Bose-Einstein Condensation

Fermions	Bosons
$S = n + \frac{1}{2}$	$S = n$
Ψ antisymetric	Ψ symetric
Pauli principle	No Pauli principle
Fermi Dirac stat.	Bose Einstein statistics
Distribution func.	Distribution func.
$f(\varepsilon,T) = \frac{1}{e^{(\varepsilon-\mu)/kT}+1}$	$n(\varepsilon,T) = \frac{1}{e^{(\varepsilon-\mu)/kT}-1}$

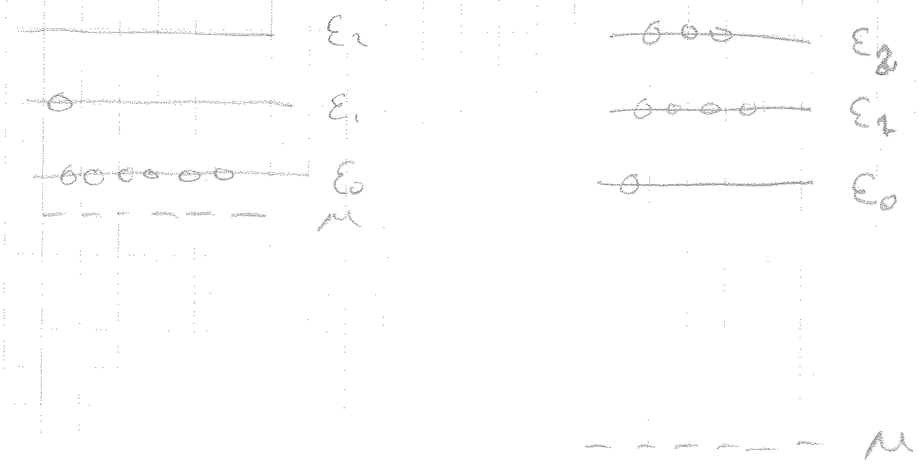
The ideal Bose gas (non interacting)

^4He atoms are Bosons

$$\sum_i n(\varepsilon_i,T) = N = \text{total number of particles}$$

many particles $N \sim 10^{23}$

Low T | High T



Energies

$$\varepsilon = \frac{\hbar^2 \vec{k}^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{\pi}{V^{1/3}}\right)^2 (l^2+m^2+n^2)$$

$$\varepsilon_0 = \varepsilon_{111} \equiv 0$$

$$\varepsilon = \varepsilon_{lmn} - \varepsilon_{111}$$

$$N(\varepsilon) = \frac{V}{4\pi^2} \frac{(2m)^{3/2}}{\hbar^3} \sqrt{\varepsilon}$$

$$\sum_{lmn} n'(\varepsilon_{lmn}, T) = N \longrightarrow \int_0^\infty N(\varepsilon) \cdot n(\varepsilon, T)\, d\varepsilon$$

integra does not include the lowes energy level sinc $N(\varepsilon=0) = 0$, we add manually

$$N = n(0,T) + \int_0^\infty N(\varepsilon)\, n(\varepsilon, T)\, d\varepsilon$$

$$= N_0 + N'(T)$$

particles in groundstate — excited particles

ρ_s — ρ_n

$$N'(T) = \frac{V}{4\pi^2} \frac{(2m)^{3/2}}{\hbar^3} \int_0^\infty \frac{\sqrt{\varepsilon}}{e^{(\varepsilon-\mu)/kT} - 1}\, d\varepsilon$$

For a fixed T $N'(T)$ is maximum if $\mu = 0$, thus

$$N'(T) \leq N'_{max}(T) = K \cdot \int_0^\infty \frac{\sqrt{\varepsilon}}{e^{\varepsilon/kT}-1} d\varepsilon$$

Substitute $x = \frac{\varepsilon}{kT}$

$$dx = \frac{d\varepsilon}{kT} \qquad d\varepsilon = kT \cdot dx \qquad \sqrt{\varepsilon} = \sqrt{kT} \cdot \sqrt{x}$$

$$N'_{max}(T) = K(kT)^{3/2} \cdot \int_0^\infty \frac{\sqrt{x}}{e^x - 1} dx$$

$$= K \cdot (k_B T)^{3/2} \cdot \underbrace{\Gamma\left(\tfrac{3}{2}\right)}_{\frac{\sqrt{\pi}}{2}} \cdot \underbrace{\zeta\left(\tfrac{3}{2}\right)}_{\text{Rieman zeta function } 2{,}612\ldots}$$

$$N'_{max}(T) = \frac{V}{4\pi^2} \frac{(2 m k_B T)^{3/2}}{\hbar^3} \frac{\sqrt{\pi}}{2} \cdot 2{,}612$$

$$= 2{,}612 \cdot V \cdot \left(\frac{m k T}{2 \pi \hbar^2}\right)^{3/2}$$

Fix N and V and increas T untill all particles are excited this happens at $T = T_B$ when $N'(T_B) = N$

$$N'(T_B) \approx N'_{max}(T_B) = N$$

$$2{,}612 \cdot V \left(\frac{m k_B T_B}{2\pi\hbar^2}\right)^{3/2} = N$$

$$T_B = \left(\frac{N}{2{,}62\, V}\right)^{2/3} \cdot \frac{2\pi\hbar^2}{m k_B}$$

Note

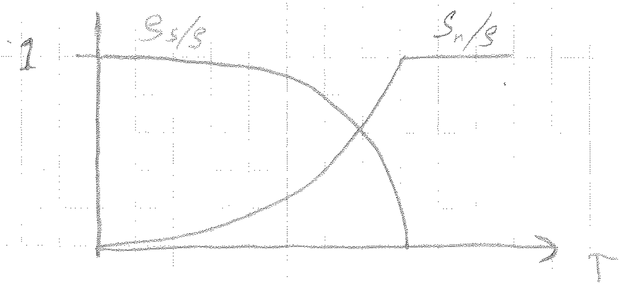
$$\rho = \frac{N}{V}$$

$$N'_{max}(T) \propto T^{3/2}$$

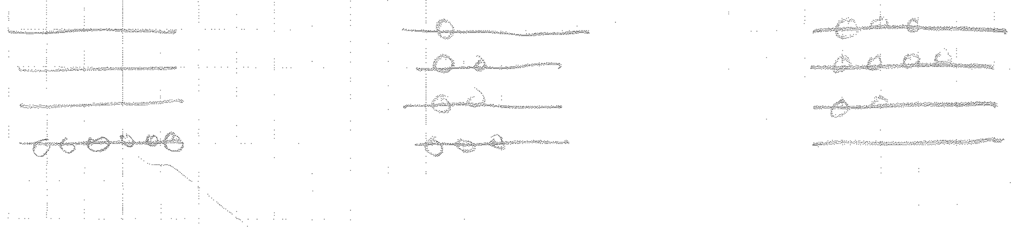
$$N'_{max}(T) = N \cdot \left(\frac{T}{T_B}\right)^{3/2} \qquad T \leq T_B$$

$$\rho_n = \frac{N'(T)}{V} \approx \frac{N'_{max}(T)}{V} = \frac{N}{V} \cdot \left(\frac{T}{T_B}\right)^{3/2} = \rho \left(\frac{T}{T_B}\right)^{3/2}$$

$$\rho_s = \frac{N_0(T)}{V} = \frac{1}{V}\left(N - N'_{max}\right) = \rho\left(1 - \left(\frac{T}{T_B}\right)^{3/2}\right)$$



$T = 0$ $\qquad$ $0 < T < T_B$ $\qquad$ $T > T_B$

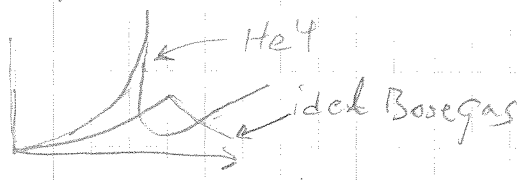


Apply to Helium 4

Gas $\left.\begin{array}{l} m = m_4 \\ \rho = \rho_{gas} \end{array}\right\}$ $T_B = 0{,}5K \quad < 4{,}2K$

no superfluid gas

Liquid $\left.\begin{array}{l} m = m_4 \\ \rho = \rho_{liquid} \end{array}\right\}$ $T_B = 3{,}1 \sim 2{,}17K$



To describe He II better we need to take interactions (van der Waals forces) into account

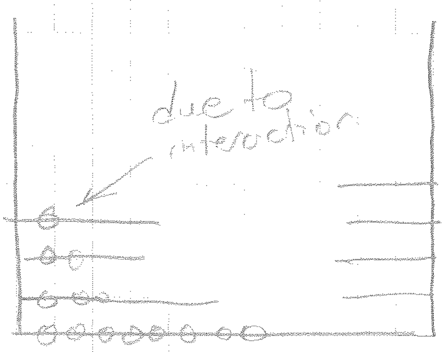
Ideal Bose gas $\rightarrow$ <u>Non-ideal Bose gas</u>

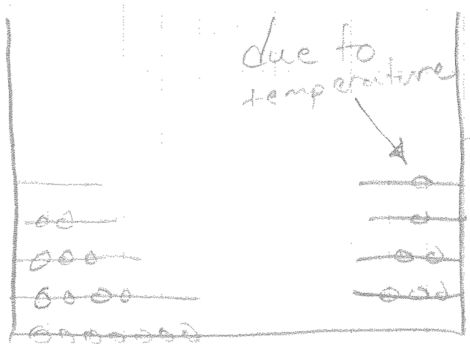
$\Rightarrow$ two consequences

1) Number of particles in E_0 reduced

2) Nature of the excited levels is modified

$T = 0$ $\qquad\qquad$ $0 < T < T_B$





The Wavefunction of the condensate

He II Similar to SCs

$$\Psi = |\Psi| e^{i\Theta} = \sqrt{\varrho_s}\, e^{i\Theta}$$

$$\bar{p}\Psi = -i\hbar\nabla\Psi = -i\hbar i\nabla\Theta\,\Psi = \hbar\nabla\Theta\,\Psi$$

$$\bar{p} = \hbar\bar{\nabla}\Theta = m_4 \cdot \bar{V}_s$$

$$\bar{V}_s = \frac{\hbar}{m_4}\nabla\Theta \qquad \bar{J}_4 = \varrho_s \cdot m_4 \cdot \bar{V}_s$$

mass flow density

Compare SC

$$\bar{J} = \varrho_s\, e\, \bar{V}_s$$

$$\Lambda\bar{J} = -\left(\frac{\hbar}{2e}\nabla\Theta + \bar{A}\right) \qquad \Lambda = \frac{m}{\varrho_s \cdot e^2}$$

$$\bar{J} = \varrho_s e \bar{V}_s = -\frac{\varrho_s e^2}{m}\frac{\hbar}{2e}\nabla\Theta - \frac{m}{\varrho_s e^2}\bar{A}$$

$$\bar{V}_s = -\frac{\hbar}{2m}\bar{\nabla}\Theta - \frac{m}{\varrho_s e^2}\bar{A}$$

He II	SC
$\nabla \times \bar{V}_s = 0$	$\nabla \times \bar{V}_s + \frac{e}{m}\bar{B} = 0$
	$\Leftrightarrow$ LE2 $\bar{\nabla}\times(\Lambda\bar{J}) = -\bar{B}$

Quantization of circulation

The circulation

$$\varkappa = \oint_L \bar{v}_s \, dl = \int_A \bar{\nabla} \times \bar{v} \, dA$$

Stokes theorem

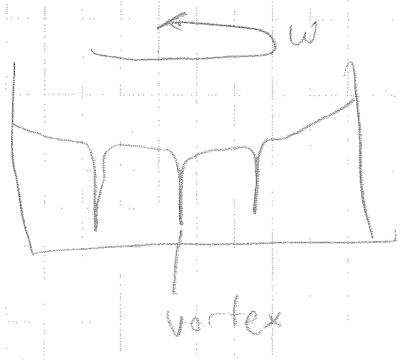
$$= \oint_L \frac{\hbar}{m_4} \cdot \nabla \theta \, dl = \frac{\hbar}{m_4} \cdot \Delta \theta = \frac{\hbar}{m_4} \cdot n \cdot 2\pi$$

$$|\bar{v}_s|(r) = \frac{\varkappa}{2\pi r}$$

$$\varkappa = n \cdot \frac{h}{m_4} = n \varkappa_0$$

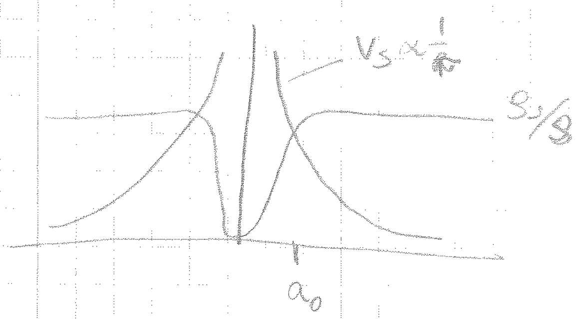
$\varkappa_0 = \frac{h}{m_4}$ = quantum of circulation

$$= 9.98 \cdot 10^{-8} \, \frac{m^2}{s}$$



Vortex density

$n_v =$ numbe



Energy of a single vortex

b — distance between vortecies

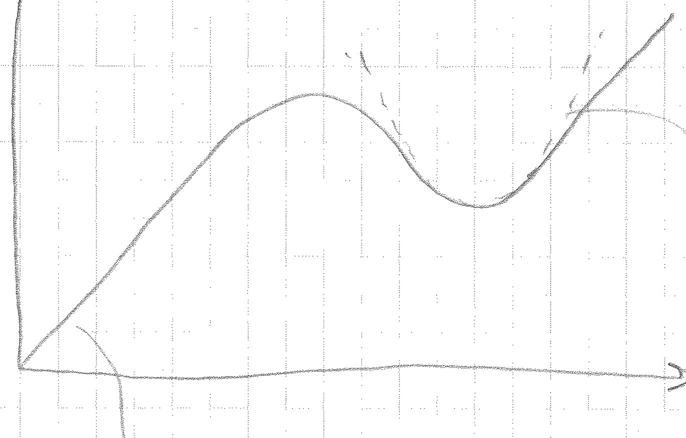
$$\mathcal{E} = \int_{a_0}^{b} \frac{1}{2} \rho_s \bar{v}_s^2 \cdot 2\pi r \, dr$$

$$= \rho_s \pi \frac{\varkappa^2}{4\pi^2} \int \frac{1}{r} dr$$

$$= \frac{\rho_s \varkappa^2}{4\pi} \ln\left(\frac{b}{a_0}\right)$$

Excitations in He II

From experiment with neutron scattering



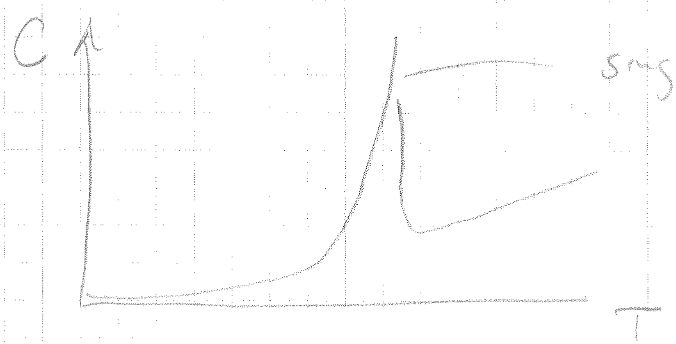
$c_1 \simeq 239\ m/s \qquad \Delta/k_B \simeq 8{,}65\ K$

$p_0/\hbar = 19{,}1/nm \qquad \mu = 0{,}16\ m_4$

$N_{ph} \propto T^3 \qquad N_{rot} \propto e^{-\Delta/kT}$

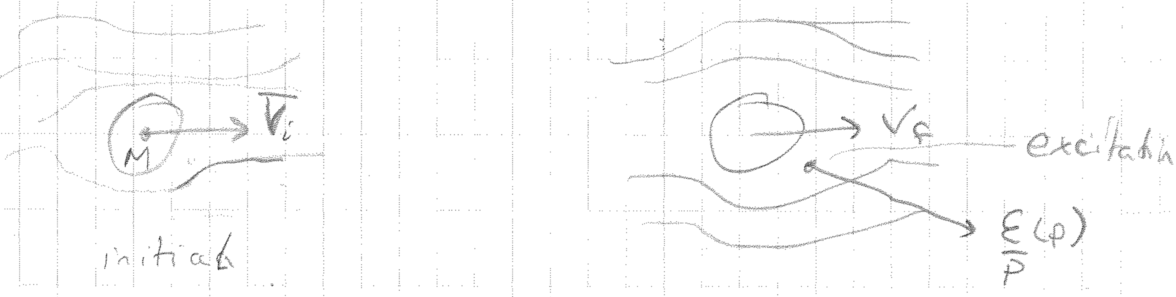
Dominates at $T < 0{,}6K$ — Dominates at $T > 0.6K$

$$C = C_{ph} + C_r = \frac{k_B}{\rho}\left[10{,}8\, N_{ph} + N_r\left(\frac{3}{4} + \frac{\Delta}{kT} + \left(\frac{\Delta}{kT}\right)^2\right)\right]$$



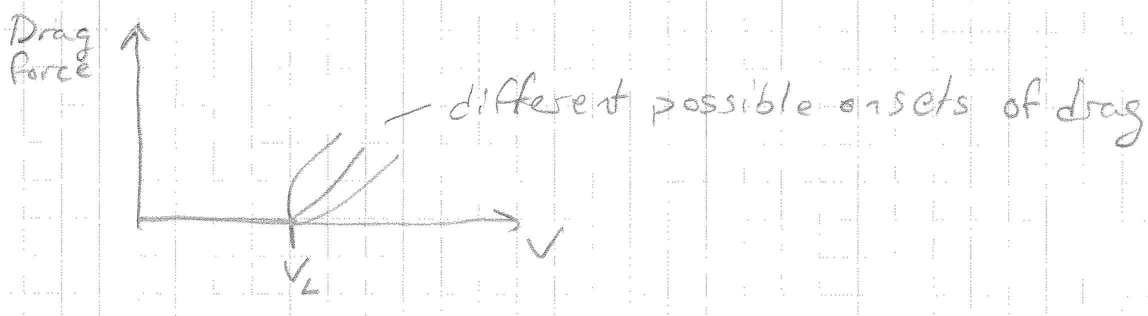
The critical velocity

Assum a massiv object moving in He II



As long as $\bar{v}$ is low there is no drag

Landau suggested that when v reaches a certain value (Later called the Landau velocity v_L, or the critical velocity) excitation will be generated



Energy conservation

$$\frac{1}{2}M\bar{V}_i^2 = \frac{1}{2}M\bar{V}_f^2 + \varepsilon(p)$$

Momentum conservation

$$M\bar{V}_i = M\bar{V}_f + \bar{p}$$

Eliminate V_f

$$\bar{V}_f = \bar{V}_i - \frac{\bar{p}}{M}$$

$$\frac{1}{2}M\bar{V}_i^2 - \frac{1}{2}M\left(\bar{V}_i - \frac{\bar{p}}{M}\right)^2 = \varepsilon(\bar{p})$$

$$\mathcal{E}(p) = \bar{V}_i \cdot \bar{P} - \frac{\bar{P}^2}{2M}$$

M Large means we can neglect last term

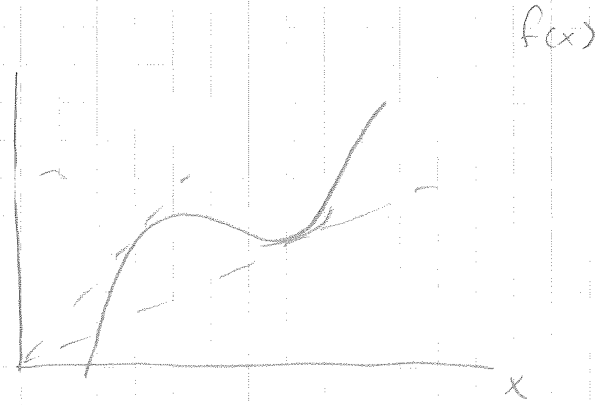
$$\mathcal{E}(p) = \bar{V}_i \cdot \bar{P} = V_i \cdot p \cdot \cos\theta$$

$$V_i = \frac{\mathcal{E}(p)}{p \cos\theta} \geqslant \frac{\mathcal{E}_p}{p}$$

Lowest velocity that can generate an excitation is

$$V_L = V_{i\,min} = \left.\frac{\mathcal{E}(p)}{p}\right|_{min}$$

Finding the minimum

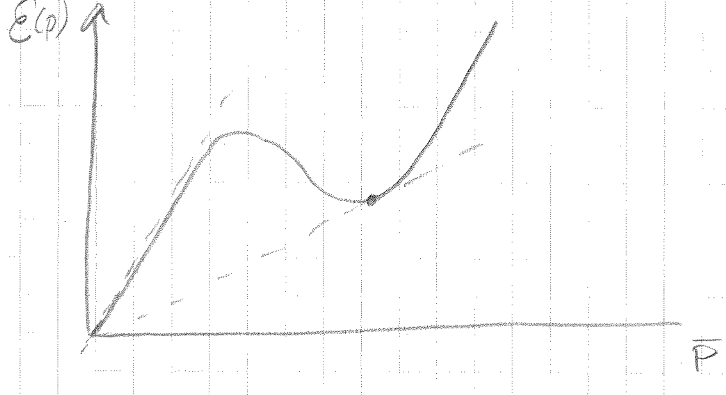


$$\frac{d}{dp}\left(\frac{\mathcal{E}(p)}{p}\right) = 0 \qquad \frac{\frac{d\mathcal{E}(p)}{dp} \cdot p - \mathcal{E}(p) \cdot 1}{\left(\frac{\mathcal{E}(p)}{p}\right)^2} = 0$$

$$\frac{d\mathcal{E}(p)}{dp} = \frac{\mathcal{E}(p)}{p}$$

equivalent of finding a line going throug the origin which tuches the curve

Apply to He II



Two possibilities

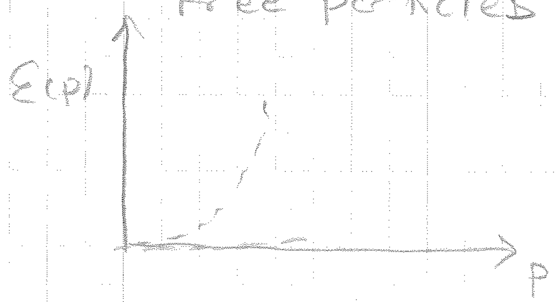
phonon branch

$$V_L = \frac{\mathcal{E}(p)}{p} = C_1 = 239\,m/s$$

rotons

$$V_L = \frac{\mathcal{E}(p)}{p} = \frac{\Delta}{p_0} \approx 58\,m/s$$ rotons generated first

Free particles



$$V_L = \left.\frac{\mathcal{E}(p)}{p}\right|_{min} = 0 \Rightarrow \text{No SF}$$