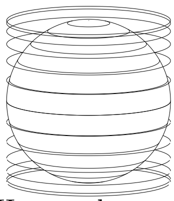
The Sphere

Sizes of Spheres

The area of a unit sphere is given by 4π and its volume by $4\pi/3$ (Think of it as a pyramid, with the apex in the center and the base the surface of the sphere, and the height the radius.). If we have a sphere of radius R those should be scaled by R^2 and R^3 respectively.

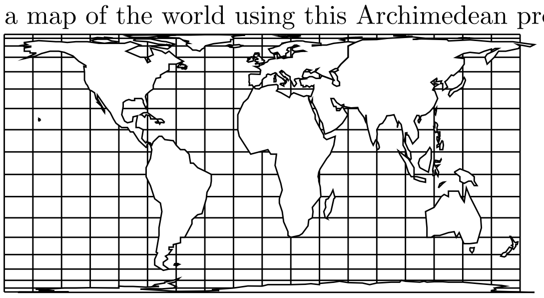
If we circumscribe a sphere with a cylinder, tangent along the equator (say, any great circle will do) and with a height equal to the diameter, its area will be that of the sphere itself. In fact if the radius is 1 the circular circumference of the cylinder is 2π and with the height $h = 2$ we are done, as areas are scaled by the square of the dimensions. This fact was discovered by Archimedes, who thus computed the area of the sphere, a highly nontrivial task before the invention of calculus.

In fact even more is true, if you project any point of the sphere to the cylinder preserving the height, this map will be area preserving.

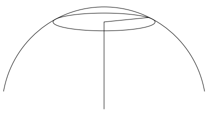


In fact the map will be given by $(\theta, \psi) \to (\theta, \sin(\psi))$ where θ gives longitude and ψ gives latitude. This map will scale as $1/\cos(\psi)$ horizontally (because all latitudes will be mapped to lines of the same length), while by $\frac{d\sin(\psi)}{d\psi} = \cos(\psi)$ vertically. Those two scalings cancel out when it comes to area.

Here we have a map of the world using this Archimedean projection.



In particular we can compute the circumference and area of a circle of radius r on a unit sphere. Clearly the circumference is that of a regular circle of radius $\sin(r)$ thus $2\pi\sin(r)$ while the area will be given by $2\pi(1-\cos(r)$. If r is small we can use the approximations r and $1 - \frac{r^2}{2}$ respectively and obtain the regular formulas $2\pi r$ and πr^2.



In fact the errors are given by $2\pi\frac{r^3}{6}$ and $\pi\frac{r^4}{12}$ respectively. We can use this archimedean property to compute the fraction of the total visual sphere a circle of radius r (radians) constitutes. In fact the fraction will be given by $\frac{1-\cos r}{2} \sim r^2/4$.

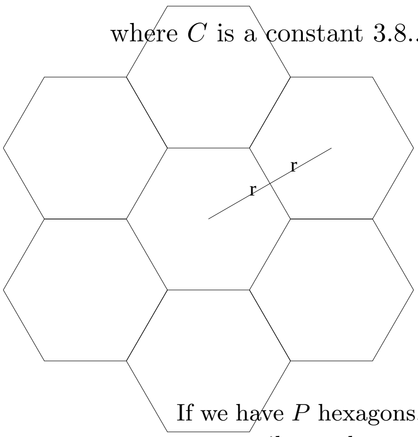
`Examples` **Earth** The radius of the earth is given by 6400 km thus its surface will be 500 million square kilometers. (We have $R = 2^8 \times 5^2$, hence $R^2 = 2^{16} \times 5^4$, furthermore $4\pi \sim 12.5$ and $2^{12} = 4096$ hence $4\pi 2^{12} \times 10^4 \sim 5 \times 10^8$). The three big Oceans, the Pacific, the Atlantic and the Indian make up 350 million km^2, and the Eurasian continent 55 million km^2.

If we have a circle of radius 64 km on the Earth, this will correspond to $r = 0.01$ above, its approximate circumference will be around 400 km and its area approximately 12000 km^2. However on the spherical surface area, those will be somewhat shorter than had the Earth been flat. In fact the errors will be given by $2\pi 10^{-6}/6 \times 6400$ km, approximately 6.4×10^{-3} km, i.e. about six meters for the circumference. For the area we will have $2\pi 10^{-8}/12 \times 6400^2$ km^2, or about 2×10^{-1} km^2, i.e. two hundred thousand square meters (20 hectars).

To get a feeling for the size of the Earth, one may compute how far it would be to the nearest neighbor of a population of P individuals, evenly distributed along it. For a sphere with radius R this distance d will be given by the approximate formula

$$d = C\frac{R}{\sqrt{P}}$$

where C is a constant 3.8.. while for a circle we will have to divide by 4



It is not so clear what is meant by a regular distribution, it ties in with so called sphere-packing. In the plane however the densest way of packing circles is known, namely by the hexagonal lattice. This means that we want to pack as many points together keeping the smallest distance as big as possible. Consider a tiling of the plane with hexagons, and assume that the normal to a side has length r then the distance will be $d = 2r$ and each hexagon will have area $6\frac{r^2}{\sqrt{3}}$

If we have P hexagons, their total area A will satisfy $A = 6\frac{r^2}{\sqrt{3}}P$ Now you cannot tile a sphere with hexagons, but approximately so, if P is large. So set A the area of a sphere with radius R and solve for r and set $d = 2r$. We get $C = \frac{2\sqrt{\pi}}{\sqrt{2\sqrt{3}}} = 3.8092..$

With a population of say 7.2 billions the formula will give about 300 meters (280 to be more exact). Conversely knowing A and d we can find the population $P = \frac{2A}{\sqrt{3}d^2}$

The volume of the Earth is given by $\frac{4}{3}\pi R^3$ or $\frac{1}{3} \times A \times R$ where A is the surface area which we have already computed. This gives roughly 10^{12} km^3. At a density of almost 5g/cm^3 or 5×10^{15}g/km^3 we are talking about a total weight of 5×10^{24} kg.

If we would spread the population of the world inside it, each individual would be given some 125 km^3 the volume of a cube of side 5km, hence there would be a distance of 5 km to the nearest neighbour (hardly visible by the naked eye).

Ganymede The radius of Ganymede is about 2634.1 ± 0.3km, this means 0.413 Earth radi. Consequently the surface area is given by 87.0 million km^2 corresponding to 0.171 of that of the Earths. This boils down to about two thirds of the land area of the Earth

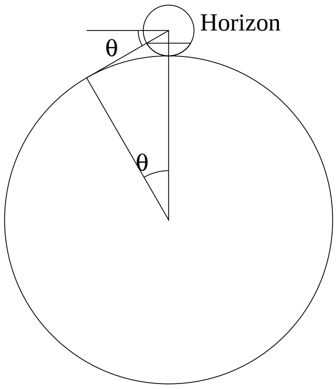
Large Spheres Jupiter is far from being a sphere but is a pronounced ellipsoid, the same for the Sun. For those surface areas are hard to compute, involving so called elliptic functions, but volume are easy. Nevertheless we will approximate them with spheres. In the case of Jupiter we are talking by a radius of 11 earth radii, hence its surface area is about 120 times that of the Earth, while the Sun is ten times as large as Jupiter and we are talking about a surface area which is 12000 times that of the Earth. As to volumes we will have factors of 1300 and 1300000 times that of the Earth, but the masses will not be proportional as the densities are lower. The mass of the Sun is about 300'000 times that of the Earth, i.e. 1.5×10^{30} kg.

But even the Sun is dwarfed by the imagined Dyson Sphere, the sphere with the radius of one A.U. i.e. 24000 earth radii. Its area will then be half a billion times that of the Earth. If it would be fashioned out of the Earth, i.e. being a shell with the same volume as that of the Earth, how thick would it be? This means that we have to divide a third of the Earth radius 64×10^5 meters by 5×10^8 getting about 4×10^{-3}m i.e about half a centimeter, talking about a soap-bubble. If a soap bubble would be about 1Å(10^{-8} cm), about one atom thick, its radius would be about 3 km! A delicate structure indeed. The apparent size of the Earth seen from the Sun compared to that of the Dyson sphere making up the entire visual field would be only a billionth, as the surface of a hemisphere is twice that of what the great circle defining it, enclosing it. This we will return to. To get an even more dramatic number to be used later, consider the human pupil, at a radius of at most 5 mm, it makes up for about 10^{-9} earth radii, and we are now talking about a fraction of 10^{-27}. The radius of the Dyson sphere is approximately 200 times that of the Sun (which is roughly the geometric mean between the Earth and the Dyson) thus it will be a about ten million times as voluminous. If it would be filled with air, one thousandth of the density of the sun, the mass of it would be about ten thousand sun masses and may quickly degenerate into a Black Hole. We will have occasion to return to the Dyson sphere, where we will have the choice of living in the inside or the outside.

Furthermore densities will scale linearly with R so if the human population would be spread over Jupiter, the distance would be 3 km, while 30 km for the Sun and 6000 km for the Dyson sphere, where each individual would have a continent size to themselves. Inside the Dyson sphere there would be no gravity (removing the Sun) and would the human population be evenly spread out inside it there would be 120000 km to the nearest neighbor corresponding to the diameter of Jupiter.

The Horizon

The most tangible way we can see that the Earth is not flat is by the finite distance to the horizon. This distance depends on our altitude, i.e. height h above the Earth, and neglecting mountains that can hide it, say on a quiet ocean, when the horizon is unobstructed, we can easily compute it as a function of h.



Let us work for simplicity on a unit sphere. The distance to the horizon will be given by the angular distance θ . Centering the visual sphere at the eye of the observer, there will be a great circle parallel to the tangent of the Earth. Had the Earth been flat then it would have coincided with the horizon, which in that case would have divided the visual sphere in two equal parts, an upper part corresponding to the sky, a lower part corresponding to the ground (earth, ocean, whatever). As it is the horizon lies slightly below that great circle, and is in fact a small circle. If it lies an angle θ below, its radius is $\cos(\theta)$ (in the visual sphere). This angle is incidentally the same as the angular distance to the horizon. Thus it is clear that when h increases the angle θ will increase approaching the limit $\pi/2$.

Thus the further away we are from the Earth, a larger and larger part will become visible, as the distance to the horizon will increase, in the limit half the Earth will become visible. On the other hand from the point of the observer the horizon will occupy a smaller and smaller part of the visual field and will in the end approach a point. Thus when look at a globe or a picture of the Earth from space, its circular circumference will be the horizon, in particular it will visibly curved being a small circle on the visual sphere. An interesting question is how far above a sphere you need to be to experience the horizon as curved, of course this being a subjective matter to draw the line between straight and curved, and to which we will return.

Distance to the horizon:

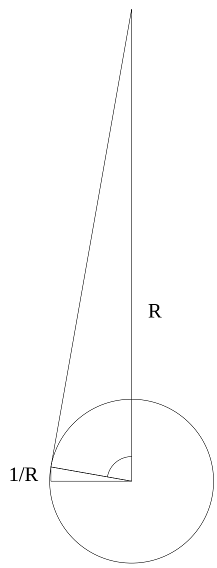
We easily get the following formula from Pythagoras

$$(1+h)\sin(\theta) = \sqrt{(1+h)^2 - 1}$$

If h is large we can write

$$\sin(\theta) = \sqrt{1 - \frac{1}{(1+h)^2}} \sim 1 - \frac{1}{2h^2}$$

this should be compared to $\sin(\pi/2 - t) = \cos(t) = 1 - t^2/2 + \ldots$. Thus the discrepancy from $\pi/2$ is given by $1/h$ for large h. This ties in with the following elementary fact below proved by similar triangles,



On the other hand if h is small we get

$$\sin(\theta) \sim \sqrt{2h}$$

and hence $\theta \sim \sqrt{2h}$. If we want to keep h fixed but scale with the radius R (in meters) we will have to replace h with h/R and multiply with R and hence get the formula $\sqrt{2Rh}$ (This could also have been derived directly from Pythagoras by considering $\sqrt{(R+h)^2 - R^2} \sim \sqrt{2Rh}$ ignoring as above h^2). Thus with given height the distance to the horizon grows like the square root of the radius, and with given radius the distance grows by the square root of h (as long as h is small compared to R).

The area encircled by the horizon (for small h) would be given by $2\pi h$ and hence a fraction $h/2$ of the total area. Once again if we scale by R, keeping h constant, we would get $2\pi(h/R)R^2 = 2\pi Rh$ thus would grow linearly both with respect to R and h as longs as h/R remains small.

`Examples` If $h = 2$ meters and $R = 6.4 \times 10^6$ meters, the radius of the Earth, we need to take the square root of $2 \times 2 \times 2^6 \times 10^5$ yielding approximately $2^4 \times 10^2\sqrt{10} \sim 5 \times 10^3$ m. Hence it is five kilometers to the horizon (provided of course that the Earth is fairly 'flat', say on an expanse of water). We can also compute the angle θ by taking the square root of $4/R = 2^{-4} \times 10^{-5}$ getting around $3/4 \times 10^{-3} = 7.5 \times 10^{-4}$.Recall that one degree corresponds to $2\pi/360 \sim 1/60 \sim 1.6 \times 10^{-2}$ and hence one minute to around 2.5×10^{-4} thus corresponding to about 5/3 km. A more careful calculation would yield 1.8 km, which is in fact the definition of the Nautical mile. (Another direct calculation would have been to divide the circumference of the earth (40000km) with 360×60 getting 1.851... km). The velocity of 1 nautical mile per hour is referred to as 1 knot.

We observe that at this distance, the horizon will lie 3 minutes of arc below the idealized one. If that would be marked on the sky, we would barely notice the separation.

Furthermore the area we would survey would be a fraction $2^{-6}10^{-5} \sim 1.6 \times 10^{-7}$ of the total area of the Earth (roughly corresponding to 80 km^2 in accordance with what we would get for a circle of radius 5 km). Multiplying with the population of the Earth we can compute how many people would be visible within the horizon, would they all be evenly spread out. About a thousand people.

We can now scale things. Assume that we are on a mountain 7200 meters tall. How far would we see? We scaled with a factor 3600 and its square root is 60 so we obtain a distance of 300 km. If we would be flying in a commercial plane between Stockholm and Gothenburg we would be able to see both of the cities simultaneously above the horizon. The first Cosmonauts, such as Gagarin were orbiting at a height of 300 km, we are now talking about a radius of 2000 km seeing about 2-3% of the Earth's surface area.

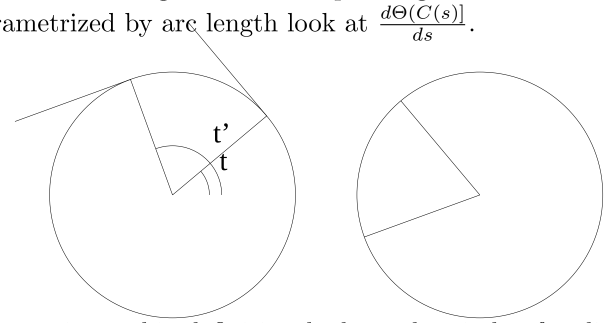
Fixing the height, the fraction the visible surface area would be of the total would vary inversely with the radius. Hence on Jupiter only a hundred people would be visible (at a fixed height of two meters), on the Sun hardly a dozen, while on the Dyson sphere (on the outside) only one in twenty horizontally bounded circles would contain a human, and it would indeed be far to the horizon, in fact 750 kilometers.

Curvature of the horizon:

Curvature measures the way a curve deviates from being a straight line. An object which is not under the influences of any forces travels in a straight line with uniform velocity. Any change of direction necessitates an intervening force. To that we will return later.

There are a variety of equivalent definitions of curvature.

1) Let $\Theta(p)$ denote the direction of the tangent at point p and let the curvature be the change of direction per length. In other words if the curve $C(s)$ is parametrized by arc length look at $\frac{d\Theta(C(s)]}{ds}$.



To motivate this definition look at the circle of radius r parametrized by $r\cos t, r\sin t)$ its tangent at a point t has the direction given by the derivative $(-r\sin t, r\cos t)$, and going from t to t' the angle changes by $t'-t$ along an arc of length $r(t'-t)$, in this case the change of angle is proportional to the length of the arc with proportionality constant $1/r$ which will be its curvature.

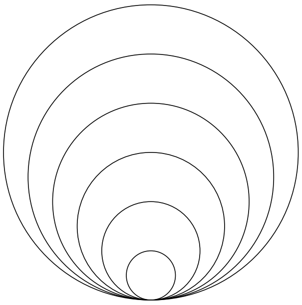
2) Let p, q, r be three points that approach P. Those three points determine a circle , and let C be the limiting circle. It is the circle which best approximates the curve, and is referred to as the osculating circle. The inverse of its radius is called the curvature. (Note that we also have in addition to the measure of curvature also a point, namely the center of the osculating circle.)

3) Let p, q approach P and let $N-p, N_q$ be the corresponding normals to the curve at P and consider the limiting intersection point O , and let the curvature be the inverse of the distance OP

Clearly 2) and 3) are intimately related, O will clearly be the center of the osculating circle.

While the tangent is a first order approximation of a curve, curvature concerns second order. Thus the second order Taylor expansion should give us a clue to curvature.

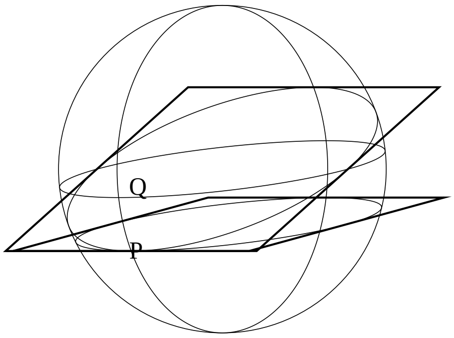
In particular consider the parabola $y = ax^2$ which has the x-axis $y = 0$ as a tangent at the origin. What is its curvature? Consider the family of circles $x^2 + (y-r)^2 = r^2$



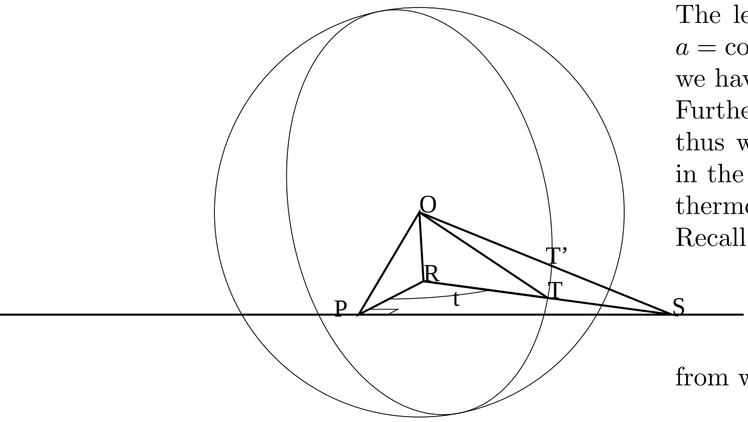
They are all tangent to to the x-axis ($y = 0$) and if we solve locally for y in the vicinity we get $y = \sqrt{r^2 - x^2} - r = r(\sqrt{1 - (\frac{x}{r})^2} - 1)$ whose first terms in the Taylor expansion is $y = r(\frac{x^2}{2r^2} + \ldots)$ or $y = \frac{1}{2r}x^2 + \ldots$. Thus comparing with the parabola above we should put $\frac{1}{2r} = a$ or the curvature $\frac{1}{r}$ being $2a$. Thus the smaller a i.e, the closer to a line, the smaller the curvature. We can also compute the (unit) tangent directions of the parabola. The derivative at x is given by $2ax$ hence with the appropriate normalization given by scaling with $K = \frac{1}{\sqrt{1+4a^2x^2}}$ we can take the scalar product of the two vectors $(1, -2ax), (1, 2ax)$ normalized by K^2 getting $\frac{\sqrt{1-4a^2x^2}}{\sqrt{1+4a^2x^2}} = \cos(\theta)$ where θ is the change of direction. Now as $\frac{1}{1+t} = 1 - t + t^2 + \ldots$ (an infinite geometric seres) we can write $\frac{1-t}{1+t} = 1 - 2t + \ldots$ and applying to $t = 4a^2x^2$ we obtain $\cos(\theta) = 1 - 8a^2x^2 + \ldots$ as $\cos(\theta) = 1 - \frac{1}{2}\theta^2 + \ldots$ we can conclude by comparison that $\theta = 4ax$ approximately while the length of the curve from $-x$ to x is approximately $2x$ thus getting a curvature of $2a$ as expected.

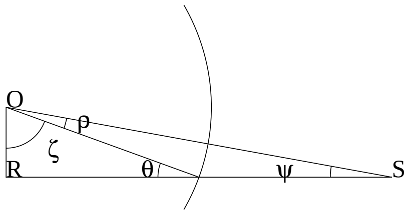
We can also work out the normals to the parabola at the points (t, at^2) as $t \to 0$. The slopes of the normals will be given by $-\frac{1}{2at}$ and hence the equations $(y - at^2) = -\frac{1}{2at}(x - t)$ setting $x = 0$ we get the point $y = at^2 + \frac{1}{2a}$ on the y-axis (the intersection of the normals, with the normal at $x = 0$. As $t \to 0$ this will approach $\frac{1}{2a}$ as expected.

Now we are in position to investigate the curvatures of various horizons, i.e. of various small circles (latitudes).



For that purpose we consider a small circle at latitude θ (the length of the great arc PQ) and then a great circle tangent to it at P. This is obtained by choosing the plane spanned by the tangent to the small circle at P and the center of the sphere, given rise to the tangent great circle at P. We now need to measure the deviation of the small circle from the great. The greater, the more curved it will appear. For that purpose we draw a couple of triangles. Note that PR is perpendicular to PS the tangent, that the angle t (PRS) is supposed to be small, and we are interested in the deviation given by TOT' and its dependence on t. Note that T, T' lie on a great circle.





The length a of RS satisfies $a\cos t = \cos\theta$ hence $a = \cos\theta\sec t$ (recall that in Anglo-Saxon literature we have the definition $\sec t = \frac{1}{\cos t} = 1 + \frac{1}{2}t^2 + \ldots$). Furthermore the angle ψ satisfies $a\tan\psi = \sin\theta$ thus we get $\tan\psi = \tan\theta\cos t$. We are interested in the angle σ. We note that $\sigma + \zeta + \psi = \pi/2$ furthermore that $\zeta + \theta = \pi/2$ thus $\sigma(t) = \theta - \psi(t)$. Recall that

$$\tan(\theta - \psi) = \frac{\tan(\theta) - \tan(\psi)}{1 + \tan(\theta)\tan(\psi)}$$

from which we get

$$\tan(\sigma(t)) = \frac{\tan(\theta)(\frac{1}{2}t^2 + \ldots)}{(1 + \tan^2(\theta))(1 - \frac{\tan^2(\theta)}{1+\tan^2(\theta)}\frac{1}{2}t^2) + \ldots}$$

which can be simplified to

$$\sigma(t) = \frac{\tan(\theta)}{2(1 + \tan^2(\theta))}t^2 + \ldots$$

If θ is small this simplifies further to $\sigma(t) = \frac{\theta}{2}t^2 + \ldots$ (which could have been observed directly above setting $\psi \sim \tan\psi, \theta \sim \tan\theta$ and hence $\psi = \theta(1 - \frac{1}{2}t^2 + \ldots)$

The observant reader may observe that if θ approach $\pi/2$ then $\tan(\theta) \to \infty$ and then the coefficient would go to zero, rather than ∞ as you expect the curvature of a small circle to be large. The problem is that the parameter t is too small when the circle is small, and should be scaled appropriately, then everything will work out as it should. The technical details are saved for later.

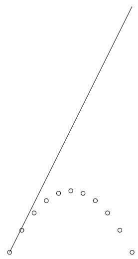
Gravity

Objects on Earth fall to the ground. They do so with a velocity that grows linearly with time. The rate of proportionality is given by g which in round figures is 10m/s^2, more exactly 9.81m/s^2, which varies with location. g is highest at the poles and lowest at the Equator, partly because the poles are closer to the center of the Earth, and partly because of the rotation of the Earth there is a slight centripedal force to some extent counterbalancing the gravitational force. To this we will return in the next section.

The velocity attained after falling freely, ignoring air-resistance, after time t is given by gt. In particular it means that after one second you have attained 10m/s, after two 20m/s and so on. The distance s travelled after t seconds will be given by $\frac{1}{2}gt^2$, thus after having fallen for four seconds

you will have covered a distance of 80m. Conversely given s we can solve for t and get $t = \sqrt{2s/g}$. Thus if you jump from the Eiffeltower, say 300 meters, it will take you $\sqrt{60} \sim 8$ seconds before you hit the ground, long enough for you to regret the impulse. The final velocity v on impact will be gt. As $2sg = (gt)^2 = v^2$ we can compute that by $v = \sqrt{2gs}$. In the case of the Eiffeltower we are talking about almost 80m/s, or almost 300km/h. Not to be recommended.

Conversely if you throw something upwards with initial velocity v, it will not stay constant but will slow down with time according to $v - gt$. When that velocity is zero, i.e. when $t = v/g$ the object will momentarily come to a standstill and thus reach its maximal height. During the time t it has travelled $S = vt - \frac{1}{2}gt^2$ and plugging in $t = v/g$ we get $H = \frac{1}{2}\frac{v^2}{g}$. We factor $S = t(v - \frac{1}{2}gt)$ nd we see that $S = 0$ also when $t = 2v/g$ i.e. the double time. This is when it reaches the ground. Had we dropped an object from age H it would have reached the ground in v/g and would have had a final velocity of v. It would have travelled half of the path of the object that was thrown upwards. An object a height h is said to have a potential energy P of mgH and if traveling with a velocity of v a kinetic energy K of $\frac{1}{2}mv^2$. If we add those two $mgS = mgvt - \frac{1}{2}g^2t^2$ and $\frac{1}{2}(v - gt)^2$ we get a constant $\frac{1}{2}mv^2$ independent of time. This can be expresses that the total energy E is constant, that potential energy is converted into kinetic energy when somethings is dropped and conversely kinetic energy can be converted to potential energy when ascending. This is a very general principle for all of mechanics. In particular it allows us to treat arbitrary movement not only linear. Note also that due to Pythagoras if we have a velocity that is decomposed into two orthogonal component, the kinetic energy is the sum of the kinetic energies of the components. This points to a deeper significance of Pythagoras.

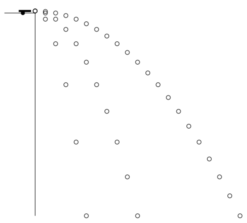


In particular we can think of projectile fired, with a horizontal velocity of v_x and a vertical velocity of v_y. We will have v_x constant and $v_y(t)$ given by $v_y - gt$. At time $t = v_y/g$ it has reached its maximal height, and at $t = 2v_y/g$ it has hit ground, and hence travelled horizontally a distance $(2v_x v_y)g$. Note that if $v_x^2 + v_y^2 = 1$ the maximal extension will occur when $v_x = v_y$ i.e. that the projectile is fired at 45 degrees. If the initial speed is $|v|$ (the length of the velocity vector $v = (v_x, v_y)$ then the maximal distance will be given by $|v|^2/g$

`Example` On Ganymede the gravitation is only about a seventh (0.146) of that of the Earth. That means that you can jump and throw seven times as far on Ganymede as you do on Earth. If you can jump from two meters on Earth without hurting yourself you can jump from fourteen meters on Ganymede. You would hit at the same velocity but it would take you seven times as long to reach it. Jumping from the Eiffel tower on Ganymede would almost 20 seconds and you would hit the ground at over 100 km/h. The scaling factor now becomes $\sqrt{7} \sim \frac{8}{3}$.

Circular Motions

If you shoot a cannon ball from a mountain it will eventually fall to the ground. The faster the ball is, the longer it will take. Could it be that there is a velocity such that it will keep falling to the ground but never reach it, because the ground itself 'falls away' due to the Earth being a sphere, not an indefinitely extended flat region.



First look at cannon balls being fired horizontally from a vertical cliff of height H. The trajectory will be a parabola and it can be described by $x = vt, y = H - \frac{1}{2}gt^2$. The ball will hit the ground when $t = \sqrt{\frac{2H}{g}}$ and the ball has then travelled a distance $x = v\sqrt{\frac{2H}{g}}$.

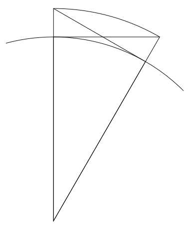
`Example` If $H = 2000m$ then $\sqrt{\frac{2H}{g}} = 20s$ (the time it would take to reach the ground after jumping neglecting air-resistance). A cannon ball that travels at $1000m/s$ would thus reach $20km$. If $H = 10000m$ and a commercial jet traveling at $300m/s$ and then losing power and falling freely would crash after 45 seconds having covered 13.5 km (gliding in the air would increase the distance and postpone the crash and prolong the agony).

We can eliminate t and write it as $y = H - \frac{1}{2}\frac{g}{v^2}x^2$ and compare with the local equation of a circle $x^2 + (y-R)^2 = R^2$ at the local tangent $y = R$ at $(0, R)$ given by $R - \frac{1}{2}\frac{x^2}{R}$ (which we derived in our discussion on curvature). Setting the coefficients equal we get $v = \sqrt{gR}$ which seems to be a critical velocity, because then the path of the cannon ball and the surface of the Earth will keep in step, i.e. agree up to second order, and have the same curvature. If it is less you will expect the ball to eventually fall on Earth, if it is more, it will follow a more flattened path. In both cases we will speak about ellipses and return to the matter in greater detail later.

The critical velocity can easily be computed, we get $v = \sqrt{10 \times 64 \times 10^5} m/s = 8000 m/s$. This is the velocity most artificial satellites travel at close to the surface of the Earth. A complete revolution around the Earth will take $40000/8 = 5000$ seconds, just about an hour an a half.

We can return to the first example and compute the velocity needed to reach the horizon. The distance to the horizon is $\sqrt{2Rh}$ as we have derived before. The time needed will be $\sqrt{\frac{2h}{g}}$ as computed initially, and this corresponds to a velocity of

$$v = \frac{\sqrt{2Rh}}{\sqrt{\frac{2h}{g}}} = \sqrt{gR}$$



independent of h and equal to the critical velocity above. The explanation is provided by the picture to the left. When the ball has travelled the required distance, it has indeed dropped the amount of h, but at the same time, the Earth has dropped the same amount, and we are back to the initial position, save that we have advanced. By induction we can continue and end up orbiting the planet indefinitely.

Newton discovering the inverse square law

The story of Newton 'discovering' gravitation by having a apple fall on his head is clearly apocryphal. A more reasonable guess is that Newton wondered why he Moon does not fall down as does the apple. The Moon has always occupied a position between the celestial world and the terrestrial, it clearly looks like a big stone. One explanation is as above, and the point is to make it more quantitative. Newton knew the distance to the Moon in terms of Earth radii, that had been known since antiquity and Ptolemy supplied a very good value. More doubtful is whether he knew the actual size of the Earth in terms of conventional measures, out of which the gravitational constant g had been computed, that is essential, without it there is no means of comparing, as in the time of Newton there were no artificial satellites orbiting close to the surface of the Earth, in particular the velocities of such were not known. Had they been known, or the time of revolution, one would have been able to figure out R knowing g. So let us assume that he did know.

The distance to the Moon is $60 \times 6400 = 384000 km$ and thus the circumference of its orbit is $2400000 km$. It makes a complete orbit in a sideral period of 27 days 7hrs and and 43 minutes. Convert that to seconds and we get 2.36×10^6. Divide by the distance and we get about $1 km$. During that second the Moon has fallen $\frac{1}{2\times 384000} = 1.3\times 10^{-6}km = 1.3mm$ while on Earth it would have fallen about $5meters$ which is about 3600 times as far. Now obviously Newton did not chose an exact value for the exponent based on the empirical evidence, but chose the simplest solution, this is a time-honored method of exact science, without which no progress would ever have been made. It is a perennial mystery to philosophers that the laws of nature are so simple and hence accessible to the aesthetic sense of the human mind.

Once we have the inverse square law, we can discover some more beautiful facts. For one thing, if $g = \frac{k}{R^2}$ where k is a constant of proportionality assumed proportional to the mass of the central body. Given the orbital velocity $v = \sqrt{gR} = \sqrt{k}\sqrt{\frac{1}{R}}$ we compute the orbital time of a circular movement to be

$$T = \frac{2\pi R}{v} = \frac{2\pi}{\sqrt{k}}R^{\frac{3}{2}}$$

and thus

$$T^2 = \frac{4\pi^2}{k}R^3$$

which is Kepler's third law verified for circular movements. Conversely assuming Kepler's law, we can derive the inverse square as we have

$$T = \frac{2\pi R}{\sqrt{gR}} = \frac{2\pi\sqrt{R}}{\sqrt{g}}$$

form which we conclude

$$g = (\frac{2\pi\sqrt{R}}{T})^2 = \frac{4\pi^2 R}{T^2}$$

and hence if $T^2 = R^3$ we are done.

This might be a more likely way for Newton deriving the inverse square law.

Artificial satellites

The first satellites put into orbit in the 50's and early 60's flew very close to the surface of the Earth. Gagarin in 1961 skirted the upper atmosphere at heights between 169 and 327 km to be exact completing one orbit in 89 minutes. Those thus travelled at a speed of 8km/s, to be compared with the speed of the rotating equator of 450 m/s. One can then ask at what distance would we have a geosynchronus satellite, whose velocity would keep in step with the rotation of the Earth. Many such satellites have been launched. We are now in position to solve it, we simply need to find an R (in terms of Earth radii) such that

$$450R = \frac{8000}{\sqrt{R}}$$

(The left hand side gives the velocity of the satellite at a distance R having a 24h rotational period, the right hand side, its velocity due to Kepler's law.) This is easily solved giving $R = 6.68(\sim 43000 \text{ km})$.

The Galilean satellites

The Four Galilean satellites follow almost circular orbits. We list the orbital characteristics

Name	Orbital radius R	Orbital period$T$	T^2/R^3
Io	421,700	1.77	42.0
Europa	671,034	3.55	41.7
Ganymede	1,070,412	7.15	41.7
Callisto	1,882,709	16.69	41.7

Comparing the $T^2/R^3 = 13181.6$ for the Earth's Moon illustrates that Jupiter is 315.8 times heavier than the Earth.

An alternate approach

A circular movement of constant velocity may be represented by $R(t) = (R\cos\omega t, R\sin\omega t)$ where the period T is determined by $\omega T = 2\pi$. We find $R'(t) = \omega(-R\sin\omega t, R\cos\omega t)$ and $R''(t) = -\omega^2(R\cos\omega t, R\sin\omega t)$ thus the velocity vector is always perpendicular to the position vector, and the acceleration vector points in the opposite direction of the latter. Letting v and a be the absolute values of the velocity and the acceleration we get $v = \omega R, a = \omega^2 R$ and hence $a = \frac{v^2}{R}$. If the inverse square is valid, then $\frac{v^2}{R} = k\frac{1}{R}$ and hence $v = \sqrt{\frac{1}{R}}$ and we will get $T^2 = R^3$. If on the other hand $\frac{v^2}{R} = kR$ as in the harmonic oscillator below, we get $v = kR$ and the ω is constant independent of R.

`Example`. When an object travels away from you, its spectrum is redshifted, the amount proportional to the fraction of the light velocity, conversely if it is approaching you it shifts towards the blue. This is part of the Doppler effect. Assume that we have a binary system of stars, one of them orbiting circularly in the same plane as our sight. This can be checked by noticing the variation of velocity relative us. By optical observation we can determine the maximal separation between the stars, i.e. the apparent size of the radius, and the orbital time. From the spectral analysis we can determine the speed of orbiting, and thus the length of the orbit. From knowing the apparent size of the orbit we can determine the distance to it, and hence the absolute magnitudes of the stars. Furthermore by determining the relation between T^2 and R^3 we can determine the mass of the major star. By its color we can determine its temperature and from its luminosity its size, knowing its mass we can compute its density. So quite a lot of information to be teased out from a few ocular observations. One can also handle more complicated movements than circular ones in the same plane of sight, but it is more technical, the simple example illustrates the principal points.

Escape velocities

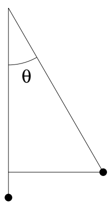
To lift an object (of mass m) from distance R_1 to R_2 in a gravitational field $\frac{k}{R^2}$ the energy needed is $m\int_{R_1}^{R_2}\frac{k}{r^2}dr = \frac{k}{R^1} - \frac{k}{R^2}$. We can let $R_2 \to \infty$ and the energy will still be finite given by $m\frac{k}{R^1}$. If all that energy is converted to kinetic energy, we obtain the velocity an object falling from infinity obtains when reaching $R_1(R)$. This is referred to as the escape velocity, because an object given that velocity will escape to infinity. If we solve for v we get $\sqrt{2gR}$ which should be compared to the figure of $\sqrt{gR}$ for a circular orbit.

`Examples` The escape velocity for the Earth at its surface is $\sqrt{2}\times 8000m/s = 11200m/s$, while that for the Sun at the orbit of the Earth is $\sqrt{2}\times 30000m/s = 42000m/s$. If we scale with R then g is scaled by R^{-2} and thus the whole expression with $R^{-\frac{1}{2}}$. The escape velocity of the Sun at its surface should hence be $\sqrt{14}$ higher, i.e. $600000m/s$ this is 1/500 of the velocity of light. If the escape velocity is equal to the velocity of light, nothing can escape, and we have a Black hole. The distance at which the escape velocity is that critical value is the so called Schwarzschild boundary, outside it, escape is still possible, inside it impossible. In order for the Sun to be a Black hole with its boundary at its present surface, it need to be $500^2 = 250000$ times heavier. Would the Sun contract to a radius 1/250000 of its present, it would become a Black hole. This would be a sphere with a radius about 3 km. Its density would be 1.6×10^{16} times its present one. One cubic millimeter would weigh 16 million tons. For the Earth to become a Black hole it would have to contract to 2×10^{-8} of its present size, increasing its density with a factor 10^{25} and its size to little more than $2.5dm$ across.

The Harmonic Oscillator

There is another very important but also simpler power law that governs motion in physics, namely the one in which the attractive force is proportional to the distance. In one dimension we are talking about the harmonic oscillator. Examples are Hooks law when the restoring force of a spring is proportional to its extension, or the pendulum (see below). In one dimension the motion is described by a function $x(t)$ which satisfies the ordinary differential equation $x'' = -k^2x$ where k is (non-zero) constant. The general solution is given by $x(t) = A\cos kt + B\sin kt$ with A, B arbitrary constants. Thus the motion will be periodic, oscillatory around a center. The period T of the motion will satisfy $kT = 2\pi$ and be independent of the initial conditions, which is an important fact that ensures the accuracies of pendulum clocks. If the initial condition is given by rest at time $t = 0$ at position L we get the conditions $x(0) = L$ and $x'(0) = 0$ which translates into $x(t) = L\cos(t)$. The potential energy is given by $k\frac{x^2}{2}$ and the kinetic by $\frac{(x')^2}{2}$ (omitting the common factor of the mass m by normalizing it into 1). By setting E to be the sum, i.e. the total energy, we find that $E' = kxx' + x'x'' = x'(x + kx'') = 0$ thus E is constant. Conversely as in the case of the pendulum if you can write down a constant energy of that form, the motion adheres to that of a harmonic oscillator.

Given a pendulum of length L whose position is given by the angle $\theta(t)$ from its equilibrium.



Its potential energy is given by $gL(1-\cos\theta)$ and its kinetic by $\frac{(L\theta')^2}{2}$. For small θ we can replace $1-\cos\theta$ by $\frac{\theta^2}{2}$ obtaining

$$E = \frac{g}{L}\frac{\theta^2}{2} + \frac{\theta^2}{2}$$

differentiating the constant we get the differential equation of the oscillator with $k = \sqrt{\frac{g}{L}}$. The period of the oscillator can be determined with great accuracy (letting in oscillate for a long time, and counting the number of oscillations), and thus k. If L can be accurately measured (by say being very long), we can get a rather precise value of g, superior to any direct measurement.

Note that by choosing suitable ψ such that $\sin(\psi) = \frac{A}{\sqrt{A^2+B^2}}$ and $\cos(\psi) = \frac{B}{\sqrt{A^2+B^2}}$ we can write $A\cos kt + B\sin kt$ as $\sqrt{A^2+B^2}\sin(kt+\psi)$, where the constant factor is the amplitude.

If we go to two dimensions, there will be no extra technical problems. The ordinary equation splits up into two independent ones, namely $x'' = -k^2x$ and $y'' = -k^2y$ and which can be solved individually, giving $(A\sin(kt+\psi), B\sin(kt+\phi))$ which is the motion of an ellipse. By using a change of co-ordinates we can bring it under standard form $(A\cos kt, B\sin kt)$ of an elipse whose axi coincide with the co-ordinate axi, and whose equation is given by $\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$. Thus the motions of objects under such a law would be ellipses with a common center, and all having the same periodicity.

There will be no point of extending the discussion to arbitrary dimensions, even if it formally works wit no problems, as motion will always take place in the plane spanned by the center of force and an initial velocity vector.

Astronomical applications

Consider a homogenous mass distribution in a sphere, say that of a gas or a globular cluster of stars. As noted in the previous section, the gravitational attraction at a point is only exerted by the shells in which the point is not included. Thus it will be given by $G\rho R^3/R^2 = \rho GR$, where G is the gravitational constant, and ρ the density. Hence the stars in such a cluster will move in ellipses centered at the center of the sphere. The times of revolution will be the same, thus stars at the edge will move much faster than those in the center. We can also compute the total potential energy of such a gas or cluster. Namely it is given by

$$\int_0^R \rho G\frac{r^2}{2}(mr^2)dr$$

where m is the constant of proportionality between the mass of a shell and r^2 which essentially gives its area, we need not concern us with it, except to note that $\int_0^R mr^2dr = \frac{1}{3}mR^3 = M$ the mass of the total ball. Integrating the first we get $\frac{1}{10}\rho GmR^5$ which can be simplified to $\frac{3}{10}\rho MGR^2$.

`Example` Applying it to the sun we get with $M = 1.5 \times 10^{30}kg, G = 6.7 \times 10^{-11}Nm^2/kg^2$ while $R = 7 \times 10^8 m$ and $\rho 10^3 kg/m^3$ getting $7 \times 10^{39} J$.

Magnitudes and Luminosity

The scale

The Greeks classified the stars into different classes, or magnitudes. One speaks informally of stars of the first, second and up to sixth magnitude, the latter barely visible to the naked human eye. This arithmetic scale corresponds to an actual geometric one. This is common for sense impressions, and known as the Weber law. Multiplying the sensation by a factor is sensed as adding a step. This makes sense as there is no natural unit, or what comes to the same thing no natural zero. When two sensations are compared all what you can do is to express one in terms of the other. One of them can be said to be the norm, then it has value zero, the other may said to have value one. Then you can compare any third to them and assign a number, a logarithm in fact. If sensation A is zero and sensation B is one (note B need not be stronger than A it is just a convention. Now if C is to B as B is to A then naturally C will be given the value 2, on he other hand if C is to A as A is to B it has the natural value -1. Increasing the magnitude with one step, means decreasing the luminosity with a certain factor. It has now been normalized such that increasing the magnitude with five steps means dividing with a hundred. Thus each step on the magnitude ladder means a factor of $10^{0.4}$ (the fifth root of a hundred). As noted magnitude 6 is about the limit the human eye can detect at night, but in full daylight the limit is -4, thus stars have to be 10'000 times brighter to be seen, and no stars qualify, apart from the Sun and the Moon, only Venus can under favorable circumstances be seen in daylight.

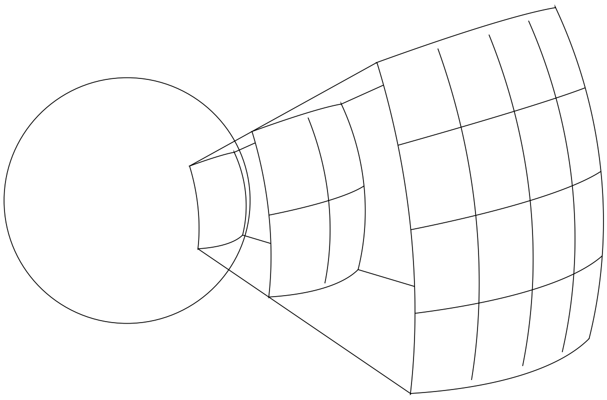
It can be handy to give a table

magnitude	0	1	2	3	4	5
luminosity	1	2.5	6.25	16	40	100

It could also be instructive to give a table of luminosities

Sun	-27	Jupiter	-2.9	Vega	0
Full Moon	-13	Mars	-2.9	Antares	1
Supernova[1]	-7.5	Mercury	-2.6	Polaris	2
Venus	-5	Sirius	-1.5	Uranus	5

Inverse Square



The intrinsic Luminosity of an object is the amount of light it emits per time unit. One can think of it as a mass of particles (photons) being sent off in all directions, and imagine an onion of concentric shells through which they pass. The further away from the source, the larger the shell, and hence less luminosity (photons) per unit area. Thus the luminosity of an object decreases by distance. This can be made precise, as the picture to the left indicates. The areas scale as squares with the distance, thus luminosity varies inversely with the square of the distance.
In particular we see that if the distance is multiplied by 4 the magnitude increases by three steps (from the table above).

It can be interesting to give a table of the magnitude of the Sun seen from different planets in the Solar system and in its vicinity. Distances are given in astronomical units, and in parenthesis distances in lightyears, one light year being about 60'000 A.U.

Mercury	0.4	-29.0	Jupiter	5.2	-23.4	Pluto	-19.0	0
Venus	0.7	-27.8	Saturn	9.5	-22.1	Oort Cloud	50000(1)	-3.5
Earth	1	-27	Uranus	19.2	-20.6	Sirius	4.510^5(8.6)	1.3
Mars	1.5	-26.1	Neptune	30.1	-19.7	Polaris	2.510^7(434)	10

Note that at the very out-skirts of the Solar System, the Sun is still the brightest star, while at the Pole Star the Sun would be very faint. Much fainter than the Polaris is seen from us. The Sun is in fact a rather unremarkable star, there are many stars significantly brighter, and larger, size and brightness being intimately related. Then during brief moments, a star can be turned into a nova, or even a supernova. SN 1006 mentioned above, shone at -7.5 at a distance of 7200 light years. At one light year the brightness of the Sun is -3.5, at one light year the magnitude of that supernova would have been -26.5 almost as bright as the Sun. At that distance it would have been lethal.

The diameter of the pupil is at its widest just short of a centimeter, at bright light it could be less than half. With a large telescope, such as the classical Hale with its 5 meter mirror, one million as much light can be collected, this makes for fifteen more magnitudes. In addition the photographic plate, as well as the more modern digital devices, also allows accumulative effects, which the human retina is incapable of.

The Luminosity of the Sun

The Sun produces during each second an amount of energy of $3.8 \times 10^{26} J$ (J standing for Joule, also known as a Newton meter, the energy needed to lift one kilo of mass one meter in the gravitational field at the surface of the Earth). This corresponds to a power of $3.8 \times 10^{26} W$. Most of that energy is converted into light. There are about 3×10^7 seconds to a year, so during a year the total energy production of the Sun during a year is about $10^{34} J$, and for simplicity assuming that the energy production has been constant during the history of the Earth spanning 5×10^9 years, we are talking about $5 \times 10^{43} J$.

An average human consumes about three thousand calories during a day, or more accurately kilo- calories. One calorie is the amount of energy needed to increase the temperature of one gram of water one degree. Most of that energy is needed as a hot-blooded mammal to maintain a steady body temperature of 37^o. Thus the need for food grows not with volume (or weight both being roughly proportional) but with surface area, so the larger the mammal the less food is needed per weight to maintain life.

> A fairly good approximation for the needed calorie intake is to measure how quickly the temperature of a milk-container filled with water at the temperature of the human body decreases with time in room-temperature, taking into account the surface area of the container, as well as its water contents, and doing the appropriate scaling. As volume and hence weight grows with the cube of the linear dimensions (assuming rough similarities), while surface area, proportional to heat loss grows like the square, the actual energy needs grows like $M^{2/3}$ while its fraction of its body weight as $M^{-1/3}$. A mouse at one promille of the weight of a human, needs to eat one percent of what the human does, and that fraction of its body weight is about ten times that of the human, and approximates its own body-weight. This in fact gives a lower bound how small a mammal can be, smaller mammals, i.e. fetuses, are immersed inside the female and thus its metabolic process integrated into a bigger one. The eggs of other hot-blooded animals, such as the birds, need to be constantly incubated. An elephant on the other hand having a weight about a hundred times of a human, need to eat 25 times as much, but the mass of the food being only a fifth of the fraction of its weight compared to humans, neglecting the fact that its vegetarian food is not as nutritious. Cold-blooded animals do not have the same energy needs making their lives simpler in this regard.

1 kcal $= 4.18 \times 10^3 J$. Thus during the day a human produces about $10^7 J$ from its food intake, as there are $24 \times 60 \times 60 = 86400$ seconds during a day this corresponds to a metabolic rate of $120W$ which is of course puny compared to the Sun. It could be instructive to compare the masses between a human and the Sun. For the first we we have an order of magnitude of 10^2 kg while for the Sun we have computed it to 1.5×10^{30} kg above. This makes for a quotient of 1.5×10^{28} to be compared to 3×10^{24} when it comes to energy.

This means that the human body produces 5000 times as much energy, kilo by kilo. Thus impressive as the energy production of the Sun is, its metabolic rate is significantly lower than that of a human. The real mystery is how it is able to maintain the rate for such a long time. The burning of coal gives $2.4 \times 10^7 J$ (of heat) per kg. Thus the output of the Sun per second is equivalent to burning 1.2×10^{19} kilos of coal. If the Sun would consist entirely of coal to be burnt (assuming a steady supply of oxygen) it would, given its mass, be able to shine at that rate for 3×10^{11} seconds (Dividing the mass of the sun, with the corresponding mass of coal). Which means 10^4 years, a very short interval compared to geological times. Another possible way of producing energy would be consider gravitational contraction. Assuming that the Sun is homogenous, we have computed its potential energy as $7 \times 10^{39} J$ from this we conclude that if we contract the sun with h meters $10^{31}(2h)J$ will be released. At the present output a contraction of $2 \times 10^{-5} m/s$ should be enough, or 6×10^2 meter a year. But this could not go on for more than a million years, which is also too short for geological times. Calculations such as those performed by the authority of a Lord Kelvin at the end of the 19th century, gave a rather strict upper bound on the age of the Sun, and hence the duration of life on Earth, too short really to allow Darwinian evolution. Darwin was considering to abandon his theory for that very reason. At the end of the century radioactivity was discovered, and the rest is history as the saying goes. Radioactivity and its various consequences have dominated the physics of the 20th century. In particular we now have an understanding on how the metabolism of stars work and why they can produce such tremendous amounts of energy, in particular putting to shame the notorious statement by Comte at the beginning of the 19th century, that mankind would never know the chemical composition of stars, and in particular what makes them shine. The short answer is of course encoded in $E = mc^2$, that matter can be converted into energy at a tremendous rate of conversion.

If we assume that all the energy of the sun is converted to (yellow) visible light (which is of course a drastic simplification) we can, by finding out the energy of a single photon, easily compute how many photons are produced a second. The energy E of a photon is given by $E = \frac{hc}{\lambda}$ where $c = 3 \times 10^8 m/s$ is the velocity of light, $h = 6.6 \times 10^{-34} Js$ the Planck constant and λ the wavelength. Thus the shorter the wave-length the higher the energy. For yellow light we are talking about $5 \times 10^{-11} m$ thus a single photon will have energy $4 \times 10^{-15} J$, doing the requisite division we end up with 10^{41} photons a second. Now only a tiny fraction 10^{-27} of those reach the Human eye, yet in absolute numbers a staggering 10^{14}. This corresponds to magnitude -27. If we add 35 magnitudes it reduces to one a second, which corresponds to a star of magnitude 8. Thus at magnitude 6 the limit of human perception, we are only talking about half a dozen photons a second. This testifies to the sensitivity of human vision. We can also use this to relate magnitudes to lamps.

Many bodies, such as the Moon and the planets do not shine on their own, only by reflected light. How to compute their borrowed luminosities?

Imagine the Dyson sphere again, and assume that it is completely reflective like a perfect mirror. All the light of the Sun will be reflected back. Remove it and leave only the Earth, and there will just be a billionth of the light left. Now this correspond to 4.5×5 magnitudes, thus -4.5. Now two things have to be considered. First this underestimates it, as wee see only half of the light that the sun emits, thus we should compensate by a factor of two. Secondly, the Earth is not a perfect mirror, it does not reflect all the light, only a percentage of it, given by its Albedo. In the case of the Earth this is given by an average of 0.3, meaning that 30% of the sunlight is reflected. Thus we should multiply by a factor of 0.6 which corresponds to $-1/2$ magnitudes, hence the Earth seen from the Sun would exhibit -4. Seen from Venus at a distance of merely 0.3 A.U., i.e. when in opposition, we should multiply by a factor of $1/0.3^2 \sim 10$ corresponding to about -2.5 magnitudes, so the Earth would appear as -6.5 at the maximal. What about the Moon? First its radius is only a fourth of that of the Earth, so we need the factor $1/4^2$, furthermore the Moon is much darker than the Earth, having a very low albedo, in fact more like dark asphalt or about 0.05, thus only a sixth of that of the Earth, thus we should multiply by $1/6$. In total about $1/100$ corresponds to 5 magnitudes, hence the Moon would appear with magnitude 1 and from Venus -1.5 as bright as Sirius. At that distance the separation of the Moon from the Earth would amount to half a degree for an observer on Venus, which is of course the separation of a lunar disc, so the twin planet system of the Earth-Moon would be quite spectacular seen from Venus.

`Example` **Jupiter and its satellites** Let us compute 1) The magnitude of Jupiter at its maximum seen from each of its major satellites and 2) the magnitude of each of those satellites as seen from Jupiter and present it at a table. The first step is to compare their magnitudes as seen from the Sun with those of the Earth, and then to use the inverse square to compute them from a much closer distance. For the first step we need to know the radii r in terms of the Earth radius, as well as their albedos a in terms of that of the Earth, and finally we need to know the distances in terms of Astronomical units. Then we get the modification in magnitudes by taking $2.5 \times^{10} \log(r^2 \times a)/d^4)$. This information is given in the table below

Name	Jupiter	Io	Europe	Ganymede	Callisto
Distance A.U.	5.2	2.8×10^{-3}	4.5×10^{-3}	7.1×10^{-3}	12.6×10^{-3}
Size E.r	11	0.28	0.24	0.41	0.38
Albedo	0.52	0.63	0.67	0.43	0.22

We give here a list of the magnitudes of Jupiter and its main satellites as seen from the Sun.

Jupiter	Io	Europe	Ganymede	Callisto
-2.6	5.1	5.4	4.7	5.6

and here we give the promised table

Io	Europe	Ganymede	Callisto
-19.0	-18.0	-16.9	-15.7
-11.2	-10.0	-9.6	-7.5