An accurate determination of the acceleration of gravity g in the undergraduate laboratory

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A rod, connected with a horizontal hinge to the vertical axle of a motor, will deviate from the vertical at angular frequencies of the motor above a certain minimum value. The acceleration of gravity is related in a simple way to the speed of the motor, the length of the rod, and its angle with the vertical. It turns out that g can be measured in this way with an accuracy of about 0.1% without the use of sophisticated technology or the application of corrections to the result. The system has some other interesting features which make it particularly appropriate for the undergraduate laboratory.

I. THE CASE OF THE SIMPLE PENDULUM

Suppose a particle with mass m is connected by a massless wire of length l to the vertical axle of a motor. Due to the centrifugal force on the particle the wire will deviate from its vertical position at a sufficiently large angular frequency of the motor and make an angle θ with the vertical. If the angular frequency of the motor is constant, the mass will move in a horizontal circle and the pendulum has only one degree of freedom, θ. Such a system is called a *conical pendulum*. It is discussed in some standard texts on mechanics, e.g., by Kibble and Berkshire.[1] Several papers on the subject have been published in this journal.[2–5] Some authors[6] use a flyball governor as an illustration of the same mechanics.

We introduce a reference system (x,y,z) which is rigidly connected to the axle of the motor (Fig. 1). The origin O is at the fixed point of the wire. The z axis points down. The horizontal y axis is in the meridian plane of the wire. If the angular frequency of the motor is constant there will be equilibrium in the rotating system (x,y,z) if the sum of the moments of the gravitational force $mg\mathbf{u}_z$ and of the centrifugal force $my\omega^2\mathbf{u}_y$ and the torque of the hinge on the pendulum equals zero. Setting the x component of the total moment equal to zero, one gets

$$ml\sin\theta(l\cos\theta\omega^2-g)=0. \tag{1.1}$$

Solutions of this equation are

1. $$\sin\theta=0, \tag{1.2}$$
2. $$\cos\theta=\frac{g}{l\omega^2}. \tag{1.3}$$

Since $\cos\theta\leqslant 1$ the solution (1.3) exists only for $\omega^2\geqslant g/l$: The pendulum can leave its vertical position provided the angular frequency is at least as large as the resonance frequency of the simple pendulum.

With (1.3) the value of g can be determined from ω, θ, and l:

$$g=l\omega^2\cos\theta. \tag{1.4}$$

II. STABILITY OF THE SOLUTIONS

We investigate the potential energy as a function of the angle θ with the vertical. Since equilibrium is established in the rotating reference system we must include the potential energy

$$V_{\mathrm{cf}}=-\tfrac{1}{2}m\omega^2y^2=-\tfrac{1}{2}ml^2\omega^2\sin^2\theta \tag{2.1}$$

of the centrifugal force.

The total potential energy V becomes

$$V=mgl(1-\cos\theta)-\tfrac{1}{2}ml^2\omega^2\sin^2\theta \tag{2.2}$$

and its derivative with respect to θ is

$$\frac{dV}{d\theta}=mgl\sin\theta-ml^2\omega^2\sin\theta\cos\theta. \tag{2.3}$$

Putting this expression equal to zero yields the solutions (1.2) and (1.3):

$$\theta=0\quad\text{or}\quad\pi \tag{2.4}$$

and

$$\theta=\arccos(g/l\omega^2). \tag{2.5}$$

The stability of these solutions depends on the sign of

$$\frac{d^2V}{d\theta^2}=mgl\cos\theta-ml^2\omega^2(\cos^2\theta-\sin^2\theta) \tag{2.6}$$

at the corresponding values of θ.

For $\theta=0$,

$$\frac{d^2V}{d\theta^2}=mgl-ml^2\omega^2. \tag{2.7}$$

If $\omega^2<g/l$ this expression is positive and the equilibrium is stable.

If $\omega^2>g/l$, the second derivative of V is negative and the equilibrium is unstable.

For $\theta=\pi$, the second derivative is always negative.

For $\theta=\arccos(g/l\omega^2)$,

$$\frac{d^2V}{d\theta^2}=-\frac{mg^2}{\omega^2}+ml^2\omega^2. \tag{2.8}$$

If $\omega^2\geqslant g/l$ (the condition for the existence of this solution) the second derivative is positive and the solution is stable.

The results are illustrated in Fig. 2 for $\omega^2<g/l$ and in Fig. 3 for $\omega^2>g/l$. The conical pendulum is thus characterized by a critical frequency

$$\omega_c=\sqrt{\frac{g}{l}} \tag{2.9}$$

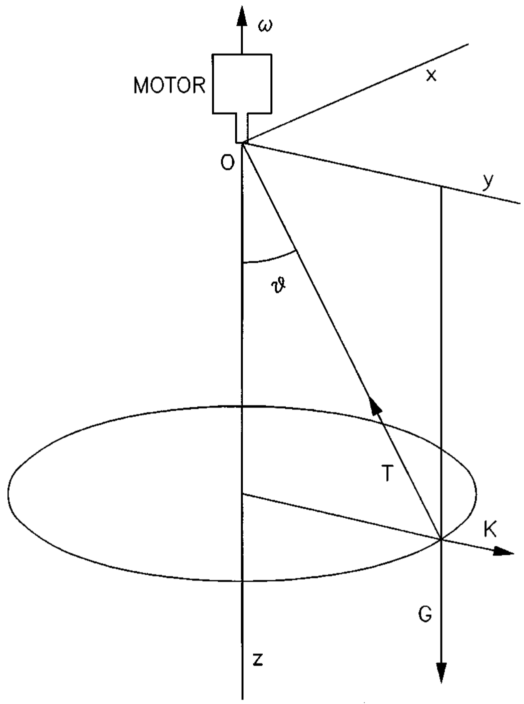

Fig. 1. The conical pendulum. G is the gravitational force, K the centrifugal force, and T the tension in the wire.

above which the pendulum can deviate from the vertical. This allows one to write (1.3) as

$$\cos\theta = \frac{\omega_c^2}{\omega^2}. \tag{2.10}$$

The system is said to show spontaneous symmetry breaking. Below the critical frequency ω_c the pendulum stays vertical, i.e., symmetrical with respect to the only special orientation, that of the gravitational force. Above ω_c the pendulum deviates from the vertical and makes an angle θ with it. In the y,z plane, the symmetry with respect to the orientation of the gravitational force is lost or ''broken.''

Spontaneous symmetry breaking is an important phenomenon in elementary particle physics, but it also occurs in several other branches of physics. A well-known example is the transition from paramagnetism to ferromagnetism. The paramagnetic state is characterized by a random orientation of the atomic magnetic moments. This arrangement changes when the temperature is lowered below the Curie temperature, the critical temperature T_c at which ferromagnetism sets in. In this state the atomic moments are oriented parallel to each other in a particular direction which could as well have been another one. The symmetry with respect to direction is broken.

Several authors[2–4,7] have treated simple mechanical systems like this one just to illustrate the principle of symmetry breaking. In this paper, however, we concentrate on a means of measuring the acceleration of gravity with a conical pendulum. Several other methods have been used for this purpose. The most obvious method is presumably the free fall experiment, but it sometimes suffers from a lack of precision. This can be remedied by slowing down the motion, either by observing the motion on a slope as was done by Galileo, or by using an Atwood's machine.

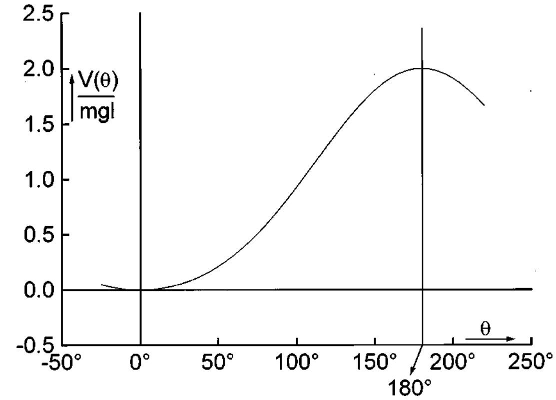

Fig. 2. Potential for $\omega^2 = \frac{1}{2}g/l$.

More precise measurements can be performed by determining the period of a plane pendulum. This was done in a very accurate way by Nelson and Olsson.[8] Jesse[9] uses a Kater pendulum, in which the swinging body can be suspended from two different points of the body.

An extremely accurate $(3\times 10^{-9}!)$ experiment was done recently by Peters *et al.*[10] using an atom interferometer.

The occurrence of a critical frequency in the conical pendulum may, however, well contribute to provoking the student's interest.

III. THE CASE OF THE PHYSICAL PENDULUM

Even if the particle in the case of the simple pendulum is replaced by a sphere, the system is not well suited to a measurement of g. A real suspension wire is not massless. The rotation of the motor will make it twist, which results in an undesirable complication of the motion. Furthermore, an accurate determination of the angle between the moving wire and the vertical is difficult. These problems can be avoided by using a physical pendulum instead of a simple pendulum.

We give a treatment of the motion of the physical conical pendulum in three different ways, thus providing an appropriate theory for students with different levels of knowledge.

The first treatment is elementary and follows the same lines as our treatment of the simple pendulum. For simplicity we use here the approximation of a thin rod. The moment of the centrifugal force is obtained by a simple integration over the length of the rod.

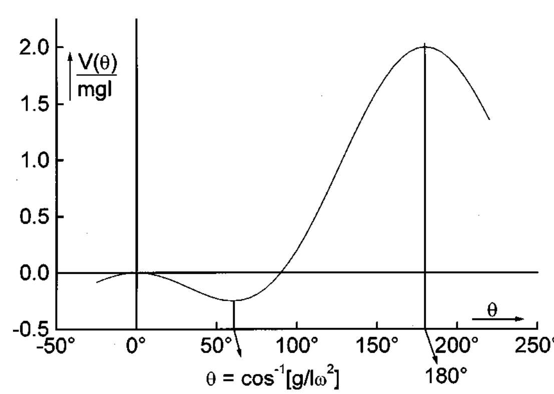

Fig. 3. Potential for $\omega^2 = 2g/l$.

In the second treatment we use Euler's equations for the motion of a rigid body with one point fixed. The moments of inertia of the rod are calculated and the solution is exact.

Finally, the motion of the conical pendulum is studied in the Lagrangian formulation.

IV. THE MOTION OF A THIN ROD

In the approximation of a thin rod, the relation between g, l, and ω remains elementary. Writing λ for the mass of the rod per unit length and defining an s axis along the rod with its origin at the hinge, the centrifugal force on a line element of the one-dimensional rod is

$$dm\,\omega^2 y\mathbf{u}_y = (\lambda\, ds)\omega^2(s \sin\theta)\mathbf{u}_y \tag{4.1}$$

and its moment with respect to the origin O is

$$s\cos\theta\mathbf{u}_z \times \lambda(ds)\omega^2 s\sin\theta\mathbf{u}_y = \lambda(ds)\omega^2 s^2 \sin\theta\cos\theta\mathbf{u}_x. \tag{4.2}$$

Integrated over the length l of the rod the moment of the centrifugal force is

$$\mathbf{u}_x\lambda\omega^2 \sin\theta\cos\theta\int_0^l s^2\,ds = -\frac{1}{3}m\omega^2\sin\theta\cos\theta l^2\mathbf{u}_x. \tag{4.3}$$

The moment of the gravitational force is

$$\tfrac{1}{2}lmg\sin\theta\mathbf{u}_x. \tag{4.4}$$

In equilibrium the total moment is zero:

$$\tfrac{1}{2}mlg\sin\theta - \tfrac{1}{3}ml^2\omega^2\sin\theta\cos\theta = 0. \tag{4.5}$$

The solutions of this equation are

$$\begin{aligned} &1.\quad \sin\theta = 0,\\ &2.\quad \cos\theta = 3g/2l\omega^2 \end{aligned} \tag{4.6}$$

or

$$g = \tfrac{2}{3}l\omega^2\cos\theta. \tag{4.7}$$

These are the solutions of a simple conical pendulum with length $2l/3$.

In this derivation it was supposed that the rod goes all the way to the fixed point which is at the center of the hinge. But the presence of the hinge itself makes this impossible (Fig. 4). If the hinge has diameter $2d$, the distance between the fixed point and the upper end of the rod is d. This means that the limits of integration of the moments of the elementary centrifugal forces must be taken to be d and $l+d$ instead of 0 and l.

For the same reason the distance between the center of mass and the fixed point is $l/2+d$. This value must be taken into account in calculating the moment of the gravitational force. With these corrections the second solution (4.7) of the equilibrium equation becomes

$$\cos\theta = \frac{3g}{2l\omega^2}\cdot\frac{1+2\dfrac{d}{l}}{1+3\dfrac{d}{l}+3\left(\dfrac{d}{l}\right)^2} \tag{4.8}$$

or

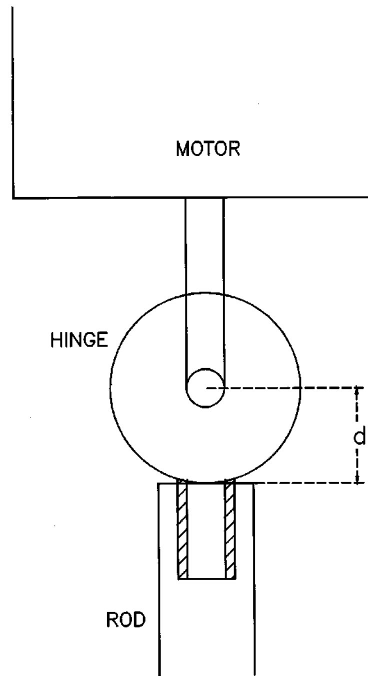

Fig. 4. The hinge.

$$g = \frac{2}{3}Cl\omega^2\cos\theta \quad \text{with} \quad C = \frac{1+3\dfrac{d}{l}+3\left(\dfrac{d}{l}\right)^2}{1+2\dfrac{d}{l}}. \tag{4.9}$$

The mass of the hinge, which is symmetrical with respect to its axis, obviously does not influence the value of θ in the equilibrium situation. It is remembered that Eq. (4.9) holds only for a rod of negligible thickness.

V. EULER'S EQUATIONS FOR THE CONICAL PENDULUM

The conical pendulum can be treated in a mathematically more elegant way using Euler's equations for the motion of a rigid body with one point fixed. We shall avoid the approximation of a thin rod and get an exact result by calculating the moments of inertia of the rod.

We use a reference system that is connected to the rod and whose origin is at the fixed point; x, y, and z are principal axes of the inertia tensor of the rod with respect to the fixed point.

The z axis is along the symmetry axis of the rod. We choose the y axis to lay in the vertical plane of the rod and to make an acute angle with the downward pointing vertical. Of the Euler angles φ, θ, and ψ, the last one can be set equal to zero because of the hinge.

The motion in an inertial reference frame is a rotation Ω whose components along the axes of the rotating frame are

$$\Omega_x = \dot{\theta}, \tag{5.1}$$

$$\Omega_y = \dot{\varphi}\sin\theta, \tag{5.2}$$

$$\Omega_z = \dot{\varphi}\cos\theta. \tag{5.3}$$

Since the z axis is an axis of symmetry, $I_x = I_y$, I_i being the moment of inertia with respect to the i axis. Therefore, the Euler equations become

$$N_x = I_x \dot{\Omega}_x - \Omega_y \Omega_z (I_x - I_z), \tag{5.4}$$

$$N_y = I_x \dot{\Omega}_y - \Omega_z \Omega_x (I_z - I_x), \tag{5.5}$$

$$N_z = I_z \dot{\Omega}_z . \tag{5.6}$$

The N_i are the components of the moments of the applied forces (gravitational force and torque of the hinge on the rod) along the axes of the moving reference system.

Only (5.4) does not contain a component of the torque of the hinge on the rod and can be written

$$-\left(\frac{l}{2}+d\right) mg \sin\theta = I_x \ddot{\theta} - \omega^2 \sin\theta \cos\theta (I_x - I_z), \tag{5.7}$$

with $\omega = \dot{\varphi}$, the constant angular frequency of the motor, and m the mass of the rod. Since we are looking for equilibrium in the y,z plane of the moving system, we can put $\ddot{\theta} = 0$. As a matter of fact, any motion in the y,z plane will soon die out owing to air resistance and friction in the hinge. In that case (5.2) reduces to

$$\left(\frac{l}{2}+d\right) mg \sin\theta = \omega^2 \sin\theta \cos\theta (I_x - I_z). \tag{5.8}$$

The solutions of (5.8) are

$$1. \quad \sin\theta = 0, \tag{5.9}$$

$$2. \quad \cos\theta = \frac{mg\left(\frac{l}{2}+d\right)}{\omega^2 (I_x - I_z)}. \tag{5.10}$$

$I_z = \frac{1}{2} m r^2$ and $I_{x'}$, with x' parallel to x and through the center of mass of the rod, is known to be[11]

$$I_{x'} = m\left(\frac{l^2}{12} + \frac{r^2}{4}\right). \tag{5.11}$$

Using Steiner's theorem

$$I_x = I_{x'} + m\delta^2 \tag{5.12}$$

with δ the distance between x and x' $(\delta = l/2 + d)$ one gets

$$I_x = m\left(\frac{l^2}{2} + ld + d^2 - \frac{r^2}{4}\right) \tag{5.13}$$

and

$$I_x - I_z = m\left(\frac{l^2}{3} + ld + d^2 - \frac{r^2}{4}\right) \tag{5.14}$$

(5.10) becomes

$$\cos\theta = \frac{g\left(\frac{l}{2}+d\right)}{\omega^2\left(\frac{l^2}{3} + ld + d^2 - \frac{r^2}{4}\right)}, \tag{5.15}$$

so that

$$g = \frac{2}{3} C' l \omega^2 \cos\theta \quad \text{with} \quad C' = \frac{1 + 3\frac{d}{l} + 3\left(\frac{d}{l}\right)^2 - \frac{3}{4}\left(\frac{r}{l}\right)^2}{1 + 2\frac{d}{l}}. \tag{5.16}$$

VI. THE CONICAL PENDULUM IN THE LAGRANGIAN FORMULATION

We write the Lagrangian in the rotating reference frame of the first paragraph.

The pendulum's kinetic energy in this frame is simply

$$T = \tfrac{1}{2} I_x \dot{\theta}^2. \tag{6.1}$$

The centrifugal force as well as the gravitational force contribute to the potential energy

$$V = -mg\left(\frac{l}{2}+d\right)\cos\theta - \frac{1}{2}\omega^2 (I_x \sin^2\theta + I_z \cos^2\theta). \tag{6.2}$$

The Lagrangian depends only on θ, ω being a constant:

$$L = \frac{1}{2} I_x \dot{\theta}^2 + mg\left(\frac{l}{2}+d\right)\cos\theta + \tfrac{1}{2}\omega^2 (I_x \sin^2\theta + I_z \cos^2\theta) \tag{6.3}$$

and the corresponding Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \tag{6.4}$$

becomes

$$I_x \ddot{\theta} + \left(\frac{l}{2}+d\right) mg \sin\theta - \omega^2 \sin\theta\cos\theta (I_x - I_z) = 0, \tag{6.5}$$

which is identical to (5.7).

VII. CORRECTIONS

Using a plane pendulum, Nelson and Olsson[8] have performed an experiment to determine the acceleration of gravity with an accuracy of 10^{-4}. To obtain this accuracy, these authors investigated the effect of some 16 possible corrections which are analyzed in detail in their paper. We shall use this analysis as a guide in determining the corrections in the case of the conical pendulum. Nine of these 16 corrections give rise to a relative error in the value of g exceeding 10^{-4} if not accounted for properly. Only one of these applies in our case: the effect of buoyancy in the atmospheric air, a correction that can be handled in a straightforward way. Other important corrections for the plane pendulum are the amplitude dependence of the period, the mass of the wire and of the suspension device, and the ''added mass'' as a result of the motion of the surrounding air which represents part of the kinetic energy of the system. This air motion is obviously also present in our system but it is the result of the work done by the motor and the torque of the motor does not enter the equilibrium equation.

There are, however, two corrections that apply in the case of the conical pendulum but not in the case of the plane pendulum. The first correction is, just like the buoyancy, caused by the atmospheric air. Because the speed of a point

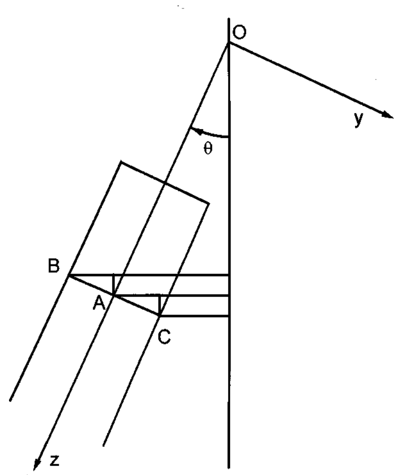

Fig. 5. Calculating the speeds of the points A, B, and C on a diameter of the rod.

on the ''outer'' side of the rod is greater than that on the ''inner'' side, a lift effect due to the atmospheric air is to be expected.[12]

A second correction is needed because of the rotation of the earth which produces a Coriolis force that influences the motion of the conical pendulum. We first derive a formula for the lift and buoyancy effects.

The magnitude of the buoyancy force is

$$\pi r^2 l \rho_a g \tag{7.1}$$

with ρ_a the density of the air.

Using coordinates x, y, and z as in the Euler equations, the moment of the buoyancy force with respect to the fixed point is (for $d \ll l$)

$$\boldsymbol{\mu}_B = (\tfrac{1}{2} l \sin\theta)(\pi r^2 l \rho_a g)\mathbf{u}_x$$

or (7.2)

$$\boldsymbol{\mu}_B = \tfrac{1}{2}\pi r^2 l^2 \rho_a g \sin\theta \mathbf{u}_x .$$

The moment of the buoyancy force has a sign opposite to that of the moment of the gravitational force. The ratio of the magnitudes of both moments is simply the ratio of the densities ρ_b of the rod (e.g., brass) and ρ_a of the air. With $\rho_b = 8500\ \mathrm{kg\,m^{-3}}$ and $\rho_a = 1.2\ \mathrm{kg\,m^{-3}}$ (20 °C and normal atmospheric pressure), $\rho_a/\rho_b = 1.4\times 10^{-4}$.

The corresponding correction is

$$\frac{\Delta g}{g} = 1.4\times 10^{-4}. \tag{7.3}$$

In order to estimate the lift force we first calculate the speed of three points A, B, and C of the rod. Using again the coordinates x, y, and z of the Euler equations, the speed of a point A on the symmetry axis can be written as

$$U_A = \omega z \sin\theta. \tag{7.4}$$

The point B has the same z value as A but lies on the outer side of the rod. It moves on a circle with radius $(z\sin\theta + r\cos\theta)$ (Fig. 5) and so its speed is

$$v_B = \omega(z\sin\theta + r\cos\theta). \tag{7.5}$$

In the same way we find for the speed of the point C on the inner side of the rod and with the same z as A and B

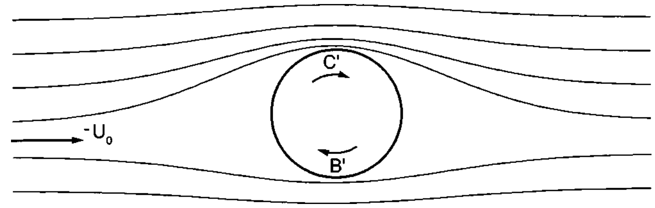

Fig. 6. Flow of air around a cylinder.

$$v_C = \omega(z\sin\theta - r\cos\theta). \tag{7.6}$$

Now we consider an arrangement of a cylinder with radius r which rotates about its axis of symmetry with an angular frequency Ω and at the same time translates with a velocity U_0 perpendicular to the rotation vector. The points B' and C' on the surface of the cylinder and on a diameter perpendicular to U_0 have velocities

$$v_{B'} = U_0 + r\Omega, \tag{7.7a}$$

$$v_{C'} = U_0 - r\Omega, \tag{7.7b}$$

if for B' the velocities due to translation and rotation are in the same direction and for C' in opposite directions.

If we now take Ω to be $\omega\cos\theta$ and U_0 to be $U_A = \omega z \sin\theta$, then

$$v_{B'} = \omega z \sin\theta + \omega\cos\theta = v_B, \tag{7.8a}$$

$$v_{C'} = \omega z \sin\theta - \omega\cos\theta = v_C. \tag{7.8b}$$

This means that *for a particular value of* z the velocities of the points B and C on the rod of a conical pendulum are the same as the velocities of B' and C' in the second arrangement. This last setup is sketched in Fig. 6, where we have assumed that the cylinder does not move, but that the wind blows with a velocity $-U_0$, which amounts to the same. This is a familiar situation in fluid dynamics. It is well known[13] that, due to the rotation of the cylinder and the viscosity of the air, the flow lines of the air are asymmetric with respect to $\mathbf{U}_0$. This causes a lift force that is perpendicular to both the rotation vector and the translational velocity. The lower point in Fig. 6 is B' because there the relative speed of the cylinder surface with respect to the air far from the rod is greater than in the upper point C', just as point B of the conical pendulum has the greater speed. In the reference system of Fig. 6 (cylinder axis at rest) the speed of the air near C' is increased by the rotation of the cylinder, whereas near B' the speed of the air is decreased. According to Bernoulli's theorem the pressure will be lower near C' than near B' and the lift force points from B' to C'. Transposing these results to the conical pendulum we see that the rod experiences a downward force.

The lift force $\mathbf{L}$ per unit length is

$$\mathbf{L} = \rho_a U_0 \Gamma \mathbf{u}_y \tag{7.9}$$

with Γ the circulation.

Assuming[14] that the magnitude of the circulation is

$$\Gamma = \oint \mathbf{v}\cdot d\mathbf{l} \tag{7.10}$$

with $\mathbf{v}$ the speed of the cylinder surface as a result of the rotation, and $d\mathbf{l}$ a line element of the circumference of the rod,

$$\Gamma=\int_0^{2\pi}\Omega r^2\,d\varphi=2\pi r^2\Omega \tag{7.11}$$

and

$$\mathbf{L}=\rho_a(\omega z\sin\theta)(2\pi r^2\omega\cos\theta)\mathbf{u}_y. \tag{7.12}$$

Substituting ω from (4.7) for a thin rod, $\mathbf{L}$ becomes

$$\mathbf{L}\equiv\frac{d\mathbf{K}}{dz}=\frac{3\pi}{l}\rho_a g r^2 z\sin\theta\mathbf{u}_y. \tag{7.13}$$

The moment of dK with respect to the fixed point has only an x component

$$-z\,dK=\frac{-3\pi}{l}\rho_a g r^2\sin\theta z^2\,dz. \tag{7.14}$$

Assuming that we can obtain the total lift force moment $\boldsymbol{\mu}_L$ by integrating this expression over the length of the rod, we get (for $d\ll l$)

$$\boldsymbol{\mu}_L=\frac{-3\pi}{l}\rho_a g r^2\sin\theta\int_0^l z^2\,dz\,\mathbf{u}_x$$

or

$$\boldsymbol{\mu}_L=-\pi\rho_a g r^2 l^2\sin\theta\mathbf{u}_x. \tag{7.15}$$

We see that $\boldsymbol{\mu}_L=-2\boldsymbol{\mu}_B$: the moment of the lift force is twice as large as the moment of the buoyancy force and has the opposite sign.

The resultant correction $\Delta g/g$ for both buoyancy and lift is therefore the same as for buoyancy alone (7.3), but with the opposite sign. Since this correction is much smaller than 10^{-3} we shall neglect it.

The last correction is the one due to the Coriolis force resulting from the rotation of the earth. It turns out that we need only an order of magnitude, which can be estimated by comparing the Coriolis force $2m\bar{v}\times\bar{\omega}$ and the centrifugal force $m\bar{\omega}\times(\bar{r}\times\bar{\omega})$ for a simple pendulum.

The ratio of the magnitudes of these forces is roughly

$$\frac{v\Omega}{r\omega^2} \tag{7.16}$$

with Ω the rotation of the earth.

Since $v=r\omega$, this ratio equals

$$\frac{\Omega}{\omega}. \tag{7.17}$$

The rotation of the conical pendulum is of the order of 10 s^{-1} and the rotation of the earth is about $10^{-4}\,\mathrm{s}^{-1}$ so that $\Omega/\omega\approx10^{-5}$.

Aiming at a precision of 10^{-3}, the influence of the rotating earth can be neglected.

VIII. THE APPARATUS

A sketch of our apparatus is shown in Fig. 7.

A rod of length l and radius r is connected to the lower side of the vertical axle of a motor by means of a hinge with a horizontal axle. A rotation of the rod around its axis of symmetry is prevented by this connection. We use a stepper motor with a 1.8° step angle.

The angle θ between the rod and the vertical is determined optically. A laser pointer projects a horizontal light beam

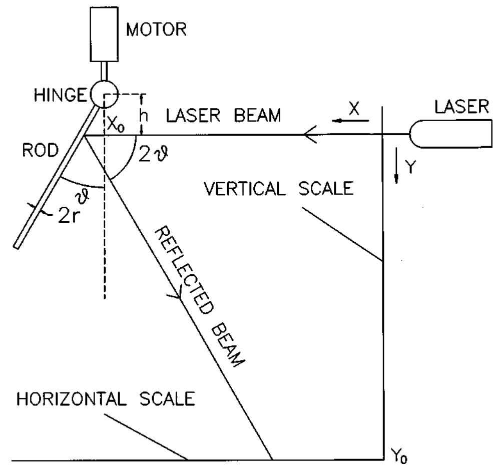

Fig. 7. Sketch of the apparatus.

through the axle of the motor. During each turn of the conical pendulum, the polished rod passes twice through the vertical plane containing the light beam and then reflects the beam. We use the reflection when the rod is on the far position from the laser (as in Fig. 7). The reflected beam traces an arc onto a vertical scale for small θ (from 0° to about 24°) and onto a horizontal scale for θ ranging from about 24° to about 62°. The point of the arc which lies in the vertical plane of the laser is read. If the spot is on the horizontal scale, it follows from the geometry of the apparatus (see Fig. 7) that the angle θ is determined by the relation

$$X=X_0+h\tan\theta-\frac{r}{\cos\theta}-\frac{Y_0}{\tan 2\theta}. \tag{8.1}$$

In this formula X is the distance of the spot from the vertical scale; X_0 is the distance from the fixed point of the rod (i.e., the center of the hinge) to the vertical scale; Y_0 is the distance from the (unreflected) laser beam to the horizontal scale; and h is the distance of the center of the hinge to the (unreflected) laser beam.

For the vertical scale the relation is

$$Y=\left(X_0+h\tan\theta-\frac{r}{\cos\theta}\right)\tan 2\theta \tag{8.2}$$

in which Y is the distance from the spot to the (unreflected) light beam. Numerical values of the constants are: $X_0=0.1657$ m; $Y_0=0.1953$ m; $h=0.0112$ m; and $d=0.0063$ m. Rods with lengths up to $Y_0+h-d\approx0.21$ m can be used. Most experiments were done with a rod with $l=0.1$ m and $r=0.003$ m.

IX. RESULTS

The value g_0 of the acceleration of gravity at sea level can to a good approximation be calculated from the ''International Gravity Formula 1967.''[15] For our campus with a North latitude of 50.864°, $g_0=9.8115\,\mathrm{m\,s^{-2}}$. The correction for the height ϵ above the sea level can to a sufficient approximation be derived from Newton's law of universal gravitation

$$g=G\frac{M}{r^2}. \tag{9.1}$$

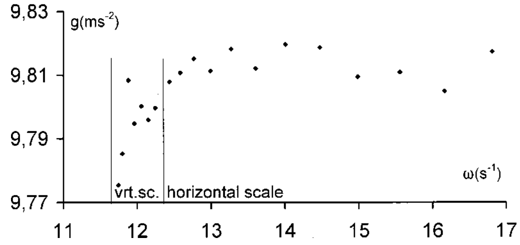

Fig. 8. g vs ω for a typical measurement.

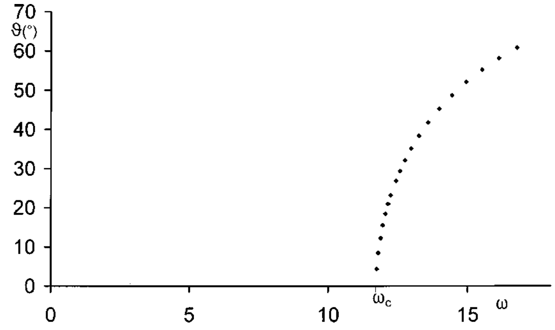

Fig. 9. θ vs ω for the measurement shown in Fig. 8.

Writing R for the radius of the earth and Δg for the correction due to ϵ, we have

$$g_0+\Delta g=\frac{GM}{(R+\epsilon)^2}=g_0\left(1+\frac{\epsilon}{R}\right)^{-2} \tag{9.2}$$

or

$$\Delta g\approx -2\frac{\epsilon}{R}g_0. \tag{9.3}$$

With $\epsilon=25$ m, $R=6.4\times10^6$ m and $g=9.8\ \mathrm{m\,s^{-2}}$, (9.3) becomes

$$\Delta g\approx -8\times10^{-5}\ \mathrm{m\,s^{-2}}.$$

The local value of the acceleration of gravity is therefore

$$g=g_0+\Delta g=9.8114\ \mathrm{m\,s^{-2}}. \tag{9.4}$$

A measurement of g with the conical pendulum involves the determination of the three quantities l, ω, and θ. The length l of the rod poses no problem; it can easily be measured to within 10^{-4}. An advantage of a stepper motor is that its speed is completely determined by the frequency of the oscillator that feeds the power supply of the motor. This frequency is measured with a timer/counter with an accuracy of at least 10^{-5} and is therefore the best known quantity of the three named above.

The main experimental difficulty in the conical pendulum setup is the accurate determination of the angle θ between the rod and the vertical. Around $\theta=45°$, a relative error of 0.1% in the value of g corresponds to an error $\Delta\theta\approx0.06°$ or about 0.5 mm on the horizontal scale. With the moderate dimensions of our apparatus (the length of the reflected beam is about 20 cm), the accuracy of our optical method is only slightly better than this.

The result of a typical measurement is shown in Fig. 8. It represents determinations of g using a polished stainless steel rod with $l=0.1$ m and $r=0.003$ m. The first 7 points correspond to readings on the vertical scale; the next 12 points are read on the horizontal scale. A plot of g vs ω should of course be a horizontal straight line. This is, within the limits of accuracy, indeed the case for the points on the graph which correspond to readings on the horizontal scale. At lower frequencies, however, the measured value of g increases with increasing ω. This may result from the motor axle not being perfectly vertical. Its effect on the measured value of the acceleration of gravity can, to first order in θ_0, be calculated to be

$$g=g'\left(1\pm\frac{2\theta_0}{\tan 2\theta}\right) \tag{9.5}$$

in which θ_0 is the angle of the motor axle with the vertical and g' is the measured value of the acceleration of gravity when the correction is not taken into account.

When calculating (9.5), it was supposed that the axle of the motor lies in the vertical plane of the laser beam; in that case the effect on the measured value of g is a maximum. The minus sign applies if the motor axle deviates from the vertical in the direction of the laser, and the plus sign if it deviates away from the laser.

It turns out that the results shown in Fig. 8 can be explained in this way if one assumes that $\theta_0\approx0.035°$ for a motor axle in the plane of the laser beam (and a larger θ_0 otherwise). Checking this value directly is, however, almost impossible for a motor axle that extends only a couple of centimeters from the body of the motor and that is covered to a large extent by the hinge. Therefore only readings on the horizontal scale—which constitutes almost 90% of the ω range—were used to calculate the value of the acceleration of gravity, the points on the vertical scale being used only to extend the range of the θ vs ω plot (Fig. 9) in the critical region.

The average value of g from the horizontal scale points is 9.813 $\mathrm{m\,s^{-2}}$ with a standard error of $5\times10^{-3}\ \mathrm{m\,s^{-2}}$. The result $g=9.813\pm0.005$ is to be compared with the value of 9.8114 $\mathrm{m\,s^{-2}}$ calculated from the International Gravity Formula.

Since the angle θ between the rod and the vertical is not calibrated but deduced from the geometry of the apparatus, it is necessary to determine the constants X_0, Y_0, and h with a high precision. Almost all inaccuracies in the construction of the apparatus and errors in the determination of these constants result in a frequency dependence of the measured values of g. Therefore the frequency independence of the measured g values constitutes a very demanding test of the apparatus.

X. CONCLUSION

As a means of measuring the acceleration of gravity, the conical pendulum combines a number of advantages which make it particularly suited for the undergraduate laboratory.

First of all, an accuracy of 10^{-3} is better than is obtained in most free fall experiments which are commonly used by students for measuring g. Furthermore, the measurement is very simple: just a reading of the frequency and of the place of a light spot on an a scale. No corrections whatsoever are needed to obtain the 0.1% accuracy. And finally, the sudden deviation of the rod from its vertical postion at the critical frequency constitutes an element of surprise which is apt to capture the student's interest.

A drawback of the conical pendulum for measuring accurately the value of g is the need for the utmost care in the construction of the apparatus, as illustrated by the results for small ω in Fig. 8 and their possible explanation.

ACKNOWLEDGMENT

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[14]See Ref. 13, p. 132.

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UNDERGRADUATE SORCERY

As an analogy, consider a quantum sorcerer's apprentice who has been given the task of constructing a chimaera. Suppose that this is undergraduate sorcery, so that only a two-component chimaera is required; a lion–goat superposition, say. The apprentice captures a pure lion state L, puts it into a cauldron with a pure goat state G, and starts stirring. After having swished the mixing ladle a certain number of times around the pot, the chimaera is declared done and is taken out for inspection. To what extent is it a lion, and to what extent a goat?

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