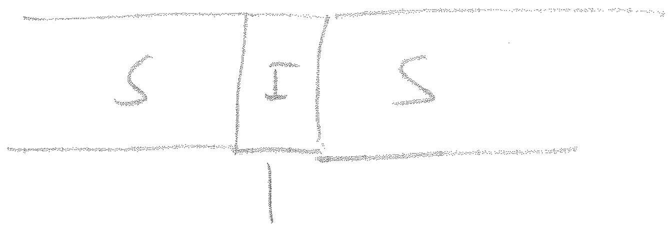
Weak Links

$\Psi = \Psi_1 e^{i\theta_1}$

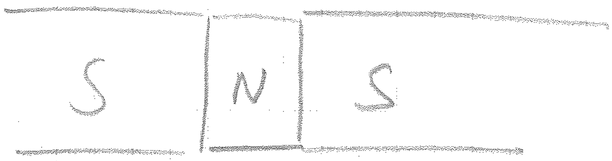
$\sqrt{\rho_1}$



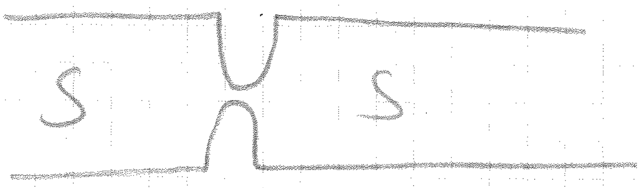
insulator

$\Psi = \Psi_2 e^{i\theta_2}$

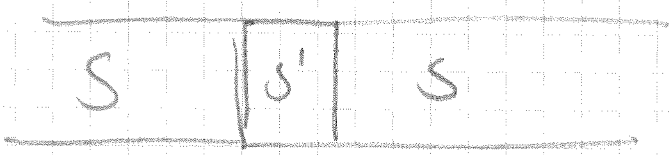
$\sqrt{\rho_2}$

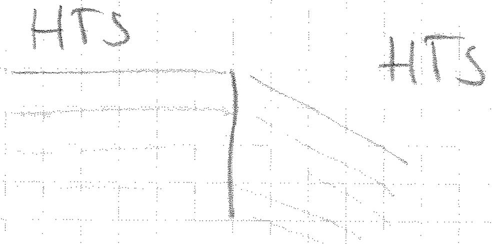


Normal



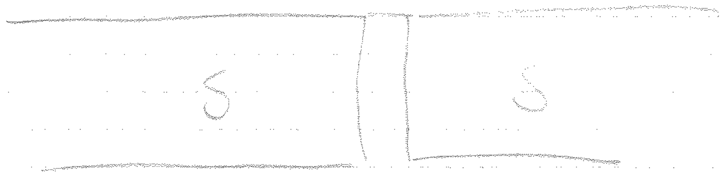
ScS





Grain Boundary

The Josepson effect



$$\Psi_1 = \sqrt{\rho_1}\, e^{i\theta_1} \qquad \Psi_2 = \sqrt{\rho_2}\, e^{i\theta_2}$$

$$i\hbar \frac{\partial}{\partial t} \Psi = H\Psi \qquad \text{time dep SE}$$

Coupling between 1 and 2

$$i\hbar \frac{\partial}{\partial t} \Psi_1 = H_1 \Psi_1 + K \Psi_2$$

$$i\hbar \frac{\partial}{\partial t} \Psi_2 = H_2 \Psi_2 + K \Psi_2$$

Voltage applied V $\quad H_1 = H_0 + \frac{2eV}{2}$

Equal SCs $\quad H_2 = H_0 - \frac{2eV}{2}$

(1) $\quad i\hbar \frac{\partial}{\partial t} \Psi_1 = (H_0 + eV)\Psi_1 + K\Psi_2 = (\mu_1 + eV)\Psi_1 + K\Psi_2$

(2) $\quad i\hbar \frac{\partial}{\partial t} \Psi_2 = + K\Psi_1 + (H_0 - eV)\Psi_1 = (\mu_2 - eV)\Psi_2 + K\Psi_1$

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} H_0 + eV & K \\ K & H_0 - eV \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$$

Insert $\Psi_1 = \sqrt{\rho_1}\, e^{i\theta_1}$, $\Psi_2 = \sqrt{\rho_2}\, e^{i\theta_2}$

$$① \quad i\hbar \frac{\partial}{\partial t}\left(\sqrt{\rho_1}\, e^{i\theta_1}\right) = (\mu_1 + eV)\, \sqrt{\rho_1} e^{i\theta_1} + K\sqrt{\rho_2}\, e^{i\theta_2}$$

$$② \quad i\hbar \frac{\partial}{\partial t}\left(\sqrt{\rho_2}\, e^{i\theta_2}\right) = (\mu_2 - eV)\, \sqrt{\rho_2} e^{i\theta_2} + K\sqrt{\rho_2}\, e^{i\theta_1}$$

$$① \quad i\hbar\left(\frac{1}{2}\frac{\dot{\rho}_1}{\sqrt{\rho_1}} e^{i\theta} + \sqrt{\rho_1}\, i\dot{\theta}_1 e^{i\theta_1}\right) = (\mu_1 + eV)\sqrt{\rho_1} e^{i\theta_1} + K\sqrt{\rho_2} e^{i\theta_2}$$

$$② \quad i\hbar\left(\frac{1}{2}\frac{\dot{\rho}_2}{\sqrt{\rho_2}} e^{i\theta} + \sqrt{\rho_2}\, i\dot{\theta}_2 e^{i\theta_2}\right) = (\mu_2 - eV)\rho_2 e^{i\theta_2} + K\sqrt{\rho_1} e^{i\theta_1}$$

$$\psi_1^* \times ① \qquad \sqrt{\rho_1}\, e^{-i\theta_1} \times ①$$

$$\delta \equiv \theta_2 - \theta_1$$

$$i\hbar\left(\frac{1}{2}\dot{\rho}_1 + \rho_1 i\dot{\theta}_1\right) = (\mu_1 + eV)\rho_1 + K\sqrt{\rho_1\rho_2}\, \underbrace{e^{i\theta_2 - \theta_1}}_{\cos\delta + i\sin\delta}$$

$$\psi_2^* \times ②$$

$$i\hbar\left(\frac{1}{2}\dot{\rho}_2 + \rho_2 i\dot{\theta}_2\right) = (\mu_2 - eV)\rho_2 + K\sqrt{\rho_2\rho_1}\, e^{i\theta_1 - \theta_2}$$

$$\text{Im } ① \qquad \frac{\hbar}{2}\dot{\rho}_1 = K\sqrt{\rho_1\rho_2}\,\sin\delta$$

$$\text{Im } ② \qquad \frac{\hbar}{2}\dot{\rho}_2 = K\sqrt{\rho_1\rho_2}\,(-\sin\delta)$$

$$1 \to 2 \qquad I = 2e\dot{\rho}_1 = 2e\rho_2 = 2e\frac{2K}{\hbar}\sqrt{\rho_1\rho_2}\,\sin\delta$$

$$\boxed{I = I_c \sin\delta}$$

The dc Josephson effect

$$I_c = \frac{4eK}{\hbar}\sqrt{\rho_1\rho_2}$$

Re ① $\quad -\hbar\varrho_1\dot{\theta}_1 = (\mu_1 + eV)\varrho_1 + K\sqrt{\varrho_1\varrho_2}\cos\delta$

Re ② $\quad -\hbar\varrho_2\dot{\theta}_2 = (\mu_2 - eV)\varrho_2 + K\sqrt{\varrho_1\varrho_2}\cos(-\delta)$

$$\frac{\text{Re}①}{\varrho_1} - \frac{\text{Re}②}{\varrho_1}$$

$$-\hbar(\dot{\theta}_1 - \dot{\theta}_2) = \mu_1 - \mu_2 + 2eV + K\left(\sqrt{\frac{\varrho_2}{\varrho_1}} - \sqrt{\frac{\varrho_1}{\varrho_2}}\right)\cos\delta$$

$$\dot{\delta} = \frac{2eV}{\hbar} + \frac{\mu_1 - \mu_2}{\hbar} + \frac{K}{\hbar}\left(\sqrt{\frac{\varrho_2}{\varrho_1}} - \sqrt{\frac{\varrho_1}{\varrho_2}}\right)\cos\delta$$

$SC1 = SC2 \Rightarrow \mu_1 = \mu_2 \quad \varrho_1 = \varrho_2$

$$\dot{\delta} = \frac{2e}{\hbar}V$$ ac Josephson effect

$$\boxed{\begin{aligned} V &= \frac{\hbar}{2e}\dot{\delta} \\ I &= I_c \sin\delta \end{aligned}}$$

Dc Josephson effect

$$I_c = \frac{\pi}{4}\,\frac{2\Delta}{eR_N}$$

R_N tunnel resistance in Normal state

Exampel

Nb: $2\Delta = 3\,\mathrm{meV}$

$A = 10\times 10\,\mu m^2$

$R = 10{,}0\,\Omega$

$I_c = 230\,\mu A$

$$J_c = \frac{I_c}{A} = \frac{2{,}3\cdot 10^{-4}}{10^{-3}\cdot 10^{-3}} = 230\,\frac{A}{cm^2}$$

Josephson coupling energy

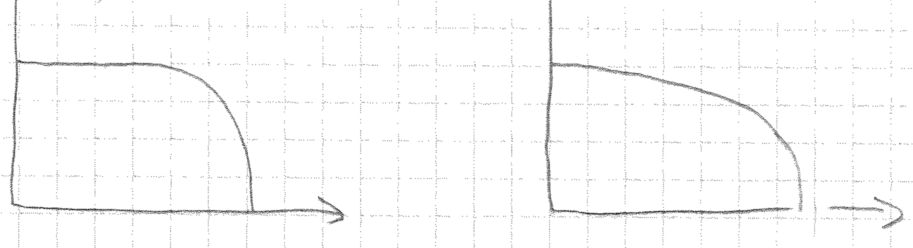
$T=0$ $$E_J = \frac{\hbar}{2e} I_c = \frac{R_Q}{R_N}\cdot\frac{\Delta}{2} \qquad R_Q = \frac{h}{4e^2} = \frac{\Phi_0}{2e}$$

$$E_J = \frac{\hbar}{2e}\,\frac{4eK}{\hbar}\sqrt{\rho_1\rho_2} = 2K$$

$$T=0 \Rightarrow \rho_1 = \rho_2 = 1$$

Temperature dependence

$$I_c(T) = \frac{R_Q}{R}\,\frac{\Delta(T)}{2}\cdot\tanh\left(\frac{\Delta(T)}{2kT}\right)$$



Ac Josephson

Fixed Voltage $\quad \delta' = \frac{2eV}{\hbar}$

$$\delta = \frac{2e}{\hbar}\int dt\, V = \frac{2e}{\hbar} \cdot Vt + \delta_0$$

$$I = I_c \sin\delta = I_c \sin\left(\delta_0 + \frac{2eV}{\hbar} t\right)$$

$$\omega = \frac{2eV}{\hbar} \qquad f = \frac{2e}{h} V = \frac{V}{\Phi_0}$$

$1\,\mu V \Longleftrightarrow 483\ MHz$

Voltage controlled oscillator

Shapiro steps

$$I = I_c \sin\delta \ , \quad V = V_0 + V_{ac} \sin \omega t$$

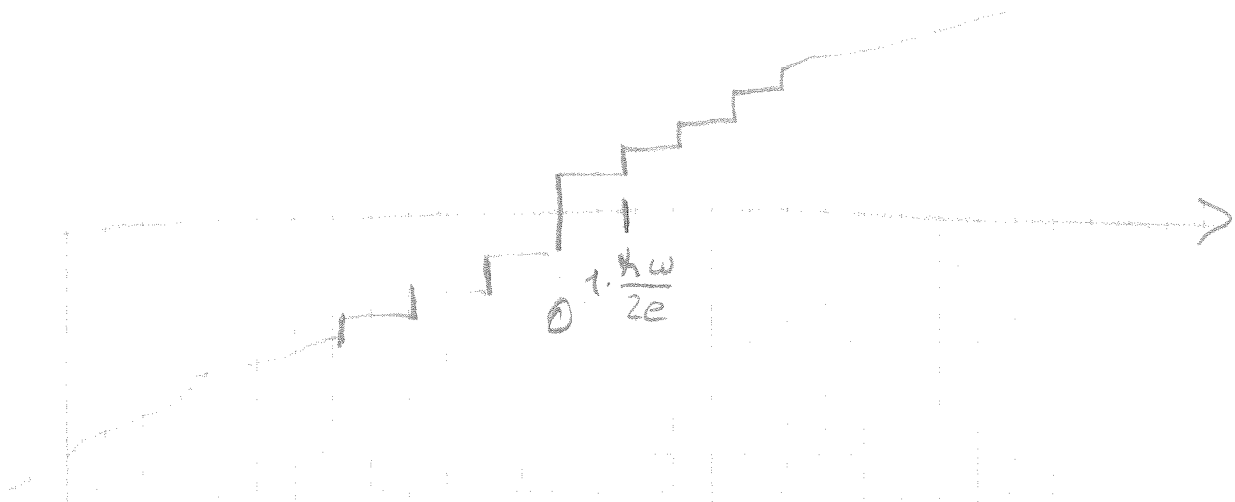
$$\delta = \frac{2e}{\hbar}\int dt \left(V_0 + V_{ac}\cos\omega t\right) = \frac{2e}{\hbar}\left(V_0 t + \frac{V_{ac}}{\omega}\sin\omega t\right) + \delta_0$$

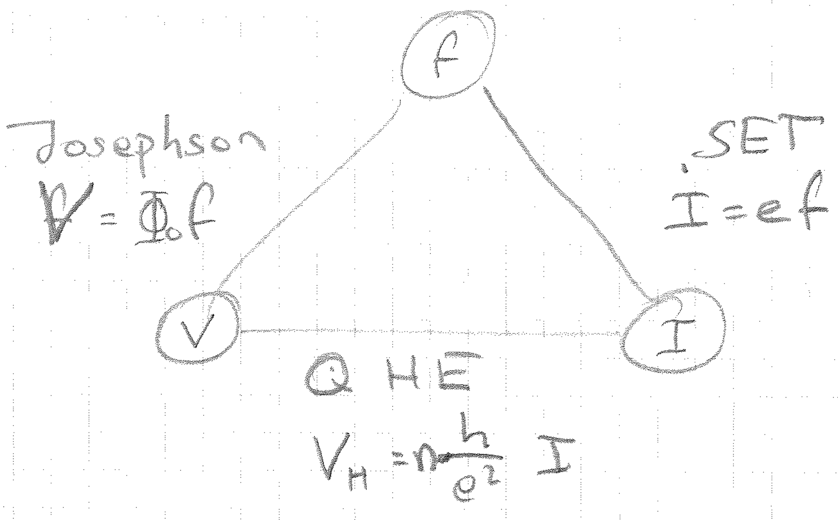
$$I = I_c \sin\left(\delta_0 + \omega_{J0} t + \frac{\omega_{J0}}{\omega_{ac}} \cdot \frac{V_{ac}}{V} \sin \omega_{ac} t\right)$$

$$\left[e^{i a \sin x} = \sum_{k=-\infty}^{\infty} J_k(a)\, \sin(i n x)\right]$$

$$I = I_0 \sum_{-\infty}^{\infty} (-1)^n J_n\left(\frac{2eV}{\hbar\omega_{ac}}\right) \sin\left[\left(\frac{2eV}{h} - n\omega_{ac}\right)t - n\Theta + \delta_0\right]$$

$$\frac{2eV}{\hbar} = n\,\omega_{ac} \Rightarrow \text{dc current}$$





Voltage Standard

Josepson Inductance and Plasma Oscillations

$$I = I_c \sin \delta$$

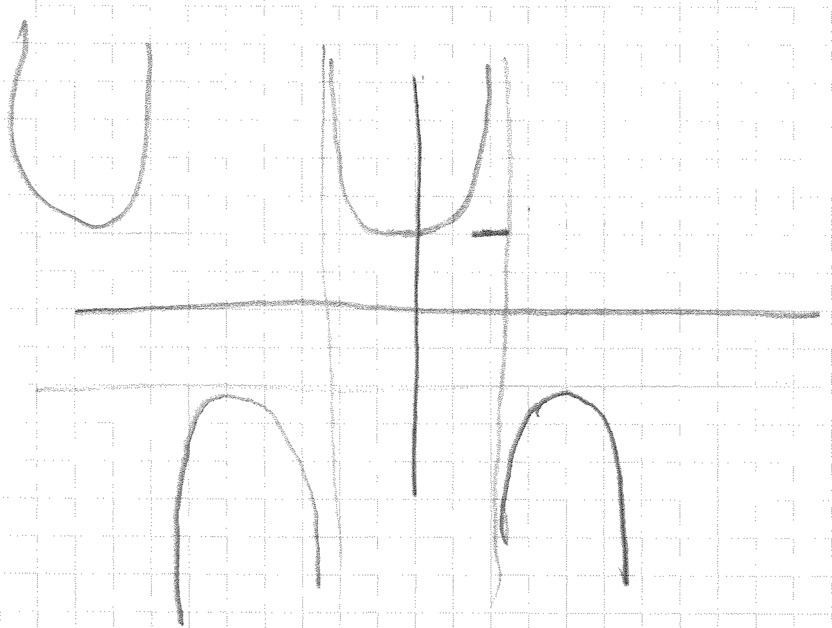
$$V = \frac{\hbar}{2e} \dot{\delta}$$

$$\frac{dI}{dt} = I_c \cos\delta \cdot \frac{d\delta}{dt} = I_c \cos\delta \cdot \frac{2e}{\hbar} V$$

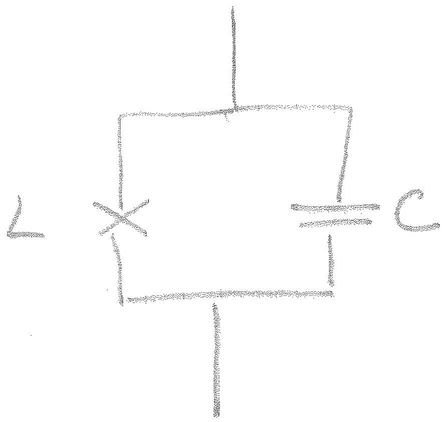
$$V = \frac{\hbar}{2e\, I_c \cos\delta} \frac{dI}{dt}$$

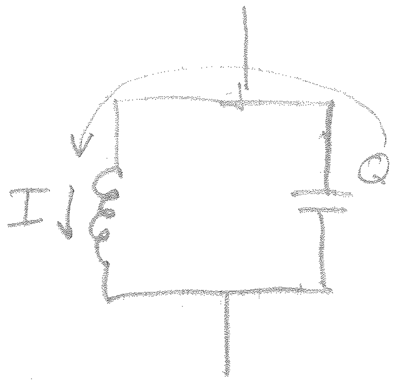
$$V = L \frac{dI}{dt}$$

$$L_J = \frac{\hbar}{2e I_c \cos\delta} = \frac{L_{J0}}{\cos\delta}$$



Plasma Oscillations



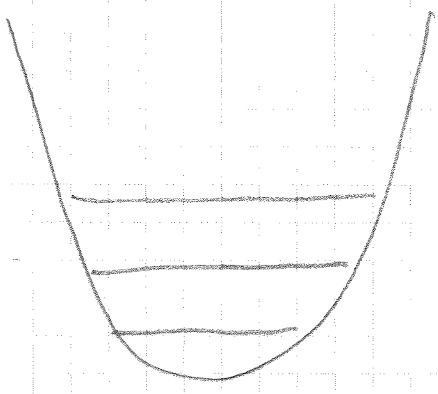


$$f = \frac{1}{2\pi\sqrt{L_J C}} \qquad f = \frac{1}{2\pi\sqrt{LC}}$$

$\delta \ll 1$ $\Downarrow$

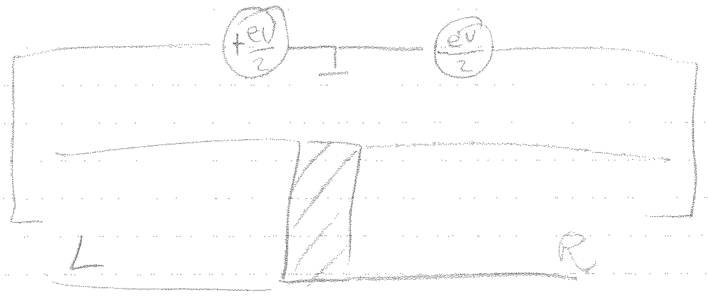
$$\frac{1}{2\pi\sqrt{\frac{\hbar C}{2eI_c}}} \iff \omega_p = \sqrt{\frac{2eI_c}{\hbar C}}$$

$$\hbar\omega_p = \sqrt{8E_J \cdot E_c} \qquad E_c = \frac{e^2}{2C}$$



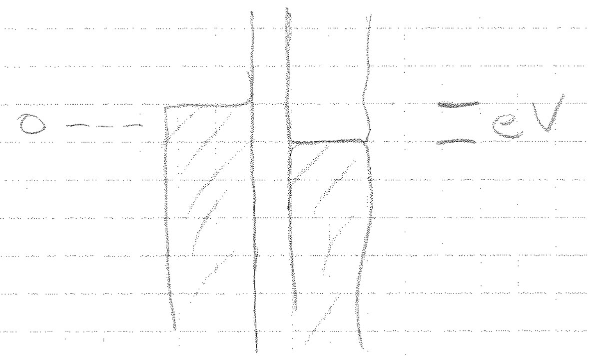
$$E = \left(n + \frac{1}{2}\right)\hbar\omega$$

Tunneling of quasiparticles



$$H = H_L + H_R + H_T$$

$$H_T = \sum_{k_L k_R} T_{LR} \left(C^\dagger_{k_L} C_{k_R} + C^\dagger_{k_R} C_{k_L} \right)$$



$$I = +e\Gamma_{L\to R} - e\Gamma_{R\to L}$$

$$\Gamma_{L\to R} = T_{LR} \int_{-\infty}^{\infty} N_L\left(E-\frac{eV}{2}\right) f\left(E-\frac{eV}{2}\right) N_R\left(E+\frac{eV}{2}\right)\left(1-f\left(E+\frac{eV}{2}\right)\right) dE$$

$$N_L(E) = N_R(E) = N(0)$$

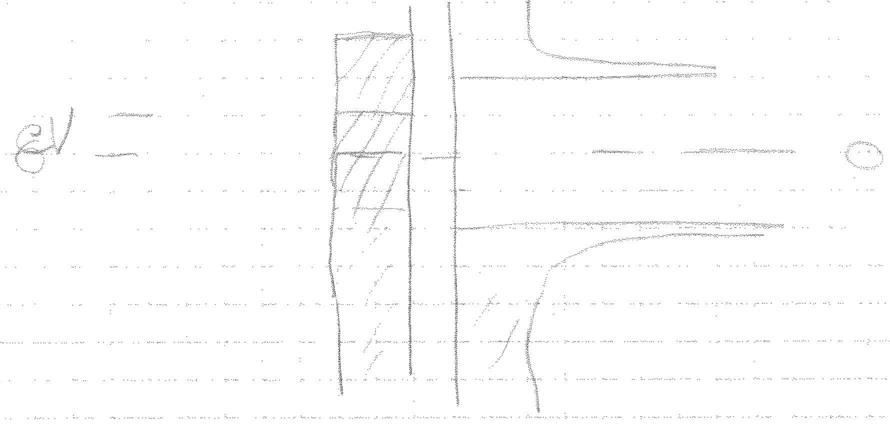
$T=0$

$$\Gamma_{L\to R} = N(0)^2 \int_{-\frac{eV}{2}}^{\frac{eV}{2}} dE = T_{LR} N(0)^2 (eV) \qquad \Gamma_{R\to L} = 0$$

$$I = +e\Gamma_{LR} = e^2 N(0)^2 T \cdot V = \frac{1}{R_N} V \qquad R_N = \frac{1}{e^2 N(0)^2 T}$$

Even if $T \neq 0$ $\quad I = \frac{1}{R} \cdot V$

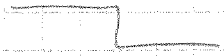
SIN - junction



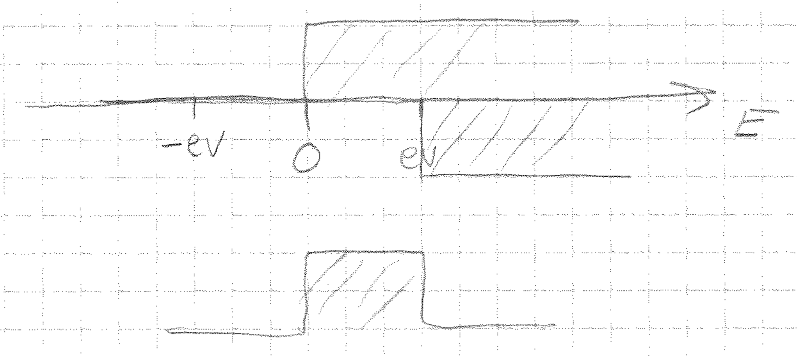
$T=0 \qquad I = e\Gamma_{LR} - e\Gamma_{RL}$

$$\Gamma_{LR} = T N^2(0) \int_{-\infty}^{\infty} f_N(E-eV) \cdot \frac{E}{\sqrt{E^2-\Delta^2}} (1-f(E))\, dE$$

$$T=0 \Rightarrow f(E) = 1-\Theta(E)$$



$$\Gamma_{LR} = T N^2(0) \int_{-\infty}^{\infty} \left[\Theta(E) - \Theta(E-eV)\Theta(E)\right] \frac{E}{\sqrt{E^2-\Delta^2}}\, dE$$



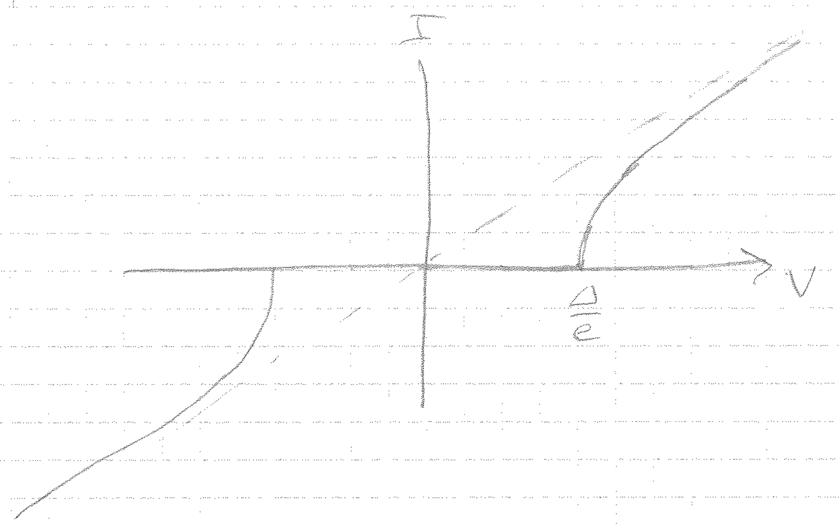
$$\Gamma_{LR} = T N^2(0) \int_{\Delta}^{eV} \frac{E}{\sqrt{E^2-\Delta^2}} = T N^2(0) \left[\sqrt{E^2-\Delta^2}\right]_{\Delta}^{eV} =$$

$$= T N^2(0) \left[\sqrt{eV^2-\Delta^2}\right]$$

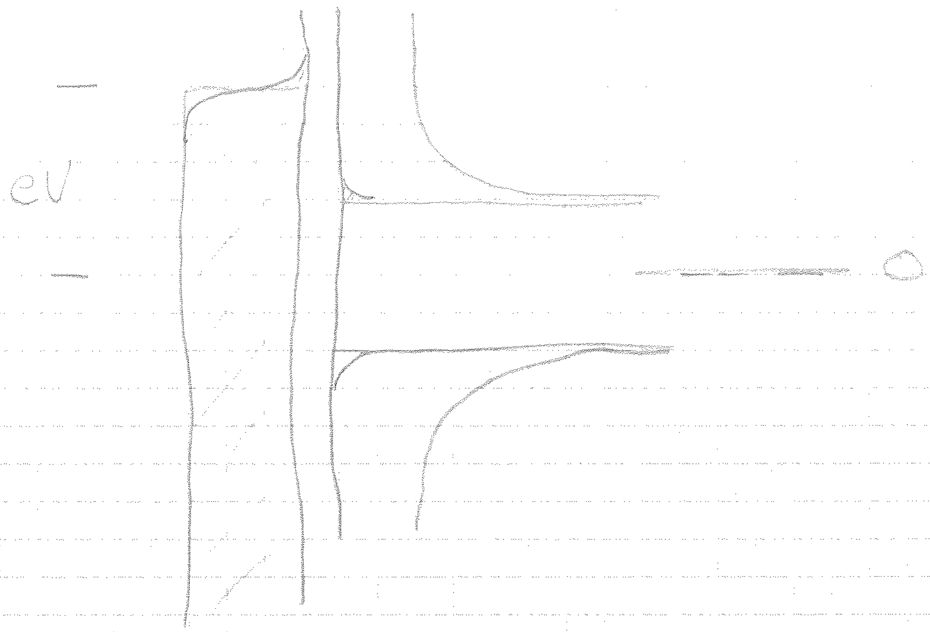
$eV > 0 \Rightarrow \Gamma_{RL} = 0$

$$I = e\Gamma_{LR} - 0 = e\Gamma N(0)\sqrt{e^2V^2 - \Delta^2}$$

Opposite when $eV < 0$



SIN junction with temperature



$$I = e\Gamma_{\rightarrow} - e\Gamma_{\leftarrow} =$$

$$= eT \cdot \int_{-\infty}^{\infty} \Big(N_N(E-eV)\, f(E-eV)\, N_S(E)\,(1-f(E)) - N_N(E-eV)\,(1-f(E-eV))\, N_S(E)\, f(E) \Big) dE$$

$$N_N(E) = N_0 \qquad N_S(E) = \begin{cases} N_0 \dfrac{E}{\sqrt{E^2+\Delta^2}} & |E| > \Delta(T) \\ 0 & |E| < \Delta(T) \end{cases}$$

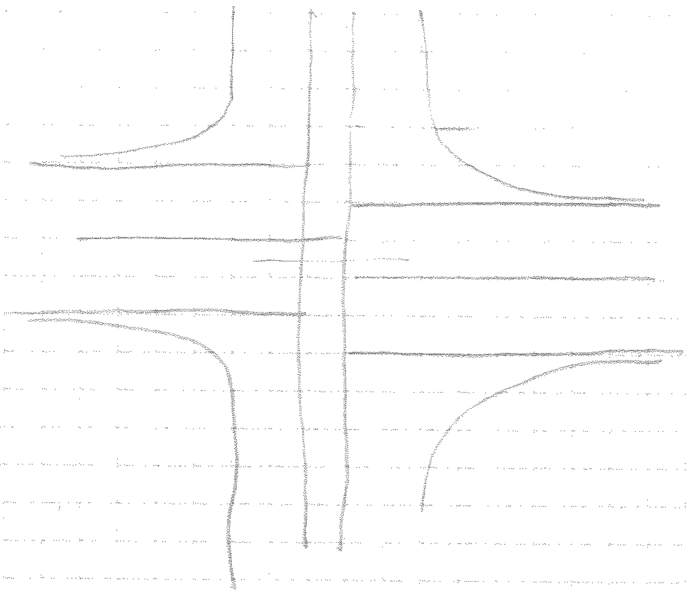
$$f(E) = \frac{1}{1 + e^{+\frac{E}{kT}}}$$

$$I = eT N_0 \int_{-\infty}^{\infty} N_S(E) \Big[f(E-eV) - f(E-eV) f(E) - f(E) + f(E) f(E-eV) \Big] dE$$

$$= eT N_0^2 \left[\int_{-\infty}^{-\Delta} \frac{E}{\sqrt{E^2+\Delta^2}} \big(f(E-eV) - f(E) \big) dE + \int_{\Delta}^{\infty} \frac{E}{\sqrt{E^2+\Delta^2}} \big(f(E-eV) - f(E) \big) \right.$$

$$f(E-eV) - f(E) = \frac{1}{1+e^{\frac{E-eV}{kT}}} - \frac{1}{1+e^{\frac{E}{kT}}} = \frac{\sinh \frac{eV}{2kT}}{\cosh\left(\frac{E - eV/2}{kT}\right) + \cosh \frac{eV}{2kT}}$$

SIS Junctions



$$\Gamma_{LR} = \int_{-\infty}^{\infty} N_L(E)\, f\!\left(E - \frac{eV}{2}\right) N_R(E) \left(1 - f\!\left(E + \frac{eV}{2}\right)\right) dE$$

limits set by Δ_L Δ_R

Δ_L Δ_R , $T = 0$