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Photon-Wavepackets as Flying Quantum Bits

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Abstract

A novel description of the interaction of quantum optical systems with a single one-photon wave packet in terms of a generalized master equation is introduced. A corresponding quantum Monte-Carlo wavefunction simulation algorithm can be obtained from the driven system approach [H. J. Carmichael, Phys. Rev. Lett. **70**, 2273 (1993); C. W. Gardiner, *ibid.* 2269 (1993)].

I. Introduction

Over the past five years quantum information [1, 2] has emerged as a novel field in quantum theory and computer science with a tremendously stimulating effect on both theoretical and experimental efforts. It now serves as an umbrella for a whole host of topics ranging from quantum cryptography to quantum computing [1–4].

In a quantum optical implementation for a quantum network photon wavepackets can serve as carriers of quantum bits for quantum information processing and communication (a "flying quantum bit"). The quantum information can be encoded, for example, in the polarization state of single photon wave packets, or can be represented by a superposition state of photon wave packets with different definite numbers of photons [1–3]. In the photonic channel as discussed in Ref. [5], for example, the qubit is encoded in a superposition of the vacuum state and a single photon wavepacket. Quantum logical operations on photon wavepackets can be realized by nonlinear optical elements which operate on the single photon level, as for example in the cavity QED implementation of the phase gate [8].

Interestingly enough relatively little has been achieved in the way of a theoretical description of the interaction of one-photon wave-packets with isolated quantum systems such as cavities or atoms. In this paper we will develop a language and formalism to describe the propagation and nonlinear interaction of photon wavepackets in a quantum optical network. Our starting point is the standard model of quantum optics, where a system of interest is coupled to a heat bath of bosonic modes representing the radiation field [7]. The radiative modes of the heatbath serve as input channels through which the system is driven, and as output channels which allow the continuous observation of the radiated fields. In the present case, a relevant example is cavity QED [8–10, 13] where an atom strongly coupled to a high-Q cavity mode represents the system, and the optical cavity mode couples to the outside radiation modes by a partially transmitting mirror. The familiar formalism developed in this context in quantum optics is tailored to the case where the incident light field

is in a coherent state (representing laser light) or a state with Gaussian statistics (thermal or squeezed light). In contrast we are interested here in a situation where the input state of the light field driving the system consists of photon wavepackets with small but *definite* numbers of photons which interact via the system and leave through the output channels. Thus, this scenario represents a particular example of a quantum optical system which is strongly driven by a non-classical light field of few-photon wavepackets. We can describe this situation from the two points of view.

First, we can specify the state of the incident radiation field directly, for example in terms of the infinite hierarchy of input correlation functions, and in a nonlinear interaction the system dynamics will be sensitive to this infinite sequence of correlation functions. In the case of N-photon wavepackets the situation is significantly simplified, however, due to the fact that the normally ordered field correlation functions of order larger than N are identically zero. As a result, this will allow us to derive a *finite* hierarchy of coupled density matrix type equations describing the system response.

A second point of view is to eliminate the incident light field in favor of the dynamics of the source which generated this field. A formalism to describe the unidirectional coupling of two quantum systems, where the first system represents the source and a second corresponds to the driven system of interest, has been developed in seminal papers by CARMICHAEL and GARDINER ([6], for precursors of this work see [11, 12]). In the present case, the sources will be generators of single photon wavepackets of a given shape [10, 13, 14]. Both of the above points of view are, of course, equivalent. In the present paper, we will develop the formalism from both of these perspectives, and compare the resulting master equations. In this paper we will mainly consider the case of single photon wave packets, and briefly present the generalization to the two-photon case. Detailed applications of this formalism will be presented elsewhere.

The paper is organized as follows: we will start out with a rederivation of a master equation for a reservoir state described by a one-photon wavepacket. We will illustrate the practicality of the method using the interaction of a two-level atom with a one-photon wavepacket. We will then proceed and proof the validity of our approach by relating it to the cascaded systems approach. Finally, we will discuss briefly an extension to the case of two-photon wavepackets which is of particular interest to quantum computation.

II. One Photon Wave Packets

Within the limits of present day technology one-photon wave-packets are very convenient carriers of quantum information between distant nodes of a quantum network. This is mainly due to the fact that optical fiber technology is extremely well developed and that photons are relatively easy to deal with. For instance quantum cryptography experiments already make use of single photons generated by parametric down conversion [1−4]. Information may be encoded in the polarization state of the photon which thus takes on the role of a *flying quantum bit*. While the practicality of sources based on parametric downconversion is somewhat limited due to insufficient external control over the generation process novel schemes which would allow the tailoring of one-photon wavepackets are already underway [13]. It has been a common practice in the past to content onseself with a rather simple description of the photons in terms of single-mode Fock-states. Clearly this suffices for many purposes but a more rigorous dynamical description of the interaction may be required for a realistic account of quantum networks.

We now consider the interaction of an isolated quantum optical system S, e.g., a resonator or a single atom, and a single one-photon wave packet whose center frequency is close to one of the resonances of S. We describe the environment our system couples to as usual by a continuum of harmonic oscillator modes. To keep things as simple as possible we

consider an interaction in one dimension. Generalization to higher dimensions is straightforward but nontrivial. Typical quantum optical system-environment couplings are characterized by a Hamiltonian of the following kind

$$\mathcal{H} = \mathcal{H}_S + \int d\omega\, \hbar\omega b^\dagger(\omega)\, b(\omega) - i(c + c^\dagger) \int d\omega\, k(\omega)\, (b(\omega) - b^\dagger(\omega))\,, \tag{1}$$

where $\mathcal{H}_S$ characterizes the system's internal unperturbed dynamics. The reservoir oscillator modes are described by bosonic creation and annihilation operators $b^\dagger(\omega)$, $b(\omega)$ satisfying standard commutation relations: $[b(\omega), b^\dagger(\nu)] = \delta(\omega - \nu)$. The reservoir couples to the system via the system operators c, $c^\dagger$. In a quantum optical context it is usually permissible to simplify the coupling by assuming a highly localized interaction which allows us to neglect the frequency dependence of the coupling constant $k(\omega)$. Since the bandwidths of the system resonances are typically much smaller than the spacing between two resonances we may omit all nonresonant terms from the interaction Hamiltonian. We then arrive at a dipole and rotating wave approximated description of the interaction which is known to give a good approximation to the real thing.

$$\mathcal{H}_{\mathrm{RWA}} = \mathcal{H}_S + \int d\omega\, \hbar\omega b^\dagger(\omega)\, b(\omega) - i \int_{-\infty}^{\infty} d\omega\, k_0(c^\dagger b(\omega) - b^\dagger(\omega)\, c)\,. \tag{2}$$

Using a coupling of this form one usually proceeds to derive a Master equation for the state of the system alone by assuming the environment to be in a simple state, e.g., the vacuum or a thermal state. Doing so implies a further approximation known as the white noise limit in which we assume a flat spectrum of the reservoir over the width of a system resonance. This then yields an equation for the system density operator which is of Lindblad form.

If we now wish to consider a state of the reservoir which is time-dependent and thus not spectrally flat we cannot apply the standard procedure and will thus have to adopt a different approach. As mentioned above we are interested in the case where a single one-photon wave-packet is incident on the system. Clearly this is quite different from the usual reservoir picture, as the single photon can be absorbed and thus the state of the reservoir will be changed. In formal terms we are interested in an initial state of the reservoir of the following kind:

$$|\Psi\rangle_R = \int_0^{\infty} d\omega\, g(\omega)\, b^\dagger(\omega)\, |\mathrm{vac}\rangle\,, \tag{3}$$

where

$$\int_0^{\infty} d\omega\, |g(\omega)|^2 = 1\,. \tag{4}$$

III. Rederivation of the Master Equation

Let us now briefly go through the standard procedure for deriving a Master equation for the system density operator. In a first step we formally integrate the time evolution equation of the reservoir modes ($\hbar = 1$):

$$\dot{b}(\omega, t) = -i\omega b(\omega, t) + k_0 c(t)\,. \tag{5}$$

Therefore we find

$$\int_{-\infty}^{\infty} d\omega\, b(\omega,\, t) = \sqrt{2\pi}\, b_{in}(t) + k_0 \int_{-\infty}^{\infty} d\omega \int_{t_0}^{t} dt'\, e^{-i\omega(t-t')}\, c(t')\,, \tag{6}$$

where the socalled input-field $b_{in}(t)$ is an explicitly time dependent Schrödinger picture operator

$$b_{in}(t) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t_0)}\, b(\omega,\, t_0)/\sqrt{2\pi}\,. \tag{7}$$

Note that the two-time commutator of the input field yields a δ-function

$$[b_{in}(t),\, b_{in}^{\dagger}(t')] = \delta(t-t')\,. \tag{8}$$

The previously made singular coupling approximation now allows us to simplify the rhs of Eq. (6). We may now break down the total field into a free field contribution and radiation reaction term:

$$k_0 \int_{-\infty}^{\infty} d\omega\, b(\omega,\, t) = \sqrt{2\varkappa}\, b_{in}(t) + \varkappa c(t)\,, \tag{9}$$

where $\varkappa = k_0^2\pi$ is the rate of radiative damping. In a next step we can now write down the equation of motion for an arbitrary system operator $X(t)$ which is defined such that at t_0: $X(t_0) = X \otimes \mathbb{1}_R$. We find

$$\begin{aligned}\dot{X}(t) = &-i[\mathcal{H}_S,\, X(t)] - [X(t),\, c^{\dagger}(t)]\,(\varkappa c(t) + \sqrt{2\varkappa}\, b_{in}(t)) \\ &+(\varkappa c^{\dagger}(t) + \sqrt{2\varkappa}\, b_{in}^{\dagger}(t))\,[X(t),\, c(t)]\,.\end{aligned} \tag{10}$$

In the standard derivation one now uses this equation to arrive at an equation of motion for the reduced density operator ϱ_s defined as

$$\varrho_s(t) = \mathrm{tr}_R\, \varrho(t)\,. \tag{11}$$

To this end we make use of the identity

$$\langle X(t)\rangle \equiv \mathrm{tr}_{S+R}\,[X(t)\,\varrho(t_0)] = \mathrm{tr}_S\,[X\varrho_s(t)]\,. \tag{12}$$

Regarding the initial condition for $\varrho(t)$ we assume a factorzed initial state of the form

$$\varrho(t_0) = \varrho_S(t_0) \otimes |\Psi_R\rangle\,\langle\Psi_R|\,, \tag{13}$$

which is clearly only meaningful if at t_0 the wavepacket is still at a sufficiently large distance from the system. Any difference in the final result for the type of Master equation obtained must necessarily arise from the terms containing the free reservoir field. Special care will thus be needed when evaluating the following two terms:

$$\mathrm{tr}_{S+R}\,([X(t),\, c^{\dagger}(t)]\, b_{in}(t)\,\varrho(t_0))\,, \tag{14}$$

$$\mathrm{tr}_{S+R}\,(b_{in}^{\dagger}(t)\,[X(t),\, c(t)]\,\varrho(t_0))\,. \tag{15}$$

E.g., substituting the rhs of Eq. (13) into Eq. (14) necessitates the evaluation of $b_{in}(t)\,|\Psi_R\rangle$ for which we find:

$$b_{in}(t)\,|\Psi_R\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, g(\omega)\, e^{-i\omega(t-t_0)}\,|\mathrm{vac}\rangle\,. \tag{16}$$

Note that $g(t)$ is actually a function of $t - t_0$ which implies that ideally $g(t)$ tends to zero as t tends to t_0. Consequently, Eq. (14) can be recast as

$$\mathrm{tr}_{S+R}\left([X(t),\, c^\dagger(t)]\, b_{in}(t)\, \varrho(t_0)\right) = g(t)\,\mathrm{tr}_{S+R}\left([X,\, c^\dagger]\,\varrho_{01}(t)\right). \tag{17}$$

Note the appearance of a new operator $\varrho_{01}(t)$ defined as

$$\varrho_{01}(t) = \mathrm{tr}_R\left(U(t,\, t_0)\,\varrho_S \otimes |\mathrm{vac}\rangle\,\langle\Psi_R|\, U^\dagger(t,\, t_0)\right), \tag{18}$$

with $U(t,\, t_0)$ denoting the propagator corresponding to the total Hamiltonian $\mathcal{H}_{\mathrm{RWA}}$. For the second term we obtain a similar result

$$\mathrm{tr}_{S+R}\left(b_{in}^\dagger(t)\,[X(t),\, c(t)]\,\varrho(t_0)\right) = g^*(t)\,\mathrm{tr}_{S+R}\left(\varrho_{01}^\dagger(t)\,[X,\, c]\right). \tag{19}$$

In order to arrive at a closed system of equations we need an evolution equation for $\varrho_{01}(t)$. This is accomplished by again exploiting the identity:

$$\mathrm{tr}_S\left(X\varrho_{01}(t)\right) = \mathrm{tr}_{S+R}\left(X(t)\,\varrho_S \otimes |\mathrm{vac}\rangle\,\langle\Psi_R|\right). \tag{20}$$

Proceeding as before we arrive at an equation for $\varrho_{01}(t)$ containing all the conventional terms of a Master equation of Lindblad form with the exception of two additional terms:

$$\mathrm{tr}_{S+R}\left([X(t),\, c^\dagger(t)]\, b_{in}(t)\,\varrho_{01}(t_0)\right) = 0\,, \tag{21}$$

$$\mathrm{tr}_{S+R}\left(b_{in}^\dagger(t)\,[X(t),\, c(t)]\,\varrho_{01}(t_0)\right) = g^*(t)\,\mathrm{tr}_S\left([X,\, c]\,\varrho_r(t)\right). \tag{22}$$

The operator $\varrho_r(t)$ is a genuine density operator and describes the state of a reference systems that evolves in the presence of an environment in the vacuum state. It is defined in the usual way as

$$\varrho_r(t) = \mathrm{tr}_R\left(U(t,\, t_0)\,\varrho_S \otimes |\mathrm{vac}\rangle\,\langle\mathrm{vac}|\, U^\dagger(t,\, t_0)\right) \tag{23}$$

The time evolution of the system S is thus governed by a 4-component system of coupled equations for two density operators $(\varrho_s,\, \varrho_r)$ of traceclass 1 and a further nonhermitian operator (ϱ_{01}) of traceclass 0. We now exploit the fact that the relations derived have to hold for an *arbitrary* system operator. Using the cyclic property of the trace operation we arrive at the following set of equations:

$$\dot{\varrho}_s(t) = \mathcal{L}\varrho_s(t) + \Omega(t)\,[\varrho_{01}(t),\, c^\dagger] - \Omega^*(t)\,[\varrho_{01}^\dagger(t),\, c]\,, \tag{24a}$$

$$\varrho_{01}(t) = \mathcal{L}\varrho_{01}(t) - \Omega^*(t)\,[\varrho_r(t),\, c]\,, \tag{24b}$$

$$\dot{\varrho}_r(t) = \mathcal{L}\varrho_r(t)\,, \tag{24c}$$

with initial conditions

$$\varrho_s(t_0) = \varrho_r(t_0) = \varrho_S\,, \quad \text{and} \quad \varrho_{01}(t_0) = 0\,. \tag{25}$$

$\Omega(t) = \sqrt{2\varkappa}\, g(t)$ is the driving field strength experienced by the system S due to the presence of a single photon. The Liouvillian $\mathcal{L}$ is given by:

$$\mathcal{L}\varrho = -i[\mathcal{H}_S,\, \varrho] - \varkappa(\{c^\dagger c,\, \varrho\}_+ - 2c\varrho c^\dagger)\,. \tag{26}$$

Eq. (24a) is almost identical to the equation we would have obtained for a system driven by a coherent field with the major exception that instead of ϱ_s the operator ϱ_{01} appears in the coupling terms. This is a signature of the fact that the single photon can be lost by absorption this altering the state of the environment.

IV. Example: Driven 2-Level Atom

To illustrate the utility of the method just presented we now consider the case of a two-level atom driven by a single one-photon wave-packet. The system is thus well characterized by two states $|g\rangle$, and $|e\rangle$, the latter of which decays with a rate of 2γ. The action of the Liouvillian is this given by

$$\mathcal{L}\varrho = -i\omega_0[c^\dagger c,\, \varrho] - \gamma(\{c^\dagger c,\, \varrho\}_+ - 2c\varrho c^\dagger)\,, \tag{27}$$

with $c = \sigma_-$ and ω_0 the atomic transition frequency. Initially the atom be in its ground state, i.e., $\varrho_s(t_0) = |g\rangle\,\langle g|$. As a convenient consequence of this we find that $\varrho_r(t)$ is time independent. This implies that we may formally integrate the equation of motion of ϱ_{01} which yields:

$$\varrho_{01}(t) = -\int_{t_0}^{t} dt'\; \Omega^*(t')\, e^{\mathcal{L}(t-t')}\, \sigma_-\,, \tag{28}$$

$$= -\int_{t_0}^{t} dt'\; \Omega^*(t')\, e^{-(\gamma - i\omega_0)\,(t-t')}\, \sigma_-\,. \tag{29}$$

The density operator thus satisfies an inhomogeneous master equation which reads

$$\dot{\varrho}_s(t) = \mathcal{L}\varrho_s(t) - \int_{t_0}^{t} dt'\; \Omega(t,\, t')\,[\sigma_-,\, \sigma_+]\,, \tag{30}$$

$$\Omega(t,\, t') = 2\,\mathrm{Re}\,(\Omega(t)\,\Omega^*(t')\, e^{-(\gamma - i\omega_0)\,(t-t')})\,. \tag{31}$$

We therefore arrive at the following formal solution for $\varrho_s(t)$:

$$\varrho_s(t) = \varrho_s(t_0) + [\sigma_+,\, \sigma_-] \int_{t_0}^{t} dt'\; e^{-2\gamma(t-t')} \int_{t_0}^{t'} dt''\; \Omega(t',\, t'')\,. \tag{32}$$

If we are interested in the interaction with a wave-packet whose spectral width is considerably smaller than the rate of spontaneous emission we may further simplify the expression

in Eq. (32). In this limit we make the following ansatz

$$\Omega(t) = \nu(t)\, e^{-i\omega_p t}\,, \tag{33}$$

where ω_p is the optical center frequency of the wave-packet whereas $\nu(t)$ is its envelope which changes slowly on the timescale given by γ^{-1}. Introducing a detuning defined by $\Delta = \omega_p - \omega_0$ we then find

$$\int_{t_0}^{t} dt'\, \Omega(t, t') \approx \frac{4\gamma^2 |g(t)|^2}{\gamma^2 + \Delta^2}\,, \tag{34}$$

and thus

$$\varrho_s(t) \approx \varrho_s(t_0) + [\sigma_+, \sigma_-]\, \frac{2\gamma |g(t)|^2}{\gamma^2 + \Delta^2}\,. \tag{35}$$

Note that this result is quite different from the one obtained for a coherent driving where there would be nonzero off-diagonal density matrix elements. It is also markedly different from what would be obtained from the interaction with a thermal reservoir with time-varying temperature due to the obvious dependence on the detuning Δ.

V. Wave Function Simulations

For sufficiently complex problems a direct integration of the Master equation may not be the most suitable method of attack for a numerical solution. For a Master equation of Lindblad type there is a straightforward prescription how a numerical quantum Monte Carlo simulation algorithm can be implemented. For the type of master equation we have just derived it is not *priori* clear whether a simulation algorithm can be found as the system evolves in a non-Markovian fashion. In principle there exists another approach to describe the interaction between a small system and a nontrivial input field state which is known as the *cascaded systems approach* [6]. The basic idea behind this method is to embed the system into a larger Hilbert space which also includes the degrees of freedom of the system which generated this particular input field state.

Suppose we label the system generating the one photon wave-packet A we then have to consider the dynamics of the compound system $A + S + R$. The coupling between A and S take place via the harmonic oscillator modes of the environment R now common to both systems. It is important to assume a onedirectional nature of the coupling between A and S which is necessary to sustain the picture of system A driving system S. A graphical illustration of a possible realization is given in Fig. 1.

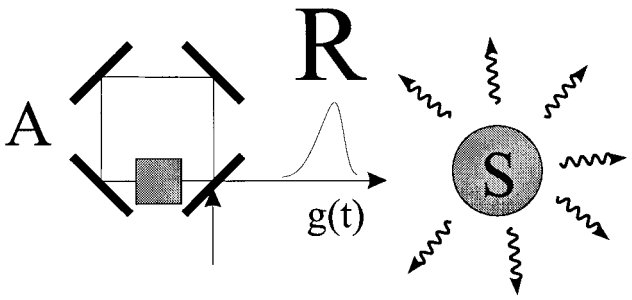

Fig. 1. Model compound system where A is a ring cavity whose output drives system S in a unidirectional fashion.

If we denote the coupling constant between system A and the reservoir by $\varkappa_a(t)$ the operator mediating the coupling by a and the reduced density operator of the enlarged system $A+S$ by ϱ the following Master equation ensues

$$\dot{\varrho}(t) = \mathcal{L}_{\text{tot}}\varrho(t) + 2\sqrt{\varkappa_a(t)\,\varkappa}([a\varrho(t),\, c^\dagger] + [c,\, \varrho(t)\,a^\dagger])\,, \tag{36}$$

$$\mathcal{L}_{\text{tot}}\varrho = \mathcal{L}\varrho(t) - i[\mathcal{H}_A(t),\, \varrho] - \varkappa_a(t)\,(\{a^\dagger a,\, \varrho\}_+ - 2a\varrho a^\dagger)\,, \tag{37}$$

with $\mathcal{H}_A(t)$ denoting the (potentially explicitly time dependent) unperturbed internal dynamics of the driving system A. The initial state of ϱ is given by the product of the initial states of systems A and S

$$\varrho(t_0) = \varrho_S \otimes \varrho_A\,. \tag{38}$$

The state ϱ_A needs to be chosen in such a way that in conjunction with the dynamics given by $\mathcal{L}_{\text{tot}}$ the desired input state for system S is generated. If we wish to numerically simulate the evolution of this system we proceed in the usual fashion by propagating the system in between state reductions with an effective nonhermitian Hamiltonian

$$\mathcal{H}_{\text{eff}} = \mathcal{H}_A(t) + \mathcal{H}_S - i(\varkappa c^\dagger c + \varkappa_a(t)\,a^\dagger a + 2\sqrt{\varkappa\varkappa_a(t)}\,c^\dagger a)\,. \tag{39}$$

State reductions are implemented by applying the operator $C(t) \equiv \sqrt{\varkappa}c + \sqrt{\varkappa_a(t)}\,a$ to the state of the compound system in conjunction with a subsequent renormalization. It now remains to be shown that we may find a driven system description for the set of master equations derived in the preceding section. This would reconcile both methods and thus underpin the validity of the presented Master equation approach.

VI. Making the Connection

When having a closer look at Eq. (36) we immediately realize that there are certain terms which are strongly reminiscent of the additional terms appearing in Eq. (24a). We will now show that the reduced density operator $\varrho_s^c = \text{tr}_A\varrho$ obtained from the cascaded systems master equation can be adapted to satisfy the same equation as ϱ_s introduced in Eq. (11). We find

$$\dot{\varrho}_s^c(t) = \mathcal{L}\varrho_s^c(t) + 2\sqrt{\varkappa_a(t)\,\varkappa}([\tilde{\varrho}_s(t),\, c^\dagger] + [c,\, \tilde{\varrho}_s^\dagger(t)])\,, \tag{40}$$

$$\tilde{\varrho}_s(t) = \text{tr}_A(a\varrho(t))\,. \tag{41}$$

Note the explicit time-dependence of the damping rate $\varkappa_a$ which can be accomplished by manipulating the coupling of system A to the environment. This is already of exactly the same structure as Eq. (24a). It now remains to find an equation of motion for $\tilde{\varrho}_s$. Assuming bosonic commutation relations for a we find:

$$\begin{aligned}\dot{\tilde{\varrho}}_s(t) = {} & \mathcal{L}\tilde{\varrho}_s(t) - \text{tr}_A((\varkappa_a(t)\,a + i[a,\, \mathcal{H}_A])\,\varrho(t)) \\ & + 2\sqrt{\varkappa_a(t)\,\varkappa}\,\text{tr}_A([c,\, a\varrho(t)\,a^\dagger] + [a^2\varrho(t),\, c^\dagger])\,.\end{aligned} \tag{42}$$

We are dealing here with an approach that allows us to consider arbitrarily complicated states to be generated by system A. In order to further simply Eq. (42) we have to make

further assumptions. The dynamics of system A be therefore such that there is only a limited supply of energy. The total amount of energy is not to exceed the equivalent of a single excitation of mode a. Under these circumstances we may drop the last term in Eq. (42). Moreover, we assume the commutator $[a,\ \mathcal{H}_A]$ to be a linear functional of a. This is a meaningful assumption as higher order nonlinear terms would not contribute due to the fact that there is at most a single excitation present in system A. We then may write:

$$\dot{\tilde{\varrho}}_s(t) = \mathcal{L}\tilde{\varrho}_s(t) - (\varkappa_a(t) + if(t))\,\tilde{\varrho}_s(t) + 2\sqrt{\varkappa_a(t)\,\varkappa}\,[c,\ \varrho_0(t)]\,, \tag{43}$$

$$\varrho_0(t) = \mathrm{tr}_A(a\varrho(t)\,a^\dagger)\,. \tag{44}$$

Note that $f(t)$ is a complex function which stems from the commutator in Eq. (42). Consequently, for $\varrho_0(t)$ we find:

$$\dot{\varrho}_0(t) = \mathcal{L}\varrho_0(t) - (2\varkappa_a(t) + i[f(t) - f^*(t)])\,\varrho_0(t) \tag{45}$$

If we now finally introduce the formal abbreviation

$$\alpha(t) = e^{-\int_{t_0}^{t} dt'\,(\varkappa_a(t') + if(t'))}\,, \tag{46}$$

we may make the following identifications:

$$\varrho_0(t)/|\alpha(t)|^2 \leftrightarrow \varrho_r(t) \tag{47}$$

$$\tilde{\varrho}_s(t)/\alpha(t) \leftrightarrow \varrho_{01}(t) \tag{48}$$

$$2\sqrt{\varkappa\varkappa_a(t)}\,\alpha(t) \leftrightarrow \Omega(t) \tag{49}$$

$$\varrho_s^c(t) \leftrightarrow \varrho_s(t) \tag{50}$$

It now basically remains to find an explicit model for the hitherto unspecified physical system capable of generating a single one-photon wave-packet.

VII. Tailoring Wave-Packets

From a purely pragmatic point of view it would suffice to consider a cavity initially prepared in a one-photon Fock state which is then allowed to leak out in a controlled fashion. The shape of the envelope and its phase could be controlled by modulating the cavity decay rate and its resonance frequency. While this suffices to generate an arbitrarily shaped wave-packet in a numerical simulation this approach is somewhat unsatisfactory from an experimental point of view as such a procedure is exceedingly difficult to implement. It has however recently been shown by Law and Kimble [13] that optical pumping can be used to produce single one-photon wave packets with near certainty. Thus there are realistic physical intra cavity systems that do give rise to essentially the same behavior as a time varying decay constant or mistuning. We may thus regard such a modulated resonator as a an effective model of a much more complicated system which is driven externally. It now remains to derive the explicit time dependence of $\varkappa(t)$ and $\Delta(t)$ for a given desired pulse shape $g(t)$, cf. also Eq. (49).

To this end we recall the equation of motion for the mode operator $a(t)$ of such a modulated cavity

$$\dot{a}(t) = -(\varkappa(t) + i\Delta(t))\, a(t) - \sqrt{2\varkappa(t)}\, a_{in}(t)\,, \tag{51}$$

with $a_{in}(t)$ describing a vacuum input to the cavity. By way of formal integration we arrive at the following expression for the field emanating from the cavity

$$a_{out}(t) = a_{in}(t) - \int_{t_0}^{t} 2\,\varkappa(t')\, \frac{g(t)}{g(t')}\, a_{in}(t')\, dt' + g(t)\, a(t_0)\,, \tag{52}$$

where

$$g(t) = \sqrt{2\varkappa(t)} \exp\left(-\int_{t_0}^{t} (\varkappa(t') + i\Delta(t'))\, dt'\right). \tag{53}$$

If the cavity is initially prepared in a one-photon Fock state we may identify $g(t)$ as the complex temporal envelope of a one-photon wavepacket emitted by the cavity.

Any given $g(t)$ may now be expressed in terms of a unique $\varkappa(t)$ and $\Delta(t)$ given by:

$$\varkappa(t) = (|g(t)|^2/2\left(1 - \int_{t_0}^{t} |g(t')|^2\, dt'\right), \tag{54}$$

$$\Delta(t) = i(\dot{g}(t)\, g^*(t) - g(t)\, \dot{g}^*(t))/2|g(t)|^2 \tag{55}$$

VIII. Correlation Functions

A full account of the properties of the driven system requires also a prescription for the calculation of higher order temporal correlation functions. For interactions describable by a single density operator alone there exists a well known approach to work out multi-time correlation functions known as the quantum regression theorem [15]. For most purposes it suffices to calculate two-time correlation functions which we shall outline briefly here-higher order correlation function are obtained by repeatedly applying the same procedure as below. Suppose we wish to calculate the correlation function

$$\langle A(t_2),\, B(t_1)\rangle = \mathrm{tr}_S(A\, \mathrm{tr}_R(U(t_2,\, t_1)\, B\varrho(t_1)\, U^\dagger(t_2,\, t_1)))\,, \tag{56}$$

where $t_2 \geq t_1$ without loss of generality. It now remains to work out the evolution equation for the operator $\varrho(t;\, t_1;\, B)$ defined as

$$\varrho_s(t;\, t_1;\, B) = \mathrm{tr}_R(U(t,\, t_1)\, B\varrho(t_1)\, U^\dagger(t,\, t_1)) \quad \text{and} \quad \varrho_s(t_1;\, t_1;\, B) = B\varrho_s(t_1)\,. \tag{57}$$

Using the same techniques as in the preceding sections one can easily show that we arrive at a set of equations identical to Eqs. (24) for a new set of operators $\{\varrho_s(t;\, t_1;\, B),\ \varrho_{01}(t;\, t_1;\, B),\ \varrho_r(t;\, t_1;\, B)\}$ with initial conditions $\varrho_\alpha(t_1;\, t_1;\, B) = B\varrho_\alpha(t_1)$. In brief this means we obtain a result analogous to the one for the interaction with standard types of reservoirs. The calculation of two-time correlation function thus reduces to the

calculation of simple one-time expectation values using the same density operator equations but with appropriately modified initial conditions. This is in essence what the quantum regression theorem states.

IX. Two Photon Wavepackets

Two-photon wavepackets are of particular interest for two bit gate operations in distributed quantum computing. A carrier and a target bit undergo local processing and are then passed on to the next node or gate [8]. Both bits could be accommodated in the polarization states of a two-photon wavepacket. It seems reasonable to assume that the approach presented above will also work for two-photon wavepackets. While this is more or less true the derivation requires some extra care. This is a consequence of the by far more complex nature of a two-photon wavepacket:

$$|\Psi_2\rangle_R = \frac{1}{\sqrt{2}} \int d\omega_1 \int d\omega_2 \, g(\omega_1, \omega_2) \, b^\dagger(\omega_1) \, b^\dagger(\omega_2) \, |\mathrm{vac}\rangle \,, \tag{58}$$

which holds in one dimension. For simplicity we have omitted polarization indices. It is reasonable to assume that g is symmetrical, i.e., $g(\omega_1, \omega_2)$ is the same as $g(\omega_2, \omega_1)$. It then follows

$$\int d\omega_1 \int d\omega_2 |g(\omega_1, \omega_2)|^2 = 1 \,. \tag{59}$$

Going through the same procedure as outlined in Sec. III produces states with an explicit time dependence such as

$$b_{in}(t) \, |\Psi_2\rangle_R = \sqrt{2} \int d\omega \, g_s(\omega, t) \, b^\dagger(\omega) \, |\mathrm{vac}\rangle \,, \tag{60}$$

where

$$g_s(\omega, t) = \int \frac{d\nu}{\sqrt{2\pi}} \, e^{-i\nu(t-t_0)} \, g(\omega, \nu) \,. \tag{61}$$

Unfortunately here the time dependence is intricately intertwined with the spectral composition of the remaining one-photon wavepacket. A treatment as simple as in the one-photon case is thus not possible. Nevertheless we may identify two tractable classes of spectral envelopes which cover practically all physically relevant cases:

1. the two-photon wavepacket was created by a single source in a way that both photons are of almost the same frequency with a distribution that is much narrower than any of the resonances of the system the packet impinges on. We may then carry out the following approximation

$$\dot{g}_s(\omega, t) \approx -i\omega_p g_s(\omega, t) \,, \tag{62}$$

   with ω_p the center frequency of the two-photon wavepacket.
2. each of the photons constituting the two-photon wavepacket was created by a different source. The two-photon spectral envelope can be factorized, i.e.,

$$g(\omega_1, \omega_2) = (g_1(\omega_1) \, g_2(\omega_2) + g_1(\omega_2) \, g_2(\omega_1))/2 \,. \tag{63}$$

For Eq. (60) this implies the following simplification:

$$b_{in}(t)\,|\Psi_2\rangle_R = \frac{1}{\sqrt{2}} \sum_{\substack{i,j=1,2 \\ i\neq j}} g_i(t) \int d\omega\, g_j(\omega)\, b^\dagger(\omega)\,|\mathrm{vac}\rangle\,, \tag{64}$$

where $g_i(t)$ denotes the Fourier transform of $g_i(\omega)$.
In order to accommodate the most general case we could also adopt a procedure similar to the one of case 2 above. By expanding $g(\omega_1, \omega_2)$ in an orthonormal set of functions we would arrive at an infinite hierarchy of coupled density operator equations. The generalization to arbitrary two-photon envelopes is thus straightforward; obtaining an analytical solution might however become quite challenging.

A. The general case

In the case of a general two-photon spectral envelope we now expand $g(\omega_1, \omega_2)$ in terms of an orthonormal set of functions $\{f_\alpha(\omega)\}$:

$$g(\omega_1, \omega_2) = \sum_{\alpha,\beta} g_{\alpha\beta} f_\alpha(\omega_1) f_\beta(\omega_2)\,, \tag{65}$$

with $g_{\alpha,\beta}$ complex and symmetric in α, β. Note that in the case of nearly Gaussian spectral distributions the harmonic oscillator eigenfunctions would in principle provide a reasonable basis set. Having done so we realize the need to introduce a set of reduced, normalized one-photon wavepackets defined by

$$|\Psi_\alpha\rangle = \int d\omega\, f_\alpha(\omega)\, b^\dagger(\omega)\,|\mathrm{vac}\rangle\,. \tag{66}$$

To simplify notation we set $|\Psi_0\rangle = |\mathrm{vac}\rangle$. This permits the definition of the following set of operators (note that $\varrho(t) \equiv \varrho_{22}(t)$)

$$\varrho_{ij}(t) = \mathrm{tr}_R(U(t, t_0))\,|\Psi_i\rangle\,\langle\Psi_j| \otimes \varrho_S U^\dagger(t, t_0))\,, \tag{67}$$

with $i, j \in \{0, \alpha, 2\}$. Using the same procedure and model as in Sec. III one obtains

$$\dot\varrho_{22}(t) = \mathcal{L}\varrho_{22}(t) + \sqrt{2}\sum_{\alpha,\beta}(f_\alpha(t)\, g_{\alpha\beta}[\varrho_{\beta 2}(t),\, c^\dagger] - f^*_\alpha(t)\, g^*_{\alpha\beta}[\varrho^\dagger_{\beta 2}(t),\, c])\,, \tag{68a}$$

$$\dot\varrho_{\alpha 2}(t) = \mathcal{L}\varrho_{\alpha 2}(t) + (f_\alpha(t)\,[\varrho_{02}(t),\, c^\dagger] - \sqrt{2}\sum_{\beta,\gamma}[\varrho_{\alpha\beta}(t),\, c]\, g^*_{\beta\gamma} f^*_\gamma(t))\,, \tag{68b}$$

$$\dot\varrho_{02}(t) = \mathcal{L}\varrho_{02}(t) - \sqrt{2}\sum_{\alpha,\beta}[\varrho_{0\alpha}(t),\, c]\, g^*_{\alpha\beta} f^*_\beta(t)\,, \tag{68c}$$

$$\dot\varrho_{\alpha\beta}(t) = \mathcal{L}\varrho_{\alpha\beta}(t) + (f_\alpha(t)\,[\varrho_{0\beta}(t),\, c^\dagger] - [\varrho_{\alpha 0}(t),\, c]\, f^*_\beta(t))\,, \tag{68d}$$

$$\dot\varrho_{0\alpha}(t) = \mathcal{L}\varrho_{0\alpha}(t) - f^*_\alpha(t)\,[\varrho_{00}(t),\, c]\,, \tag{68e}$$

$$\dot\varrho_{00}(t) = \mathcal{L}\varrho_{00}(t)\,, \tag{68f}$$

where

$$f_\alpha(t) = \sqrt{\frac{\varkappa}{\pi}} \int d\omega\, e^{-i\omega(t-t_0)} f_\alpha(\omega)\,. \tag{69}$$

In case of convergence problems we would have to issue the integral kernel with an additional cut-off term which reflects the finite bandwidth of the system resonance. The compact hierarchy of equations we have just derived is nigh insoluble unless the series expansion may be truncated after a few terms. Let us therefore turn to the two special cases mentioned above and try to see how they may be derived from the general case.

B. Narrow band two-photon source

As outlined above the assumption of narrow bandwith of the two-photon wavepackets allows a simple further treatment of the reduced wavepacket given in Eq. (60). We here promote a wholistic approach in which all partial wavepackets are dealt with in one fell swoop. The idea is to no longer discern between the contributions from the various partial wavepackets but rather to consider quantities of the following kind, as e.g.,

$$\varrho_{12}(t) = \sum_{\alpha,\beta} f_\alpha(t)\, g_{\alpha\beta} \varrho_{\beta 2}(t)\,, \tag{70}$$

which appears on the rhs of Eq. (68a). Further operators ϱ_{11}, and ϱ_{01} are defined in an analogous way. A further treatment hinges on the assumption that to good approximation the following is true for all $f_\alpha(t)$: $\dot{f}_\alpha(t) \approx -i\omega_p f_\alpha(t)$.

We then may derive the following set of coupled equations

$$\dot{\varrho}_{22}(t) = \mathcal{L}\varrho_{22}(t) + \sqrt{2}\,([\varrho_{12}(t),\, c^\dagger] - [\varrho_{12}^\dagger(t),\, c])\,, \tag{71a}$$

$$\dot{\varrho}_{12}(t) = (\mathcal{L} - i\omega_p)\,\varrho_{12}(t) + (g_s(t)\,[\varrho_{02}(t),\, c^\dagger] - \sqrt{2}\,[\varrho_{11}(t),\, c])\,, \tag{71b}$$

$$\dot{\varrho}_{02}(t) = \mathcal{L}\varrho_{02}(t) - \sqrt{2}\,[\varrho_{01}(t),\, c]\,, \tag{71c}$$

$$\dot{\varrho}_{11}(t) = \mathcal{L}\varrho_{11}(t) + (g_s(t)\,[\varrho_{01}(t),\, c^\dagger] - g_s^*(t)\,[\varrho_{01}^\dagger(t),\, c])\,, \tag{71d}$$

$$\dot{\varrho}_{01}(t) = (\mathcal{L} + i\omega_p)\,\varrho_{01}(t) - g_s^*(t)\,[\varrho_{00}(t),\, c]\,, \tag{71e}$$

$$\dot{\varrho}_{00}(t) = \mathcal{L}\varrho_{00}(t)\,, \tag{71f}$$

with the dimensionless effective complex pulse shape

$$g_s(t) = \sum_{\alpha,\beta} f_\alpha(t)\, g_{\alpha\beta} f_\beta(t)\,. \tag{72}$$

Note that the last three equations of this set are basically identically to those obtained for a single one-photon wave-packet. For the simple example of Sec. IV this set of equations is soluble within the validity of the underlying assumptions about the width of the wavepacket and the system time scales.

C. Independent sources

For two independent one-photon sources it is legitimate to restrict our considerations to those $g(\omega_1, \omega_2)$ which can be represented as a symmetrized product of the spectral envelopes of the two one-photon wavepackets. This case is of particular interest in quantum computing when two quantum bits generated by independent sources undergo processing in a two-bit quantum gate as suggested by Kimble and coworkers [8].

The corresponding set of equations can be obtained from the result for the general case by simply considering a set of functions $\{f_\alpha\}$ with only two not necessarily orthogonal members. We thus refrain from explicitly writing down the resulting master equations. It must, however, be emphasized that the complexity of these equations especially for a non-trivial Liouvillian $\mathcal{L}$ can become so great that a direct integration of the explicitly time-dependent master equation seems a hopeless task. Still these problems lend themselves to a formulation in terms of stochastic wavefunction simulations as there exists a corresponding cascaded systems picture.

X. Conclusions

In this paper we have developed a formalism to model the interaction of a quantum optical system with N-photon wave packets, as is relevant for quantum information processing and quantum communications with photons. As a result, we have obtained a hierarchy of coupled density matrix type equations which describe the system response to the incident wave packets. In the present paper we have introduced the basic ideas of our approach illustrated in the context of single photon wave packets, and we have outlined the extension of the formalism to the two-photon case. Examples of the application of this theory, in particular to quantum logical operations with photons, will be presented elsewhere.

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