}{288}\Gamma_a\Gamma^{(4)}H_{(4)}\\
&= -\frac{1}{24}H_{a(3)}\Gamma^{(3)} - \frac{1}{288}\Gamma^{(4)}\Gamma_a H_{(4)}
\end{aligned}$$

Collecting the results now gives us

$$\begin{aligned}
R_{bd} &= \frac{1}{16}\left(\mathrm{tr}\left[\Gamma_{(d}T_{b)}T_c\Gamma^c\right] - \mathrm{tr}\left[T_{(b}T_{d)}\right] + \mathrm{tr}\left[\Gamma^c T_c T_{(b}\Gamma_{d)}\right] - \mathrm{tr}\left[T_{(d}T_{b)}\right]\right)\\
&= \frac{1}{16}\left(-\frac{1}{108}\eta_{bd}H^2 - \frac{1}{27}H_d{}^{(3)}H_{b(3)} - \frac{2\cdot 8}{3\cdot 288}\eta_{bd}H^2\right.\\
&\qquad\left.+\frac{5\cdot 2}{27}H_{a(3)}H_b{}^{(3)} + \frac{5}{36}\eta_{bd}H^2 - \frac{5}{3}H_{a(3)}H_b{}^{(3)}\right)\\
&= \frac{1}{144}\eta_{bd}H^2 - \frac{1}{12}H_{a(3)}H_b{}^{(3)}
\end{aligned}$$

and

$$R = \left(\frac{11}{144} - \frac{1}{12}\right)H^2 = -\frac{1}{144}H^2$$

Einstein's equations are thus

$$\begin{aligned}
R_{ab} - \frac{1}{2}\eta_{ab}R &= \frac{1}{144}\eta_{ab}H^2 - \frac{1}{12}H_{a(3)}H_b{}^{(3)} + \frac{1}{2\cdot 144}\eta_{ab}H^2\\
&= \frac{1}{96}\eta_{ab}H^2 - \frac{1}{12}H_{a(3)}H_b{}^{(3)}
\end{aligned}$$

We can obtain another equation with dimension 2. From (3.16) we have

$$\begin{aligned} 0 &= R_{[ab|\alpha\delta|}\left(\Gamma_{c]}\right)^{\alpha\delta} \\ &= \left(D_\delta T_{[ab|\alpha|} + 2D_{[a}T_{b|\delta\alpha|} + 2T_{\delta[a}{}^{\varepsilon}T_{|\varepsilon|b|\alpha|}\right)\left(\Gamma_{c]}\right)^{\alpha\delta} \\ &= \left(\Gamma_{[a}\right)^{\alpha\delta} D_{|\delta|}T_{bc]\alpha} + 2\mathrm{tr}\left[T_{[a}T_b\Gamma_{c]}\right] \end{aligned}$$

where we used the ansatz for $T_{a\delta\alpha}$. We also have that

$$\begin{aligned} \mathrm{tr}\left[T_{[a}T_b\Gamma_{c]}\right] &= \frac{1}{288^2}\mathrm{tr}\left[\Gamma_{[a}{}^{(4)}\Gamma_b{}^{(\tilde{4})}\Gamma_{c]}\right] H_{(4)}H_{(\tilde{4})} \\ &= \frac{32}{288^2}\epsilon_{abc}{}^{(4)(\tilde{4})}H_{(4)}H_{(\tilde{4})} \end{aligned}$$

and from (3.17) we get

$$\begin{aligned} \left(\Gamma^c\right)_\beta{}^\varepsilon D_\varepsilon H_{abcd} &= -12i\left(\Gamma^c\right)_\beta{}^\varepsilon T_{[ab}{}^\eta\left(\Gamma_{cd]}\right)_{\eta\varepsilon} \\ &= -6i\left(\Gamma^c\Gamma_{c[a}T_{bd]}\right)_\beta + 6i\left(\Gamma^c\Gamma_{[ab}T_{d]c}\right)_\beta \\ &= -36i\left(\Gamma_{[a}T_{bd]}\right)_\beta \end{aligned}$$

and thus

$$D^\beta T_{[ab}\left(\Gamma_{d]}\right)_\beta = \frac{i}{36}\left(\Gamma^c\right)_\beta{}^\varepsilon D^\beta D_\varepsilon H_{abcd}$$

and

$$0 = \frac{i}{36}\left(\Gamma^c\right)_\beta{}^\varepsilon D^\beta D_\varepsilon H_{abcd} + \frac{2\cdot 32}{288^2}\epsilon_{abc}{}^{(4)(\tilde{4})}H_{(4)}H_{(\tilde{4})}$$

To get further we must use that

$$DDV^A = D\left(E^B D_B V^A\right) = T^B D_B V^A + E^B E^C D_C D_B V^A$$

and

$$\begin{aligned} DDV^A &= D\left(dV^A + V^B\wedge\omega_B{}^A\right) \\ &= d\left(V^B\wedge\omega_B{}^A\right) + dV^B\wedge\omega_B{}^A + V^C\wedge\omega_C{}^B\wedge\omega_B{}^A \\ &= -dV^B\wedge\omega_B{}^A + V^B\wedge d\omega_B{}^A + dV^B\wedge\omega_B{}^A + V^C\wedge\omega_C{}^B\wedge\omega_B{}^A \\ &= V^B\wedge R_B{}^A \end{aligned}$$

And thus

$$D_{[A}D_{B\}}V^C = V^D\wedge R_{ABD}{}^C - \frac{1}{2}T_{AB}{}^D D_D V^C$$

We now get

$$\begin{aligned} \left(\Gamma^c\right)^{\beta\varepsilon} D_\beta D_\varepsilon H_{abcd} &= \left(\Gamma^c\right)^{\beta\varepsilon}\left(-4R_{\beta\varepsilon[a}{}^e H_{|e|bcd]} - \frac{1}{2}T_{\beta\varepsilon}{}^e D_e H_{abcd}\right) \\ &= [3.16] = -i\mathrm{tr}\left[\Gamma^c\Gamma^e\right] D_e H_{abcd} \\ &= -32iD^c H_{abcd} \end{aligned}$$

and finally we have

$$D^d H_{dabc} = -\frac{3}{2 \cdot 36 \cdot 48} \epsilon_{abc}{}^{(4)(\tilde{4})} H_{(4)} H_{(\tilde{4})}$$

The last equation with dimension 5/2 gives nothing new so we are now finished with the Bianchi identities.

4

Supermembranes

When one has taken the step from point particles to strings it is only natural to continue to even higher-dimensional objects like membranes etc and Dirac originally proposed this in 1962. It was however far from clear how to incorporate supersymmetry in this theory and not until 1986 when Hughes, Liu and Polchinski published their article the supermembrane was born. They succeeded in generalizing the $\varkappa$-symmetry to membranes, a property crucial to the Green-Schwarz superparticle and superstring.

If we have a spacetime with D dimensions we can not have a p dimensional object, called a p-brane, living there for arbitrary p. It is the requirement of supersymmetry that puts constraints on the dimensions. We now have manifest spacetime supersymmetry instead of imposing the supersymmetry on the worldvolume as in chapter 2. There can be said to be four "fundamental" super p-branes and each of these gives rise to a sequence of branes obtained through dimensional reduction as suggested by Kaluza and Klein. To illustrate this we can make a *brane scan* [13]. For an introduction to supermembranes see [13].

4.1 The $p = 2$ Brane

We are now going to present the action for the $p = 2$ brane in eleven-dimensional supergravity and verify an important feature of this action, namely the $\varkappa$-symmetry. We have

$$S^{D=11}_{p=2} = \int d^3\xi \left(-\frac{1}{2}\sqrt{\gamma}\gamma^{ij}\Pi_i^a\Pi_j^b\eta_{ab} + \frac{1}{2}\sqrt{\gamma} - \frac{1}{3!}\varepsilon^{ijk}\Pi_i^A\Pi_j^B\Pi_k^C B_{CBA} \right)$$

where γ^{ij} is the metric of the world volume, $\gamma = -det\gamma_{ij}$, η_{ab} is the $d = 11$ Minkowski spacetime metric and the pullback is defined as

$$\Pi_i^A = \left(\partial_i Z^M\right) E_M{}^A$$

where

$$\begin{aligned} Z^M &= (X^m, \theta^\mu)(\xi) \\ E_M{}^A &= E_M{}^A(Z) \end{aligned}$$

The 3-form

$$B = \frac{1}{3!} E^A E^B E^C B_{CBA}$$

where

$$B_{ABC} = B_{ABC}(Z)$$

is the potential to the 4-form H, i.e.

$$H = dB \tag{4.1}$$

The $p = 2$ brane is thoroughly studied in [14] but be aware of the difference in constraints used, which for example gives rise to the differing factor 1/3! in the action. The $\varkappa$-symmetry in particular is discussed in [15].

4.2 The $\varkappa$-symmetry

In order to get an equal number of bosonic and fermionic degrees of freedom the action must have $\varkappa$-symmetry which is defined as

$$\begin{aligned} \left(\delta_\varkappa Z^M\right) E_M{}^a &= 0 \\ \left(\delta_\varkappa Z^M\right) E_M{}^\alpha &= (1+\Gamma)^\alpha{}_\beta \varkappa^\beta = \tilde{\varkappa}^\alpha \end{aligned} \tag{4.2}$$

where

$$\Gamma = \frac{1}{6\sqrt{\gamma}} \varepsilon^{ijk} \Pi_i^a \Pi_j^b \Pi_k^c \Gamma_{abc}$$

and $\varkappa^\beta(\xi)$ is an anticommuting spacetime spinor. We here use the so called "1.5 formulation" where we use the equation of motion for γ_{ij}

$$\gamma_{ij} = \Pi_i^a \Pi_j^b \eta_{ab} \tag{4.3}$$

Using (4.3) we can show that

$$\Gamma^2 = 1$$

which means that the factor $(1+\Gamma)$ occurring in (4.2) is a projection operator. The proof goes like

$$\begin{aligned}
\Gamma^2 &= \frac{1}{36\gamma}\varepsilon^{ijk}\Pi_i^a\Pi_j^b\Pi_k^c\varepsilon_{lmn}\Pi_d^l\Pi_e^m\Pi_f^n\Gamma_{abc}\Gamma^{def} \\
&= \frac{1}{36\gamma}\varepsilon^{ijk}\Pi_i^a\Pi_j^b\Pi_k^c\varepsilon_{lmn}\Pi_d^l\Pi_e^m\Pi_f^n \times \\
&\quad \left(\Gamma_{abc}{}^{def} + 9\Gamma_{[ab}\delta_{c]}{}^{[d}\Gamma^{ef]} - 18\delta_{[a}{}^{[d}\delta_b{}^e\Gamma_{c]}{}^{f]} - 6\delta_{[a}^d\delta_b^e\delta_{c]}^f\right) \\
&= \frac{9}{36\gamma}\varepsilon^{ijk}\Pi_i^{[a}\Pi_j^b\Pi_k^{c]}\varepsilon_{lmn}\Pi_{[c}^l\Pi_e^m\Pi_{f]}^n\Gamma_{ab}\Gamma^{ef} \\
&\quad -\frac{18}{36\gamma}\varepsilon^{ijk}\Pi_i^{[a}\Pi_j^b\Pi_k^{c]}\varepsilon_{lmn}\Pi_{[a}^l\Pi_b^m\Pi_{f]}^n\Gamma_c{}^f \\
&\quad -\frac{6}{36\gamma}\varepsilon^{ijk}\Pi_i^{[a}\Pi_j^b\Pi_k^{c]}\varepsilon_{lmn}\Pi_{[a}^l\Pi_b^m\Pi_{c]}^n \\
&= -\frac{6}{36\gamma}\varepsilon^{ijk}\varepsilon_{ijk} = \left[\varepsilon_{ijk} = -\gamma\varepsilon^{ijk}\right] = \frac{6}{36}\varepsilon^{ijk}\varepsilon^{ijk} = 1
\end{aligned}$$

where we have used that $i, j, k, \ldots$ only can take three values. We are now ready to verify the $\varkappa$-symmetry.

$$\delta_\varkappa S = \int d^3\xi \left(\underbrace{-\sqrt{\gamma}\gamma^{ij}\left(\delta_\varkappa\Pi_i^a\right)\Pi_j^b\eta_{ab}}_{I} - \underbrace{\delta_\varkappa\left(\frac{1}{3!}\varepsilon^{ijk}\Pi_i^A\Pi_j^B\Pi_k^C B_{CBA}\right)}_{II}\right)$$

To evaluate term I we need

$$\begin{aligned}
\delta_\varkappa\Pi_i^A &= \left(\partial_i\delta_\varkappa Z^M\right)E_M{}^A + \partial_i Z^M\left(\delta_\varkappa E_M{}^A\right) \\
&= \partial_i\tilde{\varkappa}^A - \left(\delta_\varkappa Z^M\right)\partial_i E_M{}^A + \partial_i Z^M\left(\delta_\varkappa E_M{}^A\right) \\
&= -\left(\delta_\varkappa Z^M\right)\partial_i E_M{}^A + \partial_i Z^M\delta_\varkappa Z^N\partial_N E_M{}^A \\
&= -\tilde{\varkappa}^M\partial_i E_M{}^A + \partial_i Z^M\tilde{\varkappa}^N\partial_N E_M{}^A \\
&= -\tilde{\varkappa}^M\Pi_i^N\partial_N E_M{}^A + \partial_i Z^M\tilde{\varkappa}^N\partial_N E_M{}^A \\
&= -\tilde{\varkappa}^M\partial_i Z^N\partial_N E_M{}^A + \partial_i Z^M\tilde{\varkappa}^N\partial_N E_M{}^A \\
&= 2\partial_i Z^M\tilde{\varkappa}^N\partial_{[N}E_{M\}}{}^A \\
&= 2\Pi_i^B E_B{}^M\tilde{\varkappa}^N\partial_{[N}E_{M\}}{}^A
\end{aligned}$$

We must now use that (3.4) gives

$$\frac{1}{2}dZ^{M_1}dZ^{M_2}T_{M_2M_1}{}^A = dZ^{M_1}\partial_{M_1}dZ^{M_2}E_{M_2}{}^A + dZ^{M_1}E_{M_1}{}^B dZ^{M_2}\omega_{M_2B}{}^A$$

$$\begin{aligned}
&\Rightarrow \quad \frac{1}{2} T_{M_2 M_1}{}^A = \partial_{[M_2} E_{M_1\}}{}^A - (-)^{BM_2}\, E_{[M_2}{}^B \omega_{M_1\}B}{}^A \\
&\Rightarrow \quad \partial_{[M_2} E_{M_1\}}{}^A = \frac{1}{2} T_{M_2 M_1}{}^A + (-)^{BM_2}\, E_{[M_2}{}^B \omega_{M_1\}B}{}^A
\end{aligned}$$

and we then get

$$\delta_\varkappa \Pi_i^A = \underbrace{\Pi_i^B E_B{}^M \tilde{\varkappa}^N T_{NM}{}^A}_{III} + \underbrace{(-)^{BN}\, 2\Pi_i^C E_C{}^M \tilde{\varkappa}^N E_{[N}{}^B \omega_{M\}B}{}^A}_{IV}$$

By now using

$$T_{NM}{}^A = (-)^{N(M+C)}\, E_M{}^C E_N{}^B T_{BC}{}^A$$

we get

$$\begin{aligned}
III &= (-)^{N(M+C)}\, \Pi_i^D E_D{}^M \tilde{\varkappa}^N E_M{}^C E_N{}^B T_{BC}{}^A \\
&= \Pi_i^C \tilde{\varkappa}^\beta T_{\beta C}{}^A
\end{aligned}$$

By using (3.1) and that in our case $A = a$, we get

$$\begin{aligned}
IV &= -(-)^{N(B+M)}\, \Pi_i^C E_C{}^M \tilde{\varkappa}^N E_M{}^B \omega_{NB}{}^a \\
&= -\Pi_i^c \tilde{\varkappa}^N \omega_{Nc}{}^a
\end{aligned}$$

which vanish when we take into account the symmetries of the additional terms $\gamma^{ij}\Pi_j^b \eta_{ab}$ and use that $\omega_{MB}{}^A$ is antisymmetric in the last two indices. Collecting the results so far we get

$$I = -\sqrt{\gamma}\gamma^{ij}\Pi_i^C \Pi_j^a \tilde{\varkappa}^\beta T_{\beta C a}$$

From the constraints discussed in chapter 3 we see that we get a non-vanishing contribution only when $C = \gamma$ and thus

$$I = -2i\sqrt{\gamma}\gamma^{ij}\Pi_i^\gamma \Pi_j^a \tilde{\varkappa}^\beta \left(\Gamma_a\right)_{\beta\gamma}$$

To get further we must analyze

$$\begin{aligned}
\tilde{\varkappa}^\beta \left(\Gamma_d\right)_{\beta\gamma} &= (1+\Gamma)^\beta{}_\alpha \varkappa^\alpha \left(\Gamma_d\right)_{\beta\gamma} \\
&= \left(1 + \frac{1}{6\sqrt{\gamma}} \varepsilon^{ijk} \Pi_i^a \Pi_j^b \Pi_k^c \Gamma_{abc}\right)^\beta{}_\alpha \varkappa^\alpha \left(\Gamma_d\right)_{\beta\gamma}
\end{aligned}$$

We now divide this expression into terms proportional to $\Gamma^{(n)}$. We don't get a term proportional to $\Gamma^{(4)}$ since this term must be antisymmetric in four world-volume coordinates, and the world-volume is only three-dimensional. We get two types of terms

$$\frac{1}{6\sqrt{\gamma}}\varepsilon^{ijk}\Pi_i^a\Pi_j^b\Pi_k^c\left(3\eta_{ad}\Gamma_{bc}\varkappa\right)_\gamma$$

and

$$\left(\varkappa\Gamma_d\right)_\gamma$$

By using the equation of motion for γ_{ij} (4.3) we now get

$$\begin{aligned}
I &= -2i\sqrt{\gamma}\gamma^{lm}\Pi_l^\gamma\Pi_m^d\left(\frac{1}{2\sqrt{\gamma}}\varepsilon^{ijk}\Pi_i^a\Pi_j^b\Pi_k^c\eta_{ad}\Gamma_{bc}\varkappa+\varkappa\Gamma_d\right)_\gamma \qquad (4.4)\\
&= -2i\sqrt{\gamma}\Pi_l^\gamma\left(\frac{1}{2\sqrt{\gamma}}\varepsilon^{ljk}\Pi_j^b\Pi_k^c\Gamma_{bc}\varkappa+\gamma^{lm}\Pi_m^d\varkappa\Gamma_d\right)_\gamma
\end{aligned}$$

In order to analyze II we will use some concepts of topology [16]. We first note that the term we are varying in II is actually a 3-form. One can also see that the $\varkappa$-variation can be written as a Lie derivative. We thus have

$$\delta_\varkappa\hat{B}=\mathcal{L}_\varkappa\hat{B}\equiv\left(di_\varkappa+i_\varkappa d\right)\hat{B}=di_\varkappa\hat{B}+i_\varkappa\hat{H}$$

where the hat denotes pullback. The first term is just a total derivative and by using that

$$i_\varkappa\hat{H}=\frac{1}{3!}d^3\xi\varepsilon^{ijk}\Pi_i^A\Pi_j^B\Pi_k^C\tilde{\varkappa}^\gamma H_{\gamma CBA}$$

we get

$$II=\frac{1}{3!}\varepsilon^{ijk}\Pi_i^A\Pi_j^B\Pi_k^C\tilde{\varkappa}^\gamma H_{\gamma CBA}$$

From the constraints we see only one term of H survive and we have

$$II=\frac{3\cdot 2i}{3!}\varepsilon^{ijk}\Pi_i^a\Pi_j^b\Pi_k^\delta\tilde{\varkappa}^\gamma\left(\Gamma_{ba}\right)_{\gamma\delta}$$

We now get three types of terms

$$\begin{aligned}
&\frac{i}{6\sqrt{\gamma}}\varepsilon^{ijk}\varepsilon^{lmn}\Pi_i^a\Pi_j^b\Pi_k^\delta\Pi_l^d\Pi_m^e\Pi_n^f\left(\Gamma_{ba}\Gamma_{def}\varkappa\right)_\delta\\
=\;&\left[\text{keep terms } \sim\Gamma_{(3)}\right]\\
=\;&\frac{i}{\sqrt{\gamma}}\varepsilon^{ijk}\varepsilon^{lmn}\Pi_i^a\Pi_j^b\Pi_k^\delta\Pi_l^d\Pi_m^e\Pi_n^f\left(\eta_{ad}\Gamma_{bef}\varkappa\right)_\delta\\
=\;&\frac{i}{\sqrt{\gamma}}\varepsilon^{ijk}\varepsilon^{lmn}\gamma_{il}\Pi_j^b\Pi_k^\delta\Pi_m^e\Pi_n^f\left(\Gamma_{bef}\varkappa\right)_\delta\\
=\;&\frac{i}{\sqrt{\gamma}}\left(\gamma^{jn}\gamma^{km}-\gamma^{jm}\gamma^{kn}\right)\Pi_j^b\Pi_k^\delta\Pi_m^e\Pi_n^f\left(\Gamma_{bef}\varkappa\right)_\delta\\
=\;&0
\end{aligned}$$

$$i\varepsilon^{ijk}\Pi_i^a\Pi_j^b\Pi_k^\delta\varkappa^\gamma\left(\Gamma_{ba}\right)_{\gamma\delta}$$

and

$$\begin{aligned}
& \frac{i}{6\sqrt{\gamma}}\varepsilon^{ijk}\varepsilon^{lmn}\Pi_i^a\Pi_j^b\Pi_k^\delta\Pi_l^d\Pi_m^e\Pi_n^f\left(\Gamma_{ba}\Gamma_{def}\varkappa\right)_\delta \\
= & \left[\text{keep terms } \sim \Gamma_{(1)}\right] \\
= & \frac{-i}{\sqrt{\gamma}}\varepsilon^{ijk}\varepsilon^{lmn}\Pi_{ai}\Pi_{bj}\Pi_k^\delta\Pi_l^d\Pi_m^e\Pi_n^f\left(\Gamma_f\delta_{de}^{ba}\varkappa\right)_\delta \\
= & \frac{-i}{\sqrt{\gamma}}\varepsilon^{ijk}\varepsilon^{lmn}\gamma_{im}\gamma_{jl}\Pi_k^\delta\Pi_n^f\left(\Gamma_f\varkappa\right)_\delta \\
= & -2i\sqrt{\gamma}\gamma^{kn}\Pi_k^\delta\Pi_n^f\left(\Gamma_f\varkappa\right)_\delta
\end{aligned}$$

where we used

$$\varepsilon^{ijk}\varepsilon^{lmn}\gamma_{kl} = \gamma^{in}\gamma^{jm} - \gamma^{im}\gamma^{jn}$$

$$\varepsilon^{ijk}\varepsilon^{lmn}\gamma_{jm}\gamma_{kn} = -2\gamma\gamma^{il}$$

and

$$\Gamma^{ba}\Gamma_{def} = \Gamma^{ba}{}_{def} + 6\Gamma^{[b}{}_{[ef}\delta^{a]}{}_{d]} - 6\Gamma_{[f}\delta^{ba}_{de]}$$

By comparing to (4.4) we now see that $I - II = 0$ and thus the action is $\varkappa$-symmetric.

A

Spinor Algebra in Two Dimensions

We are here going to review some of the necessary formalism used to carry out the calculations in chapter 2. We are only going to consider the two-dimensional case and for a more general treatment see [17],[18],[19] and [20].

A.1 Conventions and Definitions

The Dirac matrices satisfy the Clifford algebra

$$\left\{\rho^{\alpha}, \rho^{\beta}\right\} = 2g^{\alpha\beta}$$

and as a consequence they are traceless, which is shown in chapter 3. Since we have a curved two-dimensional space the "curved" Dirac matrices ρ^{α} are related to the constant "flat" Dirac matrices ρ^{a} through the zweibein

$$\rho^{\alpha} = e^{\alpha}{}_{a}\rho^{a}$$

where

$$\left\{\rho^{a}, \rho^{b}\right\} = 2\eta^{ab}$$

and η^{ab} is the flat metric. The zweibein satisfy

$$\begin{aligned} e^{\alpha}{}_{a}e^{\beta}{}_{b}g_{\alpha\beta} &= \eta_{ab} \\ e^{\alpha}{}_{a}e^{\beta}{}_{b}\eta^{ab} &= g^{\alpha\beta} \end{aligned}$$

and the inverse is defined through

$$e_{\alpha}{}^{a}e^{\alpha}{}_{b} = \delta^{a}_{b}$$

If one wants an explicit representation for the constant Dirac matrices one can use

$$\begin{aligned}\rho^0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \bar{\rho} &= \rho^0\rho^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\end{aligned}$$

We define the charge conjugation matrix as $C = \rho^0$ and then have

$$(\rho^a)^T = -C\rho^a C^{-1} \tag{A.1}$$

The spinors we will use are Majorana spinors and thus satisfy

$$\bar{\lambda} \equiv \lambda^\dagger \rho^0 = \lambda^T C$$

which means that Majorana spinors are real. We can also use the notation of spinor indices, e.g.

$$\Lambda\Gamma\psi = \Lambda^A \Gamma_A{}^B \psi_B$$

where Γ is some combination of Dirac matrices. The spinor indices are raised and lowered using

$$\epsilon^{AB} = \epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

according to

$$\begin{aligned}\Lambda^A &= \Lambda_B \epsilon^{BA} \\ \Lambda_A &= \epsilon_{AB}\Lambda^B\end{aligned}$$

If we let $A = \pm$ we have

$$\begin{aligned}\psi^+ &= -\psi_- \\ \psi^- &= +\psi_+\end{aligned}$$

A.2 Some Identities

We have

$$\rho^\alpha \rho_\beta \rho_\alpha = 0$$

which follows trivially from the Dirac algebra. Also

$$\rho^\alpha \rho^\beta = g^{\alpha\beta} + \frac{1}{e}\epsilon^{\alpha\beta}\bar{\rho}$$

where e is the determinant of the zweibein $e_\alpha{}^a$ and $e = \sqrt{g}$. To prove this identity we first note that

$$\rho^\alpha \rho^\beta = g^{\alpha\beta} + \frac{1}{2}\left[\rho^\alpha, \rho^\beta\right]$$

and it thus remains to show that

$$\frac{1}{2}\left[\rho^\alpha, \rho^\beta\right] = \frac{1}{e}\epsilon^{\alpha\beta}\bar{\rho}$$

Proof.

$$\begin{aligned}
\frac{1}{2}\left[\rho^\alpha, \rho^\beta\right] &= \frac{1}{2}\left[\rho^a, \rho^b\right] e^\alpha{}_a e^\beta{}_b \\
&= \epsilon^{ab}\bar{\rho} e^\alpha{}_a e^\beta{}_b \\
&= \frac{1}{e}\epsilon^{\alpha\beta}\bar{\rho}
\end{aligned}$$

For anti-commuting Majorana spinors we have

$$\bar{\lambda}_1 \rho^{\alpha_1} \cdots \rho^{\alpha_n} \lambda_2 = (-1)^n \bar{\lambda}_2 \rho^{\alpha_n} \cdots \rho^{\alpha_1} \lambda_1$$

which is easily shown by using $(A.1)$. Another identity for anti-commuting spinors is the Fierz identity

$$\left(\bar{\psi}\lambda\right)\left(\bar{\phi}\chi\right) = -\frac{1}{2}\left[\left(\bar{\psi}\chi\right)\left(\bar{\phi}\lambda\right) + \left(\bar{\psi}\bar{\rho}\chi\right)\left(\bar{\phi}\bar{\rho}\lambda\right) + \left(\bar{\psi}\rho^\alpha\chi\right)\left(\bar{\phi}\rho_\alpha\lambda\right)\right]$$

which we will use many, many times in chapter 3.

Bibliography

[1] M.B. Green, J.H. Schwarz and E. Witten, Superstring theory, 2 Vols., Cambridge Monographs on Mathematical Physics, Cambridge University Press (1987).

[2] D. Lüst and S. Theisen, Lectures on String Theory, volume 346 of *Lecture Notes in Physics*, Springer-Verlag (1989).

[3] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite Conformal Symmetry in Two-Dimensional Quantum Field Theory*, Nucl. Phys. B241 (1984) 333–380.

[4] M. Henningson, *Lecture Notes in Conformal Field Theory*, Unpublished (1991).

[5] J.L. Petersen, *Notes on Conformal Field Theory*, Unpublished (1994).

[6] P. Ginsparg, *Applied Conformal Field Theory*, Les Houches Lectures (1988) 1–175.

[7] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity, John Wiley & Sons (1972).

[8] B.E.W. Nilsson, *Lecture Notes in String Theory*, Unpublished.

[9] M.E. Peskin and D.V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley (1995).

[10] N. Sandström, *BRST Quantization*, Master's thesis, Chalmers University of Technology and Göteborg University (1996).

[11] D. Friedan, E. Martinec and S. Shenker, *Conformal Invariance, Supersymmetry and String Theory*, Nucl. Phys. B271 (1986) 93–165.

[12] L. Brink and P. Howe, *Eleven-dimensional Supergravity on the Mass Shell in Superspace*, Phys. Lett. B91 (1980) 384–386.

[13] M.J. Duff, *Supermembranes*, hep-th/9611203 (1996).

[14] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Properties of the Eleven-Dimensional Supermembrane Theory*, Ann. Phys. 185 (1988) 330–368.

[15] E. Bergshoeff, E. Sezgin and P.K. Townsend, *Supermembranes and Eleven-Dimensional Supergravity*, Phys. Lett. B 189 (1987) 75–78.

[16] M. Nakahara, GEOMETRY, TOPOLOGY AND PHYSICS, Graduate Student Series in Physics, IOP Publishing (1990).

[17] J.F. Cornwell, GROUP THEORY IN PHYSICS, VOL III, volume 10 of *Techniques in Physics*, Academic Press (1989).

[18] F. Gliozzi, J. Scherk and D. Olive, *Supersymmetry, Supergravity Theories and the Dual Spinor Model*, Nucl. Phys. B122 (1977) 253–290.

[19] C. Wetterich, *Massless Spinors in more than Four Dimensions*, Nucl. Phys. B211 (1982) 177–188.

[20] M.P. Blencowe and M.J. Duff, *Supermembranes and the Signature of Spacetime*, Nucl. Phys. B310 (1988) 387–404.