An accurate measurement of *g* using falling balls

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We describe an experiment to determine the acceleration due to gravity, *g*, with an accuracy of about 1 part in 10^4. The experiment was designed to expose students to critical thinking in collecting, selecting, and analyzing data, and interpreting the results.

I. INTRODUCTION

In teaching statistical analysis of data in the laboratory, it is useful to make measurements on a simple system which is familiar to the student. The measurement of the acceleration due to gravity, *g*, by observations on a falling object is such a system, and we have found it instructive in exposing students in our junior-level laboratory to χ^2 analysis. We have designed an experiment with sufficiently high precision that students cannot easily look up the expected result; this overcomes a common problem in beginning laboratories, i.e., the ''how close did I get to the right answer'' syndrome.

In principle, *g* can be determined by measuring the time *t* for an object to fall through a known distance *y* and by applying the simple formula $y=\frac{1}{2}gt^2$. With simple electronics, it is straightforward to measure time intervals to a high precision. A problem arises, however, if an object is to be started from rest: There is no simple means of measuring a start time with sufficient accuracy, or of making a release mechanism which will respond instantaneously to an electronic signal. For precision results, we have found that it is necessary to measure the fall time between two different heights during the motion. We have found that it is possible to measure the time for a spherical ball to fall between two light beams up to 1 m apart with a precision of a few tens of microseconds. It follows that, by measuring the distance between the two heights to better than 0.1 mm, it is possible to measure *g* to about 1 part in 10^4.

In an experiment with this precision, the effects of air buoyancy and drag on a falling ball become significant. By making observations on identically sized balls with different masses (we have used stainless steel and nylon balls) it is possible to derive a correction for these effects.

II. APPARATUS

The main components of the apparatus are shown in Fig. 1. They consist of a caliper, a ball release mechanism, two pairs of light emitter sensors, an electronic amplifier and trigger circuit for the sensors, and a commercial period measuring instrument. The modified, commercial 100-cm caliper has a nominal resolution of 0.02 mm and forms the backbone of the apparatus. The balls are dropped through holes in a pair of aluminum blocks which are rigidly attached to the jaws of the caliper; the lower jaw is fixed while the upper jaw can be adjusted over the entire length of the caliper, allowing the balls to be dropped over a range of heights from 5 to 100 cm. The optical emitter-sensor pairs are mounted on opposite sides of the blocks. Greater accuracy could have been provided by a screw-type system but the caliper accuracy is compatible with the achievable timing accuracy; we believe that it is also educational for the students to learn how to read the vernier scale on the caliper. The caliper is mounted on a solid base with adjusting screws to permit tilt adjustment, and additional support structure provides vibration damping. The entire apparatus is metal and is grounded to prevent electrostatic charging, which can affect the motion of falling balls.

The ball release mechanism, shown in Fig. 2, uses a piston-cylinder arrangement to create a partial vacuum to hold the 1.9-cm-diam balls. The device is armed by pressing a ball against the lower air inlet and pulling the piston back until it ratchets. The mechanism is then placed in a fixed position on the movable upper jaw, approximately 0.5 cm above the first light beam. A small hole in the cylinder wall allows the air pressure inside to increase slowly, and the ball is released after a few seconds, when the pressure has increased sufficiently. The advantage of this suction device is that it does not introduce any observable vibrations or perturbations when a ball is released. In addition, it works with both plastic and metal balls so that significantly different masses can easily be used, an important consideration in making drag corrections.

The light beams are provided by infrared light-emitting diodes (LEDs) (FD1QT) shining through 0.75-mm-diam holes and they are detected by infrared photodiodes (PN334PA) located behind 0.75-mm-diam holes. As a ball falls between the two light beams, the resulting pulses generated by the photodiodes are strongly amplified, and a circuit consisting of two comparators is used to generate a pulse from an RS-type flip-flop. The width of this pulse is equal to the time between the start and stop pulses and this can be measured, for example, by a Philips PM2525 meter operated in its period measuring mode. We have found that the typical uncertainty in fall times is about 40 μs, independent of the distance fallen.

Vertical alignment of the apparatus is necessary; the two light beams must lie within $\sim$1 mm to the vertical to avoid a systematic error of more than a few parts in 10^5. Adequate alignment can be accomplished with a plumb bob and a target, both of which can be mounted in the holes through which the balls fall. Leveling screws in the base provide for this adjustment.

We have found it useful to provide a sand-filled box to catch the balls. This not only prevents the laboratory being filled with bouncing balls, but also preserves the surface of the balls. (Brass balls were found to be too easily deformed in hard bounces).

III. THEORY

The equation of motion for a ball of mass *m* falling in air is

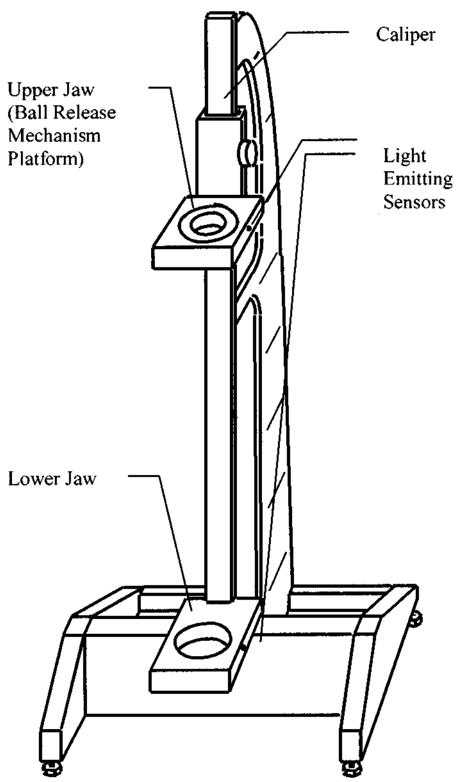

Fig. 1. The apparatus (the release mechanism is not shown).

$$m\frac{d^2y}{dt^2}=(m-m_{\mathrm{air}})g-F_{\mathrm{drag}}, \tag{1}$$

where m_{air} is the mass of displaced air and F_{drag} is the drag force which is taken to be proportional to the square of velocity, i.e., $F_{\mathrm{drag}}=k(dy/dt)^2$. The quantity $k=0.5C\rho_{\mathrm{air}}A$ where ρ_{air} is the density of air, A is the cross-sectional area of the falling ball, and C is the drag coefficient, which is essentially constant for the velocities of concern here. (C varies between 0.50 and 0.45 for Reynolds numbers R in the range 10^3-10^5; for 2-cm-diam balls falling up to 1 m in air, the maximum value of R is $\sim 5\times 10^3$.[1])

Defining

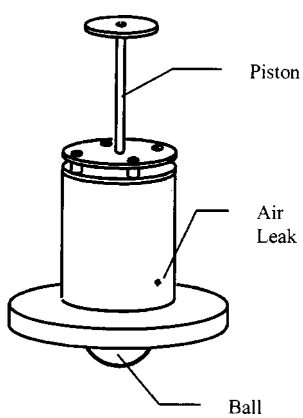

Fig. 2. The ball release mechanism.

$$g_B=g\left(1-\frac{m_{\mathrm{air}}}{m}\right), \tag{2}$$

the equation of motion becomes

$$\frac{d^2y}{dt^2}=g_B-\frac{k}{m}\left(\frac{dy}{dt}\right)^2, \tag{3}$$

which can be integrated to give:

$$y(t)=\frac{v_T^2}{g_B}\ln\left(\cosh\left(\frac{g_B}{v_T}t\right)\right), \tag{4}$$

where v_T is the terminal velocity of the ball, $v_T=\sqrt{mg_B/k}$.

This expression can be inverted to give t as a function of y:

$$t=\frac{v_T}{g_B}\cosh^{-1}\left(\exp\left(\frac{g_By}{v_T^2}\right)\right), \tag{5}$$

which can then be expanded to a more useful form:

$$t=\sqrt{\frac{2y}{g_B}}\left(1+\frac{1}{6}\left(\frac{g_By}{v_T^2}\right)+\frac{1}{120}\left(\frac{g_By}{v_T^2}\right)^2+\cdots\right). \tag{6}$$

We need only consider the first two terms of this series, which contain the effect of drag to first order. (The maximum value of the second term amounts to $\sim$ a few times 10^{-3} in the present experiment.) Substituting g_B from Eq. (2) and keeping terms to first order yields

$$t=\sqrt{\frac{2y}{g}}\left(1+\frac{1}{m}\left(\frac{m_{\mathrm{air}}}{2}+\frac{ky}{6}\right)\right). \tag{7}$$

This gives the time taken for a ball to fall from an initial height $y=0$ and demonstrates more explicitly the effects of both buoyancy and drag.

Now we consider two separate balls of mass m_A and m_B with identical shape (same k) which take times t_A and t_B to fall between heights y_1 and y_2. Thus, for example,

$$t_A=\sqrt{\frac{2y_2}{g}}\left(1+\frac{1}{m}\left(\frac{m_{\mathrm{air}}}{2}+\frac{ky_2}{6}\right)\right)-\sqrt{\frac{2y_1}{g}}\left(1+\frac{1}{m}\left(\frac{m_{\mathrm{air}}}{2}+\frac{ky_1}{6}\right)\right), \tag{8}$$

which can be written as

$$t_A=t_\infty+\frac{1}{m_A}f(y_1,y_2), \tag{9}$$

where we have defined a quantity t_∞ as

$$t_\infty=\sqrt{\frac{2y_2}{g}}-\sqrt{\frac{2y_1}{g}}, \tag{10}$$

which would be the time taken for a ball of infinite mass to fall between heights y_1 and y_2, i.e., where there are no effects of drag and buoyancy. After writing a similar expression for t_B, it readily follows that

$$t_\infty=\frac{t_Am_A-t_Bm_B}{m_A-m_B}. \tag{11}$$

Thus for each value of y_2 we can determine t_∞ from the two measured fall times and the masses of the balls. (We note that the effect of buoyancy drops out in the final analysis.)

Some simple manipulation then yields the familiar equation

$$\Delta y = y_2 - y_1 = \tfrac{1}{2} g t_\infty^2 + v_1 t_\infty, \tag{12}$$

where $v_1 = \sqrt{2gy_1}$ is the velocity of the ball after it has fallen a distance y_1, i.e., its initial velocity when it passes the first light beam. Finally, this equation can be linearized

$$\frac{\Delta y}{t_\infty} = v_1 + \frac{1}{2} g t_\infty, \tag{13}$$

so that a graph $\Delta y/t_\infty$ vs t_∞, which is determined from measured quantities, should be a straight line with slope $g/2$ and a standard least-squares fitting program can be used to analyze the data.

IV. DATA COLLECTION AND ANALYSIS

First, students make several measurements of the time required for a ball to drop some fixed Δy in order to assess the timing uncertainty. It is then suggested that they make an initial error analysis to determine an optimum strategy for data taking. The balls are weighed and their diameters measured. Timing measurements for each ball, one steel and one nylon, are made at 20–30 values of Δy over the full range, up to 100 cm, with 2 or 3 measurements made for each point to detect any immediately obvious systematic timing error. The value of t_∞ can be calculated from these data.

A problem appears in the determination of Δy, since the vernier scale readings on the caliper give only the relative displacement between the two light beams, because of offsets that cannot easily be determined directly. In addition, the absolute value of Δy also depends on the exact positions where the ball intercepts the light beams and on the threshold settings of the comparators. To determine the offset δ between the actual Δy and that obtained from the caliper readings, trial values of δ can be added to the observed values of Δy. The best estimate of δ is that which minimizes χ^2 in the straight line fit of $\Delta y/t_\infty$ vs t_∞. χ^2 is defined as

$$\chi^2 = \sum_{i=1}^{n} \chi_i^2 = \sum_{i=1}^{n} \frac{((y/t)_i - (y/t)_{\text{fit}})^2}{(\sigma_{y/t})_i^2}, \tag{14}$$

where $(\sigma_{y/t})_i$ is calculated from the estimated uncertainties in y and t.

All the fitting is done using an Excel spreadsheet in which students use equations as given in *Numerical Recipes in C*,[2] for example. Not only does this teach students the utility of spreadsheets, but setting up their own spreadsheets allows them to examine the χ_i^2 of each point, to check for bad data points, and to check for trends. These features are not generally available in inexpensive commercial data analysis software packages.

It is particularly instructive for the students to inspect the plots of $\Delta y/t_\infty$ vs t_∞ as a function of δ, along with the corresponding value of χ^2. When the total χ^2 of the fit has been minimized by choosing the best value of δ, the fit automatically gives the value of g with its associated uncertainty. Observing the absolute value of the minimum χ^2 and comparing this with the number of data points (or, more precisely, the number of degrees of freedom) gives the student an appreciation of the importance of error assignment.

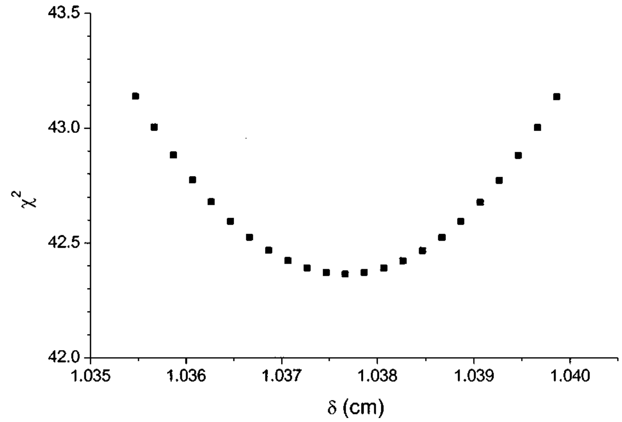

Fig. 3. χ^2 vs assumed caliper offset δ in centimeters. (43 data points.)

A typical graph of χ^2 vs δ is shown in Fig. 3. From these data, $\delta = 1.0377 \pm 0.0014$ cm. We then find $v_1 = 29.817 \pm 0.007$ cm/s and $g = 980.38 \pm 0.06$ cm/s^2, where the quoted errors are statistical only.

V. SYSTEMATIC ERRORS

When an experiment aims for high precision, it is inevitable that systematic errors become significant unless special care is taken. This is certainly the case here. Any systematic uncertainty in the time measurements is negligible compared to the statistical spread, but there are possible systematic uncertainties in the measurements of the distance fallen by the balls.

We have already mentioned that the apparatus must be accurately aligned vertically so that the centers of the balls pass through both light beams. Analysis shows that the fractional error in g arising from a misalignment of the apparatus by a small angle θ from the vertical is

$$\frac{\Delta g}{g} = \frac{y}{R} \frac{\theta^2}{2}, \tag{15}$$

where R is the radius of the ball and y is the height fallen. Thus for $R = 0.95$ cm (the present case), $y = 100$ cm, and $\theta = 10^{-3}$, i.e., assuming that the center of the ball passes through the upper beam and that the lower beam is displaced by a nominal 1 mm, then the estimated systematic error in g from a possible vertical misalignment is $+0.05$–0.00 cm/s^2. (Such a misalignment leads to a value of g which is low since the ball must fall a greater distance before it interrupts the lower light beam). We note that this uncertainty is approximately equal to the statistical uncertainty in our result, and that doubling the nominal displacement of the lower beam, to 2 mm, would result in our result for g being low by 0.20 cm/s^2. Accurate vertical alignment of the apparatus is essential! (We note that the Coriolis deflection of a ball falling 1 m is only 0.018 mm and is of no consequence in the present case).

Determining the uncertainty in the absolute length of the caliper presents a more difficult problem. Calipers are generally initially calibrated at a temperature of 20 °C. Our caliper is of uncertain origin, but assuming a nominal temperature coefficient of expansion of 1×10^{-5} °C and assigning an uncertainty of ± 3 °C to the temperature at which the measurements were made, we can estimate a systematic uncertainty in g of ± 0.03 cm/s^2. We have attempted to calibrate the caliper using a coordinate measuring machine[3] which has a nominal resolution of ~0.01 mm and have shown that there are likely slight distortions of the caliper in the region where it is supported by a screw at its center. Such distortions have only a small effect on the determination of g, compared to uncertainties in the full length of the caliper. We are uncertain of the absolute calibration of the coordinate measuring machine, but assign a nominal uncertainty of 0.1 mm in the absolute (100 cm) length of the caliper, roughly the size of the local distortions. This corresponds to a possible systematic uncertainty of $\pm 0.10\ \mathrm{cm/s^2}$ in our measurement of g. This latter uncertainty is almost twice as large as the statistical uncertainty in our measurement; it could be reduced by using a caliper of more certain pedigree (and expense).

We have found that, even for the smallest y, nylon balls take a measurably longer time than steel balls to fall between the two light beams, typically 300 μs for $\Delta y = 2$ cm, implying that the balls must pass through the first light beam with slightly different initial velocities v_1. This is likely to have only a very small effect on t_∞ because of the dominance of the much more massive steel balls in calculating this quantity, but the magnitude of the effect must be evaluated. The effect is such that it appears that the nylon balls fall from a slightly smaller height than the steel balls, so that their initial velocity v_1 is slightly smaller. Fits to the data for y less than 10 cm, where drag is negligible, and allowing both δ and v_1 to vary, indeed show that both steel and nylon data yield the same δ but that the nylon v_1 is significantly smaller (corresponding to an effective difference in the height y_1 of 0.013 cm in this case). Since the diameters of the balls have been measured to be the same within 0.003 cm, we assume that the effect is due to the existence of a partial vacuum (Bernoulli effect) which exists just after the ball is released, acting to reduce g effectively over some short distance. The effect would be greater for the lighter nylon ball. A simple correction for this effect can be obtained by differentiating the equation of motion for a falling ball

$$y = v_1 t + \tfrac{1}{2} g t^2 \tag{16}$$

to give

$$dt = -\frac{dv_1}{g\left(1 + \frac{v_1}{gt}\right)}. \tag{17}$$

After correcting the fall times for the nylon balls in this manner we find that the effect on the fitted g is to increase it by an amount $0.03\ \mathrm{cm/s^2}$, well within the statistical error.

From this particular experiment, then, after applying this small correction, we have determined a best value $g = 980.41 \pm 0.06 \pm 0.11\ \mathrm{cm/s^2}$, where the quoted errors are statistical and systematic, respectively. And yes, we have succumbed to the temptation alluded to in Sec. I: We have contacted the Minnesota Geological Survey[4] and have found that the local value of g is known to be $980.583\,22\ \mathrm{cm/s^2}$ with a systematic uncertainty on the order of $0.000\,08\ \mathrm{cm/s^2}$. That measurement was made close to ground level, approximately 100 m from our laboratory, using a pendulum technique. Although our laboratory is approximately 10 m below ground, we estimate that this would reduce our value for g by only about $0.003\ \mathrm{cm/s^2}$. Our result is thus significantly lower than the accepted. As we have commented above, we suspect that the main cause of this disagreement is the uncertainty in the absolute length of our caliper.

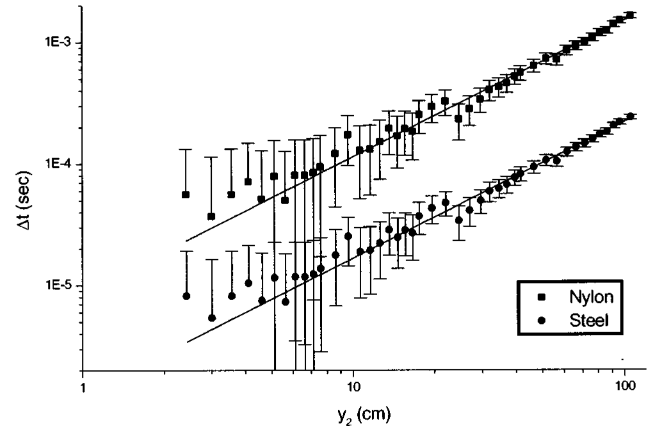

Fig. 4. Δt, excess fall time due to drag, vs y_2, the total height fallen. The solid lines show the fits to $y_2^{3/2}$.

VI. DETERMINING THE DRAG COEFFICIENT

From equations developed earlier, we can obtain an expression for the difference between t_∞ and the measured t at each height, demonstrating directly the effects of buoyancy and drag on the fall times, i.e.,

$$\Delta t = t - t_\infty = \frac{m_{\mathrm{air}}}{2m} t_\infty + \frac{k}{6m}\sqrt{\frac{2}{g}}\,(y_2^{3/2} - y_1^{3/2}), \tag{18}$$

where the first term gives the effect of the buoyant force and the second the effect of drag. The result of subtracting the small buoyancy term from the values of Δt and plotting the remainder versus $y_2 (= \Delta y + y_1)$ is shown in Fig. 4. The lines show the predicted variation as $y_2^{3/2}$. While this graph shows that the data prefer a variation less than the power 3/2, an acceptable χ^2 is obtained for the fit to the predicted behavior. Using only the data beyond $y_2 = 50$ cm and the measured masses, we find $k = (8.43 \pm 0.11) \times 10^{-5}$ and $8.45 \pm 0.48 \times 10^{-5}\ \mathrm{N/m^2/s^2}$ for the nylon and steel balls, respectively. Taking the density of air to be $1.18 \pm 0.01\ \mathrm{kg/m^3}$ (at 20 °C and 50% humidity)[5] we determine the drag coefficient $C = 0.50 \pm 0.01$.

ACKNOWLEDGMENTS

We wish to thank Bill Voje and Rob Leeson of our departmental machine shop for constructing this sturdy apparatus and especially for developing the ingenious ball-dropping mechanism. Marty Stevens provided the drawings for the paper. Finally, we thank the many students who have conducted this experiment over the past several years; their experiences have been invaluable.

[1] Dwight E. Gray, *American Institue of Physics Handbook* (McGraw–Hill, New York, 1972), 3rd ed., pp. 2–268.

[2] W. H. Press *et al.*, *Numerical Recipes in C* (Cambridge U.P., New York, 1994), 2nd ed., pp. 656–666.

[3] ''Microval'' coordinate measuring machine, Browne and Sharpe Mfg. Co., North Kingstown, RL.

[4] V. Chandler, Minnesota Geological Survey (private communication).

[5] C. W. C. Kaye and T. H. Laby, *Tables of Physical and Chemical Constants* (Longman, New York, 1968), 15th ed., p. 18, ed. 1995.