Theoretical Physics
Chalmers | Göteborg University

Written Exam in Quantum Mechanics, FKA081

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Examiner: Henrik Johannesson, phone: 3185.

No references are allowed.

Please put your name on *each solution sheet*, and don't forget to also put your e-mail address on the cover sheet.

Structure your solutions carefully. State precisely which assumptions, theoretical results, approximations, etc. you use. The logic of your arguments must be transparent, and you should strive for optimal readability! All problems are equally weighted (6p/problem.)

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1. Simple harmonic oscillator

(a) Consider a one-dimensional simple harmonic oscillator. Calculate the expectation value of the operator X^4 in the state with energy $(n+1/2)\hbar\omega$.
Hint: $X = \sqrt{\hbar/2m\omega}(a + a^\dagger)$ where a $(a^\dagger)$ is a destruction (creation) operator.

(b) Use the result in (a) to calculate the energy spectrum for a particle whose Hamiltonian is

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2 - \lambda X^4$$

with λ an arbitrary constant.

2. Density matrices, pure and mixed ensembles, and all that...

(a) Consider a pure ensemble consisting of identically prepared states $|\psi\rangle = a_1 |\phi_1\rangle + a_2 |\phi_2\rangle + ... + a_k |\phi_k\rangle$ where $|\phi_i\rangle$ are the eigenkets to some observable $(i = 1, 2, ..., k)$. Is it possible to reinterpret this ensemble as a mixed ensemble of states $|\phi_1\rangle, |\phi_2\rangle, ... |\phi_k\rangle$ with relative probabilities $a_1, a_2, ..., a_k$?

(b) A beam of light is propagating along the z-direction. Using the basis

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ for x-polarization, and } \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ for y-polarization,}$$

construct the density matrices for 45° and 135° polarizations. What is the density matrix for a 40% - 60% mixture of 45° and 135° polarized light?

(c) Consider an ensemble of "three-level" atoms at thermal equilibrium, with states $|g\rangle$ (groundstate) $|e_1\rangle$ (first excited state) and $|e_2\rangle$ (second excited state) of energies 0, $\hbar\omega_0$ and $2\hbar\omega_0$ respectively. Construct the density matrix. Use your result to calculate the mean energy of an atom at temperatures $T = 0, 300$ and 10^4 K.

3. Spin

(a) Find the eigenvalues and eigenstates of the spin operator $\boldsymbol{S}$ of an electron in the direction of a unit vector $\boldsymbol{n}$, where $\boldsymbol{n}$ is arbitrary.

(b) Find the probability of measuring $S_z = -\hbar/2$.

(c) Assuming that the eigenstates of the spin calculated in (a) correspond to $t = 0$, find these states at a later time t in the case where the electron moves in a uniform magnetic field directed along the $z-$axis.

4. Transition probabilities

A one-dimensional charged-particle harmonic oscillator with fundamental frequency ω and mass m is in a time-dependent homogeneous electric field given by

$$E(t) = \frac{A}{\sqrt{\pi}\tau} \mathrm{e}^{-(t/\tau)^2} ,$$

where A and τ are constants, and where the field is directed along the line of displacement of the oscillator. If, at $t = -\infty$, the oscillator is in its groundstate, find, to a first approximation, the probability that it will be in its first excited state at $t = \infty$. For what frequencies ω is your perturbative solution guaranteed to be valid?
Hint: $\int_{-\infty}^{\infty} \mathrm{e}^{i\beta x - \alpha x^2} dx = \sqrt{\pi/\alpha}\, \mathrm{e}^{-(\beta^2/4\alpha)}$.

5. Understanding quantum mechanics

Imagine that a few years from now you're lecturing a quantum mechanics course (Sakurai style!), and it's getting close to the written examination. You have to come up with a good *theory question*, something that tests your students understanding of some important conceptual issue in quantum physics. Draft a good question, and provide a suggested answer!