Second Quantization

Appendix!

Describe occupation instead of describing the state of each particle

$$\Psi(0,1,0,0,1,0,1,0\ldots)$$

Fermi exclusion principle 0, 1

Box 0, 1, 2, 3...

$$\phi_2, \phi_5, \phi_7 \quad (= \phi_0 e^{i\vec{k}\cdot\vec{r}})$$

not just product, Ψ has to be antisym for fermions

$$\Psi(0,1,0,0,1,0,1,0\ldots) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \phi_2(a) & \phi_2(b) & \phi_2(c) \\ \phi_5(a) & \phi_5(b) & \phi_5(c) \\ \phi_7(a) & \phi_7(b) & \phi_7(c) \end{vmatrix}$$

10^{23} particle determinant!

$$C_i\,\Psi(\ldots,1,0\ldots) = (-1)^{S(i)}\,\Psi(\ldots,0,\ldots)$$

$$C_i\,\Psi(\ldots,0,0\ldots) = 0$$

$$C_i^+\,\Psi(\ldots,0,\ldots) = (i)^{S(i)}$$

$$C^+\,\Psi(\ldots,1,\ldots) = 0$$

$S(i)$ number of filled rows below (states)

Commutation relations

anticommutation

$$c_i c_j = -c_j c_i \quad \Rightarrow \quad \{c_i, c_j\} = 0$$

$$c_i^\dagger c_j = -c_j c_i^\dagger$$

$$c_i^\dagger c_j^\dagger = -c_j^\dagger c_i^\dagger$$

$$c_i c_i = c_i^\dagger c_i^\dagger = 0$$

Number operator

$$\hat{n}_i = c_i^\dagger c_i = \begin{cases} 1 & \text{if filled} \\ 0 & \text{if empty} \end{cases}$$

$$1 - \hat{n}_i = c_i c_i^\dagger = \begin{cases} 1 & \text{if empty} \\ 0 & \text{if filled} \end{cases}$$

$$\{c_i, c_j^\dagger\} = \delta_{ij}$$

$$c_i c_j^\dagger + c_j^\dagger c_i = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Bosons

Comutation instead of anticomutation

$c \psi_n = \sqrt{n}\, \psi_{n-1}$

$c^\dagger \psi_n = \sqrt{n+1}\, \psi_{n+1}$

$c^\dagger c = n$

$c c^\dagger = n+1$

Fermi Ground st

$$\Psi = \prod_{k<k_F,\sigma} c_{k\sigma}^\dagger |0\rangle$$

mixed occupation # $\Psi = (c_2^\dagger c_3^\dagger + c_4^\dagger)|4\rangle$

Operators

Single particle

$$\hat{F}^{(1)} = \sum_{ij} c_j^+ f_{ji} c_i$$

Example kinetic energy

$$E_{kin} = \sum c_{k\sigma}^+ c_{k\sigma} \varepsilon_k = \sum n_{k\sigma} \varepsilon_k$$

Two Particle

$$\hat{F}^{(2)} = \frac{1}{2} \sum_{ij,kl} c_i^+ c_j^+ f_{ij,kl} c_l c_k$$

Coulomb interaction

describe scattering from state kl, to state ij

Fourier / Localized Operators

$$\psi_\sigma(r) = \frac{1}{\sqrt{V}} \sum_k c_{k\sigma} e^{-ikr}$$

$$\psi_\sigma^+(r) = \frac{1}{\sqrt{V}} \sum_k c_{k\sigma}^+ e^{+ikr}$$

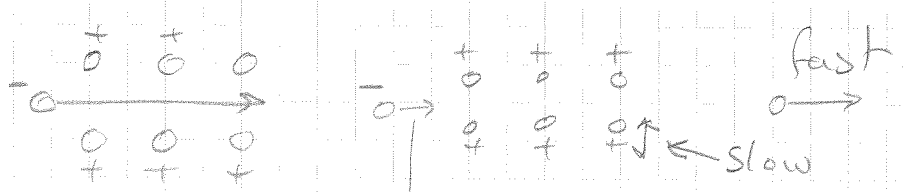
$$\{\psi_\sigma, \psi_\sigma^+\} = \delta(r - r') \quad \text{Pauli principle}$$

$$F^{(1)} = \sum_\sigma \int \psi_\sigma^+ f(r) \psi_\sigma \, dr$$

similar to the expectation value

Simple minded picture

Normally electrons repell each other



attractive force due to excess positive charge

This attractive force can be more rigorously derived describing the polarization of the crystal due to lattice distortion

i.e electron phonon interaction

See sec 7.2 in the Book

This attractive force leads to Cooper pairing

Opposite k and σ

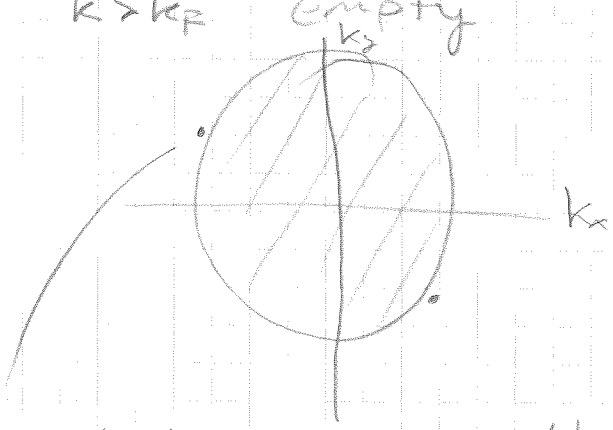
$$\vec{k}_1 = -\vec{k}_2 \qquad \vec{\sigma}_1 = -\vec{\sigma}_2$$

$$H = H_0 + V(\vec{r}_1 - \vec{r}_2)$$

such that $V(\vec{r}_1 - \vec{r}_2) < 0$

The Cooper-pair model

Assume a filled Fermi sea
with all statse for which $k < k_F$ are filled
if $k > k_F$ empty



Two electrons are added

Hamiltonian for the added pair

$$H = H_0 + V(\bar{r}_1 - \bar{r}_2)$$

H_0: non-interacting pair; $V(\bar{r}_1 - \bar{r}_2)$: interaction

Energies are measured with respect to E_F

SE

$$H\psi = W\psi$$

W: energy

$$(W - H_0)\psi = V(\bar{r}_1 - \bar{r}_2)\psi$$

non interacting wave function

$$\phi_1 = e^{i\bar{k}_1\bar{r}_1} \quad \phi_2 = e^{i\bar{k}_2\bar{r}_2}$$

New coordinates

Center of mas

$$\bar{R} = \frac{1}{2}(\bar{r}_1 + \bar{r}_2)$$

$$\bar{K} = (\bar{k}_1 + \bar{k}_2)$$

relative motion

$$\bar{r} = \bar{r}_1 - \bar{r}_2$$

$$\bar{k} = \frac{1}{2}(\bar{k}_1 - \bar{k}_2)$$

$$\bar{r}_1 = \bar{R} + \frac{\bar{r}}{2}$$

$$\bar{r}_2 = \bar{R} - \frac{\bar{r}}{2}$$

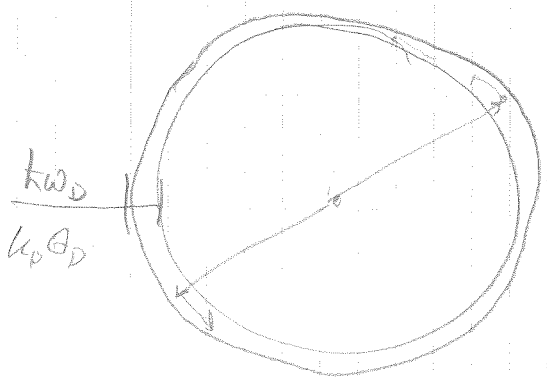
$$\bar{k}_1 = \frac{1}{2}\bar{K} + \bar{k}$$

$$\bar{k}_2 = \frac{1}{2}\bar{K} - \bar{k}$$

$$\phi = \phi_1 \cdot \phi_2 = e^{i\bar{k}_1\bar{r}_1} \cdot e^{i\bar{k}_2\bar{r}_2} = e^{i(\bar{k}_1\bar{r}_1 + \bar{k}_2\bar{r})}$$

$$= e^{i\left(\left(\frac{\bar{K}}{2} + \bar{k}\right)\left(\bar{R} + \frac{\bar{r}}{2}\right) + \left(\frac{\bar{K}}{2} - \bar{k}\right)\left(\bar{R} - \frac{\bar{r}}{2}\right)\right)}$$

$$= e^{i(\bar{K}\bar{R} + \bar{k}\bar{r})}$$



States with opposite $\bar{k}$

$\Rightarrow \bar{K} = 0$

i.e

$$\phi = e^{i\bar{k}(\bar{r}_1 - \bar{r}_2)}$$

relative coordinates

Stationary state is a superposition of different $\bar{k}$ states

$$\Psi = \sum_k a_k e^{ik(r_1 - r)}$$

$$(W - H_0)\,\Psi = V(\bar{r}_1 - \bar{r}_2)\,\Psi$$

$$(W - H_0) \sum_k a_k e^{i\bar{k}(\bar{r}_1 - \bar{r}_1)} = V(\bar{r}_1 - \bar{r}_2) \sum a_k e^{i\bar{k}'(\bar{r}_1 - \bar{r}_2)}$$

Multiply by $e^{-i\bar{k}(\bar{r}_1 - \bar{r}_2)}$ and $\int$ over Volume Ω

Lefthand side

$$\frac{1}{\Omega} \int_\Omega e^{-i\bar{k}(\bar{r}_1 - \bar{r}_2)} (W - H_0) \sum_k a_k e^{i\bar{k}'(\bar{r}_1 - \bar{r}_2)}\, d(\bar{r}_1 - \bar{r}_2)$$

$$- \frac{1}{\Omega} W \sum_k a_k \int_\Omega e^{-i\bar{k}(\bar{r}_1 - \bar{r}_2)} e^{i\bar{k}(\bar{r}_1 - \bar{r}_2)}\, d(\bar{r}_1 - \bar{r}_2)$$

$$- \frac{1}{\Omega} \sum a_k \int_\Omega e^{-i\bar{k}'(\bar{r}_1 - \bar{r}_2)} H_0\, e^{i\bar{k}(\bar{r}_1 - \bar{r}_2)}\, d(\bar{r}_1 - \bar{r}_2)$$

$$- \frac{1}{\Omega} W a_k \Omega - \frac{1}{\Omega} \sum_{ak} a_k \varepsilon_k \int_\Omega e^{i(\bar{k}' - \bar{k})(\bar{r}_1 - \bar{r}_2)}\, d(\bar{r}_1 - \bar{r}_2)$$

$$W a_k - 2\varepsilon_k a_k = (W - 2\varepsilon_k) a_k$$

The right hand side

$$\frac{1}{\Omega}\int_{\Omega} e^{-ik(r_1-r_2)} \sum_{k'} a_{k'}\; V(r_1-r_2)\cdot e^{+ik'(r_1-r_2)}\; d(r_1-r_2)$$

$$\left\{ V_{kk'} = \text{matrix element} = \frac{1}{\Omega}\int e^{-ik(r_1-r_2)}\, V\, e^{ik'(r_1-r_2)}\, d(r_1-r_2) \right\}$$

$$\sum a_{k'} \cdot V_{kk'}$$

$$\boxed{(W-2\varepsilon_k)\, a_k = \sum_{k'} a_{k'} \cdot V_{kk'}}$$

Assume $V_{kk'}$ independent of $kk' = -V$

Calculate W = Energy of the Cooper pair

$$\boxed{(W-2\varepsilon_k)\, a_k = -V \sum_{k'} a_{k'}}$$

$$a_k = \frac{-V \sum_{k'} a_{k'}}{W-2\varepsilon_k}$$

$$\sum_{k'} a_{k'} = \sum_{k'} \left[\frac{-V \sum_{k'} a_{k'}}{W-2\varepsilon_{k'}} \right]$$

$$1 = -V \sum_{k'} \frac{1}{W-2\varepsilon_{k'}}$$

convert sum to integral

$$1 = -V \int_0^{\hbar\omega_D} \frac{N(\varepsilon)}{W - 2\varepsilon} \, d\varepsilon$$

$\hbar\omega_D \ll \varepsilon_F \Rightarrow N(\varepsilon) \sim N(0)$

$$1 = -VN(0) \int_0^{\hbar\omega_D} \frac{1}{W - 2\varepsilon} \, d\varepsilon$$

$$x = W - 2\varepsilon$$

$$dx = -2 \, d\varepsilon$$

$$d\varepsilon = -\frac{1}{2} dx$$

$$1 = -VN(0) \left(-\frac{1}{2}\right) \int_0^{\hbar\omega_D} \frac{1}{x} \, dx$$

$$1 = \frac{1}{2} VN(0) \left[\ln x \right]_{W}^{W - 2\hbar\omega_D}$$

$$\frac{2}{VN(0)} = \ln\left(\frac{W - 2\hbar\omega_D}{W}\right)$$

$$e^{\frac{2}{VN(0)}} = 1 - \frac{2\hbar\omega_D}{W}$$

$$e^{\frac{2}{VN(0)}} - 1 = \frac{2\hbar\omega_D}{W}$$

$e^{\frac{2}{VN(0)}} \gg 1$

$$\boxed{W = -2\hbar\omega_D e^{-\frac{2}{VN(0)}}}$$

Size of the Cooper pair

Limited number of $\vec{k}$ states is equivalent

$$\Delta x \cdot \Delta p \sim \hbar$$

$$\Delta r \sim \frac{1}{\Delta k}$$

$$\Delta k = \frac{\partial k}{\partial \mathcal{E}} \cdot \Delta E$$

$$\mathcal{E} = \frac{\hbar^2 k^2}{2m} \qquad \frac{\partial \mathcal{E}}{\partial k} = \hbar^2 \frac{2k}{2m}$$

$$\Delta r = \frac{1}{\Delta k} = \frac{\partial \mathcal{E}}{\partial k} \frac{1}{\Delta E} = \frac{\hbar^2 k}{m} \frac{1}{\Delta E} = \frac{\hbar v_F}{\Delta E} \sim$$

$$\simeq \frac{\hbar v_F}{k_B T_c} \sim 1\,\mu m$$