Theoretical Physics
Chalmers & Göteborg University

Written Exam in Quantum Mechanics, FKA081/FYN190

Wednesday January 15 2003, 14.15 - 19.15

Examiner: Henrik Johannesson, phone: 3185.

No reference/calculator is allowed.

Please put your name on *each solution sheet*, and don't forget to include also your e-mail address on the cover sheet.

Please structure your solutions carefully. State precisely which assumptions, theoretical results, approximations, etc. you use. The logic of your arguments must be transparent, and you should strive for optimal readability! All problems are equally weighted (6p/problem).

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1. Measurements: observables and uncertainties

Consider a system whose Hamiltonian H and an operator A are given by the matrices

$$H = \epsilon_0 \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad A = a \begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where ϵ_0 and a are constants.

a) If we measure the energy, what values can we obtain?
b) Suppose that when we measure the energy, we obtain a value of $-\epsilon_0$. Immediately afterward, we measure A. What values can we obtain for A and what are the probabilities corresponding to each value?
c) Calculate the uncertainty ΔA.

2. Electron in a magnetic field

An electron is subject to a uniform, time-independent magnetic field of strength B in the positive z-direction. At $t = 0$ the electron is known to be in an eigenstate of $\boldsymbol{S} \cdot \boldsymbol{n}$ with eigenvalue $\hbar/2$, where $\boldsymbol{n}$ is a unit vector in the xz-plane that makes an angle β with the z-axis.

a) Obtain the probability for finding the electron in the $s_x = \hbar/2$ state as a function of time.
b) Find the expectation value of S_x as a function of time.
c) To check your results, show that your answers in a) and b) make good sense in the extreme cases (i) $\beta \to 0$ and (ii) $\beta \to \pi/2$.

3. Scalar operators and angular momentum states

An operator $\mathcal{O}$ is said to be *scalar* if it commutes with the three components of the angular momentum operator J_i, $i = x, y, z$. Consider the $(2j+1) \times (2j'+1)$ matrix elements

$$\langle \alpha, j, m \mid \mathcal{O} \mid \beta, j', m' \rangle \,, \qquad m = -j, ...j, \qquad m' = -j', ...j'$$

where α and β represent the set of quantum numbers required to completely specify the states in addition to the angular momentum quantum numbers j, m and j', m'.

a) Using $[J_z, \mathcal{O}] = 0$, show that $\langle \alpha, j, m \mid \mathcal{O} \mid \beta, j', m' \rangle$ can be nonzero only if $m = m'$.
b) Using $[J_+, \mathcal{O}] = 0$, show that

$$\sqrt{j(j+1) - m(m+1)}\mathcal{O}_m = \sqrt{j'(j'+1) - m(m+1)}\mathcal{O}_{m+1} \,.$$

Show also that

$$\sqrt{j(j+1) - m(m+1)}\mathcal{O}_{m+1} = \sqrt{j'(j'+1) - m(m+1)}\mathcal{O}_m \,.$$

Here $\mathcal{O}_m \equiv \langle \alpha, j, m \mid \mathcal{O} \mid \beta, j', m \rangle$, and J_+ is the usual "step-up" ladder operator.
c) Deduce from b) that $\mathcal{O}_m$ vanishes if $j \neq j'$, and that all these matrix elements are equal if $j = j'$.
d) Given your results in a) - c), formulate a *selection rule* for matrix elements of scalar operators in an angular momentum basis.

4. Time evolution of a harmonic oscillator

Consider a one-dimensional harmonic oscillator with the Hamiltonian $H = P^2/2m + m\omega^2 X^2/2$ and its first two normalized eigenfunctions $\phi_0(x)$ and $\phi_1(x)$. A system has at time $t = 0$ the wave function

$$\psi(x, t = 0) = \cos\theta \, \phi_0(x) + \sin\theta \, \phi_1(x), \qquad \text{where } 0 \leq \theta < \pi.$$

a) What is the wave function $\psi(x, t)$ at time t?
b) Calculate the expectation values $\langle E \rangle$, $\langle E^2 \rangle$ and $\Delta E^2 = \langle E^2 \rangle - \langle E \rangle^2$. Explain why they are all time-independent!
c) Calculate the time evolution of $\langle x \rangle$ and $\langle x^2 \rangle$.

5. ABC of quantum mechanics

Imagine that a few years from now you are asked to give a popular science talk at your company's annual "come together" event. Topic: "What is quantum physics?". Your colleagues are all pretty well educated - some of them with an advanced degree in engineering - but none of them knows much about quantum physics. Your task clearly needs some preparation!

Sketch a synopsis of your talk, and develop one of your themes in some detail.