The BCS Groundstate

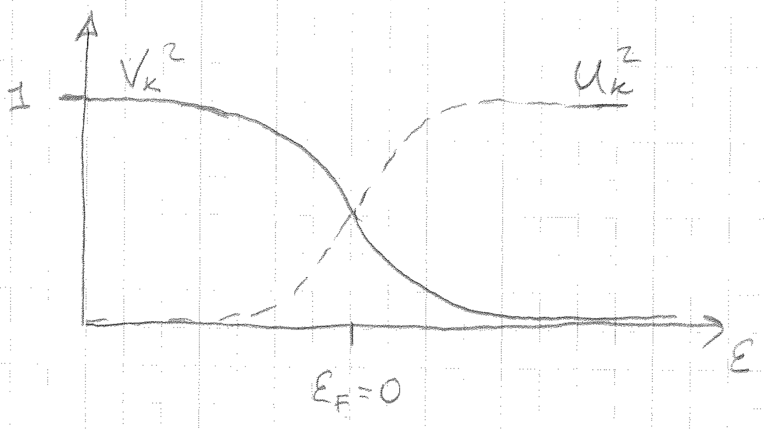
$$|\text{Fermi}\rangle = |\Psi\rangle = \prod_{k \le k_F} c_{k\uparrow} c_{k\downarrow} |0\rangle$$

BCS Suggested a groundstate of pairs

$$|BCS\rangle = |\Psi_{BCS}\rangle = \prod_{k} \left(u_k + e^{i\theta} v_k \, c^{+}_{\bar{k}+\bar{s},\uparrow} \, c^{+}_{-\bar{k}-\bar{s},\downarrow}\right)|0\rangle$$

pairwise creation

u_k^2 probability of empty state
v_k^2 probability of occupied state
θ phase of the superfluid
$\hbar \bar{s} = 2m \bar{v}_s$ Center of mass momentum for the pair



Fuzzy Fermi surface even at $T=0$

$u_k^2 + v_k^2 = 1$

The BCS Hamiltonian

$$H_{BCS} = \underbrace{\sum_{k,\sigma} \varepsilon_k c^{+}_{k\sigma} c_{k\sigma}}_{\text{Kinetic energy}} + \underbrace{\frac{1}{2} \sum_{k_1\sigma_1, k_2\sigma_2, q} c^{+}_{k_1+q,\sigma_1} c^{+}_{k_2-q,\sigma_2} V_{k_1,k_2,q} \, c_{k_1\sigma_1} c_{k_2\sigma_2}}_{\text{Interaction energy}}$$

Scattering from $k_1, k_2 \rightarrow k_1+q, k_2-q$

The expectation value

$$\langle E \rangle = \langle \Psi_{BCS} | H_{BCS} | \Psi_{BCS} \rangle = \overbrace{\sum_k 2 v_k^2 \varepsilon_k}^{\text{kinetic}} + \overbrace{\sum_{k,k'} v_k u_{k'} u_k v_{k'} V_{kk'}}^{\text{interaction}}$$

Note: independent of θ

Note Ψ_{BCS} is not an eigenfunction to H_{BCS} but close enough.

Let's minimize the energy by variation of u_k, v_k

$$d\langle E \rangle = \frac{\partial \langle E \rangle}{\partial v_k} \cdot dv_k + \frac{\partial \langle E \rangle}{\partial u_k} du_k = 0$$

First we rewrite

$$\langle E \rangle = \sum_k 2 v_k^2 \varepsilon_k + \sum_k v_k u_k \cdot 2 \underbrace{\sum_{k'} u_{k'} v_{k'} V_{kk'}}_{-\Delta_k}$$

$$= \sum_k \left(2 v_k^2 \varepsilon_k - 2 v_k u_k \Delta_k \right)$$

$$d\langle E \rangle = \sum_k (4 v_k \varepsilon_k - 2 u_k \Delta_k) dv_k - 2 v_k \Delta_k du_k = 0$$

$$\left[\begin{aligned} u_k^2 &= 1 - v_k^2 \\ u_k &= \sqrt{1 - v_k^2} \\ du_k &= \frac{1}{2\sqrt{1 - v_k^2}} (-2 v_k) dv_k = -\frac{v_k}{u_k} dv_k \end{aligned} \right]$$

Substitute du_k

$$4 v_k \varepsilon_k dv_k - 2 u_k \Delta_k dv_k + 2 \frac{v_k^2}{u_k} \Delta_k dv_k = 0$$

Multiply by $\frac{u_k}{2 dv_k}$

$$2 v_k u_k \varepsilon_k - (u_k^2 - v_k^2) \Delta_k = 0$$

Thus

$$\boxed{\Delta_k = \frac{2 V_k U_k \varepsilon_k}{U_k^2 - V_k^2}}$$ Energy gap!

Introduce new parameter E_k

$$V_k^2 = \frac{1}{2}\left(1 - \frac{\varepsilon_k}{E_k}\right)$$ probability occupied

$$U_k^2 = \frac{1}{2}\left(1 + \frac{\varepsilon_k}{E_k}\right)$$ empty

Still $$U_k^2 + V_k^2 = 1$$

$$U_k^2 - V_k^2 = \frac{\varepsilon_k}{E_k}$$

$$U_k^2 \cdot V_k^2 = \frac{1}{4}\left(1 - \frac{\varepsilon_k^2}{E_k^2}\right)$$

$$U_k \cdot V_k = \frac{1}{2}\sqrt{1 - \frac{\varepsilon_k^2}{E_k^2}}$$

$$\Delta_k = \frac{2\,\varepsilon_k \frac{1}{2}\sqrt{1 - \frac{\varepsilon_k^2}{E_k^2}}}{\frac{\varepsilon_k}{E_k}} = E_k\sqrt{1 - \frac{\varepsilon_k^2}{E_k^2}} = \sqrt{E_k^2 - \varepsilon_k^2}$$

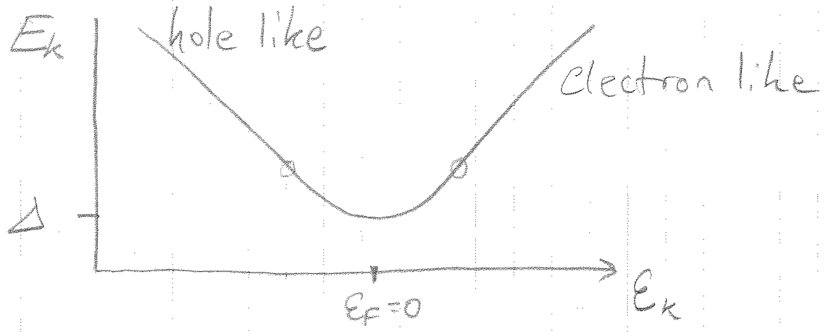
$$E_k = \sqrt{\varepsilon_k^2 + \Delta_k^2}$$ excitation energy

Weakly coupled SC

i.e weak electron phonon coupling
all SC except Pb and Hg

$$\left.\begin{aligned} V_{kk'} &= -V \\ \Delta_k &= \Delta \end{aligned}\right\} \quad \text{for } k, k' < k_D \leftarrow \text{Debey}$$

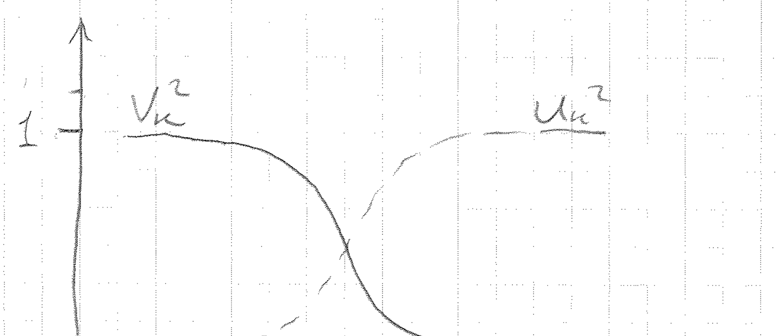
$$E_k = \sqrt{\varepsilon_k + \Delta^2}$$



V_k and U_k in terms of ε_k and Δ

$$V_k^2 = \frac{1}{2}\left(1 - \frac{\varepsilon_k}{E_k}\right) = \frac{1}{2}\left(1 - \frac{\varepsilon_k}{\sqrt{\varepsilon_k^2 + \Delta^2}}\right)$$

$$U_k^2 = \frac{1}{2}\left(1 + \frac{\varepsilon_k}{E_k}\right) = \frac{1}{2}\left(1 + \frac{\varepsilon_k}{\sqrt{\varepsilon_k^2 + \Delta^2}}\right)$$



The size of Δ

$$\Delta_k = -\sum_{kk'} u_{k'} v_{k'} V_{kk'}$$

Weakly coupled SC

$$\Delta = V \sum_k u_k v_k$$

$$u_k v_k = \frac{1}{2}\sqrt{1 - \frac{\varepsilon^2}{\varepsilon^2+\Delta^2}} = \frac{1}{2}\frac{\Delta}{\sqrt{\varepsilon^2+\Delta^2}}$$

$$\sum_k \rightarrow \int d\varepsilon$$

$$\Delta = \frac{V}{2}\int_{-\hbar\omega_D}^{\hbar\omega_D} N(\varepsilon)\frac{\Delta}{\sqrt{\varepsilon^2+\Delta^2}}\,d\varepsilon$$

$$1 = \frac{V \cdot N(0)}{2}\int_{-\hbar\omega_D}^{\hbar\omega_D}\frac{1}{\sqrt{\varepsilon^2+\Delta^2}}\,d\varepsilon = \frac{VN(0)}{2}\left[\ln\left(\varepsilon+\sqrt{\varepsilon^2+\Delta^2}\right)\right]_{-\hbar\omega_D}^{\hbar\omega_D}$$

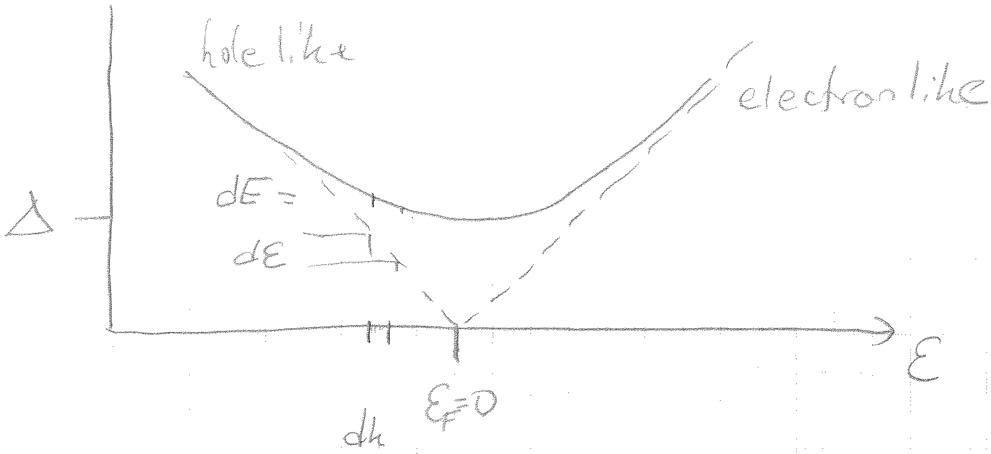
$$\frac{2}{VN(0)} = \ln\left[\frac{\sqrt{\hbar^2\omega_D^2+\Delta^2}+\hbar\omega_D}{\sqrt{\hbar^2\omega_D^2+\Delta^2}-\hbar\omega_D}\right] \quad \cdots\cdots$$

$$\sinh\left(\frac{1}{VN(0)}\right) = \frac{\hbar\omega_D}{\Delta} \qquad \underbrace{V \cdot N(0) < 0.3}_{\text{small}}$$

$$\boxed{\Delta = \hbar\omega_D \cdot e^{-\frac{1}{N(0)V}}}$$

The density of states

Excitation energies



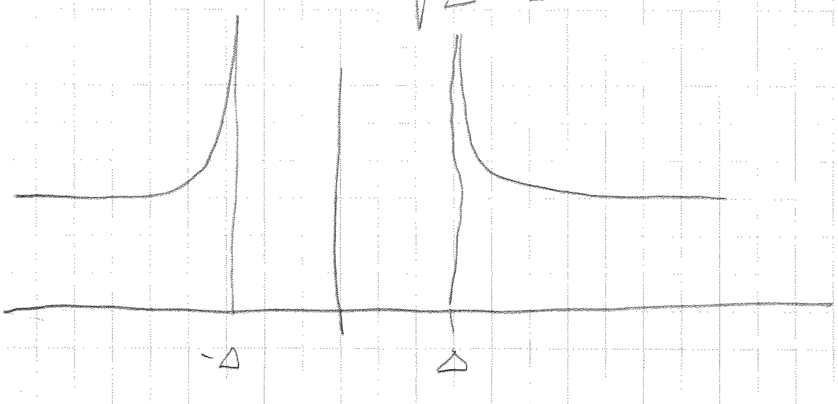
$$N_s(E)\,dE = N(\mathcal{E})\,d\mathcal{E}$$

$$N_s(E) = N_n(0)\frac{d\mathcal{E}}{dE}$$

$$\mathcal{E} = \sqrt{E^2-\Delta}$$

$$\frac{d\mathcal{E}}{dE} = \frac{1}{2}\frac{2E}{\sqrt{E^2-\Delta^2}} = \frac{E}{\sqrt{E^2-\Delta^2}}$$

$$N_s(E) = \frac{E}{\sqrt{E^2-\Delta^2}} \qquad |E|>\Delta$$



The isotope effect

$$k_B T_c = 1.14\, \hbar \omega_D \cdot e^{\frac{1}{N_0 V}}$$

$$\hbar \omega_D \propto \sqrt{\frac{k}{M}}$$

if N_0 and V indep of M

then

$$T_c \propto \omega_D \propto \frac{1}{\sqrt{M}}$$

The size of a Cooper pair

Limited number of k states involved

$$\Delta x \cdot \Delta p = \hbar$$

$$\Delta x = \frac{\hbar}{\hbar \Delta k}$$

$$\Delta k = \frac{\partial k}{\partial \varepsilon} \cdot \Delta \varepsilon \qquad \Delta x = \frac{\frac{\partial \varepsilon}{\partial k}}{\Delta E}$$

$$\varepsilon = \frac{\hbar^2 k^2}{2m} \Rightarrow \frac{\partial \varepsilon}{\partial k} = \frac{\hbar^2 k}{m}$$

$$\Delta x \sim \frac{\hbar^2 k}{m \cdot \Delta} \sim \frac{\hbar m v_F}{m \Delta} = \frac{\hbar v_F}{\Delta} \sim 1 \mu m$$

<u>More rigorously we get</u>
<u>the BCS coherence length</u>

$$\xi_0 = \xi_{BCS} = \frac{\hbar v_F}{\pi \Delta(0)} = 0{,}18 \frac{\hbar v_F}{k_B T_c}$$

Close to T_c $\quad \xi_{GL}(T) = 0{,}74\, \xi_0 \sqrt{1 - \frac{T}{T_c}}$

For Dirty SC

$$\xi_{Pippard} = \frac{1}{\frac{1}{\xi_0} + \frac{1}{\ell}}$$

prefactor close to 1

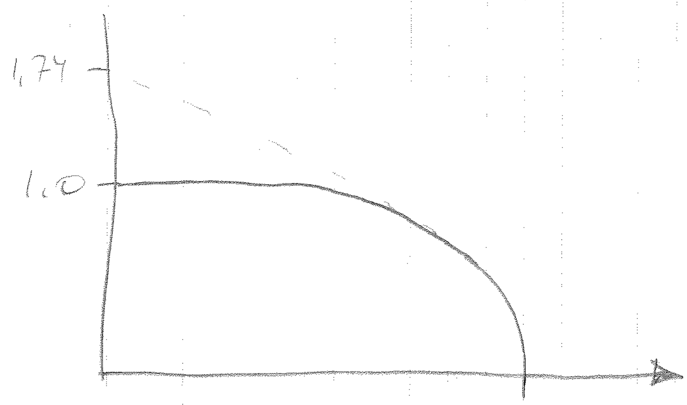
Important results

① $\Delta(0) = 2 \cdot k_B \Theta_D e^{-1/NV}$

② $kT_c = 1{,}14\, k_B \Theta_D\, e^{-\frac{1}{NV}}$

③ $\Delta(0) = \frac{2}{1{,}14} k_B T_c = 1{,}76\, k_B T_c$

④ $\Delta(T) = 1{,}74 \cdot \Delta_0 \sqrt{1 - \frac{T}{T_c}}$ near T_c



⑤ $F_n(0) - F_s(0) = \frac{1}{2} N_0 \cdot \Delta_0^2 = \frac{\mu_0}{2} H_c^2$

⑥ $\Delta C_p = 1{,}43 \cdot \gamma \cdot T_c$

⑦ $T_c \propto \frac{1}{\sqrt{M}}$

⑧ $\xi_0 = \frac{\hbar v_F}{\pi \Delta_0}$ BCS coherence length

$= \xi_{GL} = 0{,}74\, \xi_0 \frac{1}{\sqrt{1 - \frac{T}{T_c}}}$ clean SC $l \gg \xi_0$