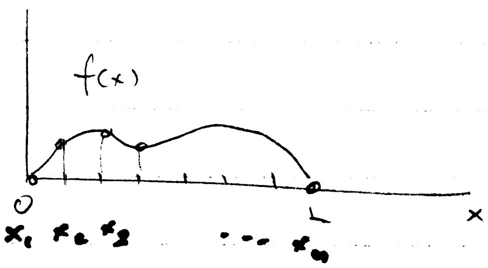
QUANTUM MECHANICS FKA 081, FALL 2003

SELECTED LECTURE NOTES Henrik Johannesson

LINEAR VECTOR SPACES: GENERALIZATION TO INFINITE DIMENSIONS

<u>EX</u>. displacement of a string



$f_n(x)$: discrete approx. of $f(x)$

$$|f_n\rangle = \begin{pmatrix} f_n(x_1) \\ f_n(x_2) \\ \vdots \\ f_n(x_n) \end{pmatrix} = f_n(x_1)|x_1\rangle + f_n(x_2)|x_2\rangle + \ldots$$

To each point x_i : $\overset{\text{BASIS KETS}}{|x_i\rangle} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th}$ position

$\langle x_i | x_j \rangle = \delta_{ij}$ ORTHOGONALITY

$\sum_{i=1}^{n} |x_i\rangle\langle x_i| = \mathbb{1}$ COMPLETENESS ("resolution of the identity")

$\Downarrow$

$$|f_n\rangle = \sum_{i=1}^{n} |x_i\rangle \underbrace{\langle x_i | f_n\rangle}_{f_n(x_i)} = \sum_{i=1}^{n} f_n(x_i)\,|x_i\rangle$$

INNER PRODUCT

$$\langle f_n | g_n \rangle = \sum_{i=1}^{n} f_n^*(x_i)\, g_n(x_i)$$

$\langle f_n | g_n \rangle = 0 \Rightarrow |f_n\rangle, |g_n\rangle$ ORTHOGONAL

$n \to \infty$ "maximal specification"

$$f_n \to f \,, \quad g_n \to g \,, \quad \sum_{i=1}^{n} \ldots \longrightarrow \int_0^L \ldots \, dx$$

INNER PRODUCT $\langle f | g \rangle = \int_0^L f^*(x)\, g(x)\, dx$

BASIS KETS ?

EACH POINT x GETS A BASIS VECTOR $|x\rangle$!

Note difference from HM.1 where there were only two basis kets, one for each mass. Why not try something similar here and introduce a single ket defining the direction along which we measure the displacement of the string? Works OK in classical mechanics but not in quantum mechanics (as we shall see)!

$\langle x | x' \rangle = \delta(x - x')$ ORTHOGONALITY

$\int dx \, |x\rangle\langle x| = \mathbb{1}$ COMPLETENESS

$X|x\rangle = x|x\rangle$, X = "POSITION OPERATOR"

Dirac delta function

$\delta(x - x') = 0$ if $x \neq x'$

$$\int_a^b \delta(x - x')\, dx' = 1 \,, \quad a < x < b$$

$$\underbrace{\frac{d}{dx}}_{\text{acts on the } \delta\text{-function}} \delta(x - x') = \underbrace{\delta'(x - x') = \delta(x - x') \frac{d}{dx'}}_{\text{WHEN BOTH SIDES APPEAR IN AN INTEGRAL OVER } x'}$$

acts on any function to the right

$$\langle x | f \rangle = f(x)$$

$$|f\rangle = \int_0^L dx\, |x\rangle\langle x|f\rangle = \int_0^L f(x)\, |x\rangle\, dx$$

OPERATORS

since the kets are in 1-1 correspondence to the functions

$$\Omega |f\rangle = |\tilde{f}\rangle \quad \Rightarrow \quad \Omega : f(x) \to \tilde{f}(x)$$

IMPORTANT EXAMPLE: $\Omega = D$ DIFFERENTIAL OPERATOR

$$D|f\rangle = \left|\frac{df}{dx}\right\rangle$$

↑ ket corresponding to the function $\frac{df}{dx}$ in the $|x\rangle$-basis

Matrix elements of D? Let's find out!

COMPLETENESS $\int dx' |x'\rangle\langle x'| = \mathbb{1}$

$$\langle x|D|f\rangle = \left\langle x \left| \frac{df}{dx} \right.\right\rangle = \frac{df(x)}{dx} = \int_0^L \langle x|D|x'\rangle\langle x'|f\rangle\, dx'$$

$$\Downarrow$$

$$\langle x|D|x'\rangle = D_{xx'} = \delta'(x-x') = \delta(x-x')\frac{d}{dx'}$$

Nota bene: usually one doesn't think of D as a matrix since the delta function makes the integration trivial.

DEFINE $K = -iD$

$$K^*_{x'x} = \left(-i\delta'(x'-x)\right)^* = i\delta'(x'-x) = -i\delta'(x-x') = K_{xx'}$$

K thus looks Hermitian, but we actually have to test for one more thing!

Consider two kets $|f\rangle$ and $|g\rangle$. If K is Hermitian we must also have

$$\langle g|K|f\rangle = \langle g|Kf\rangle = \langle Kf|g\rangle^* = \langle f|K^+|g\rangle^* = \langle f|K|g\rangle^*$$

Let's check it out by inserting $\underbrace{\int dx\, |x\rangle\langle x|}_{=1\!\!1}$ and $\underbrace{\int dx'\,|x'\rangle\langle x'|}_{=1\!\!1}$ on both sides above:

$$\int_0^L\int_0^L dx\,dx'\,\langle g|x\rangle\langle x|K|x'\rangle\langle x'|f\rangle \stackrel{?}{=} \left(\int_0^L\int_0^L dx\,dx'\,\langle f|x\rangle\langle x|K|x'\rangle\langle x'|g\rangle\right)^*$$

By partial integration it's easy to see that this is true, i.e. K is HERMITIAN only if the space consists of functions satisfying

$$\boxed{\left[-ig^*(x)\,f(x)\right|_0^L = 0}$$

VANISHING BOUNDARY TERM

Sense morale: $K_{xx'} = K^*_{x'x}$ is not a sufficient condition for K to be Hermitian (in contrast to finite-dimensional vector spaces)

THE EIGENVALUE PROBLEM FOR K

$$K|k\rangle = k|k\rangle$$

$$\langle x|K|k\rangle = k\underbrace{\langle x|k\rangle}_{\equiv \psi_k(x)} = k\psi_k(x)$$

MATRIX ELEMENTS OF K IN THE $|k\rangle$ BASIS : $\langle k|K|k'\rangle = k'\langle k|k'\rangle = k'\delta(k-k')$

$\Downarrow$ insert a complete set

$$\int_0^L \underbrace{\langle x|K|x'\rangle}_{-i\delta(x-x')\frac{d}{dx'}} \underbrace{\langle x'|k\rangle}_{\psi_k(x')} dx' = k\psi_k(x)$$

$$\Vert$$

$$-i\int_0^L \delta(x-x')\frac{d}{dx'}\psi_k(x')\,dx' = -i\frac{d\psi_k(x)}{dx} = k\psi_k(x)$$

$\Downarrow$ SOLUTION

$$\psi_k(x) = A e^{ikx}$$

k: any real number

A: free parameter

NOTE IF k IMAGINARY $\Rightarrow$ $\Rightarrow K$ NON-HERMITIAN (SURFACE TERMS BLOW UP)

From now on, let's look at functions defined on the full real line: $-\infty < x < \infty$

$$\langle k|k'\rangle = \int_{-\infty}^{\infty} \langle k|x\rangle\langle x|k'\rangle\, dx' = |A|^2\int_{-\infty}^{\infty} e^{i(k-k')x}\,dx$$

Fourier's integral $\quad = |A|^2 \cdot 2\pi\,\delta(k-k')$

Choose $A = 1/\sqrt{2\pi} \quad \Rightarrow \quad \langle k|k'\rangle = \delta(k-k')$

VECTORS, LIKE $|k\rangle$, WHICH CAN ONLY BE NORMALIZED TO A DELTA FUNCTION ARE CALLED "IMPROPER VECTORS".
"PROPER VECTORS" <u>CAN</u> BE NORMALIZED TO <u>UNITY</u> : $\langle\phi|\phi\rangle = 1$

(PHYSICAL) HILBERT SPACE

$$= \{\text{PROPER } \underline{\text{OR}} \text{ IMPROPER VECTORS}\}_{d=\infty}$$

EXPANSIONS

INSERT $\int |x\rangle\langle x|\, dx = \mathbb{1}$

$$f(k) = \langle k|f\rangle = \int_{-\infty}^{\infty} \underbrace{\langle k|x\rangle}_{\frac{1}{\sqrt{2\pi}}e^{-ikx}} \underbrace{\langle x|f\rangle}_{f(x)}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx$$

INSERT $\int |k\rangle\langle k|\, dk$

$$f(x) = \langle x|f\rangle = \int_{-\infty}^{\infty} \underbrace{\langle x|k\rangle}_{\frac{1}{\sqrt{2\pi}}e^{ikx}} \underbrace{\langle k|f\rangle}_{f(k)}\, dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx} f(k)\, dk$$

Thus, the well-known Fourier transform in the Dirac formalism is simply the passage from one basis $\{|x\rangle\}$ to another $\{|k\rangle\}$! It's as simple as that!

BACK TO THE POSITION OPERATOR X

$$X|x\rangle = x|x\rangle$$

$$\langle x'|X|x\rangle = x\underbrace{\langle x'|x\rangle}_{\delta(x'-x)} = x\,\delta(x'-x)$$

$$\boxed{X|f\rangle = |\tilde{f}\rangle} \Rightarrow X|f(x)\rangle = x|f(x)\rangle = |x f(x)\rangle$$

ket corresponding to the function $f(x)$ in the $|x\rangle$-basis

$$\underbrace{\langle x|X|f\rangle}_{|\tilde{f}\rangle} = \int_{-\infty}^{\infty} \underbrace{\langle x|X|x'\rangle}_{x'\delta(x-x')} \underbrace{\langle x'|f\rangle}_{f(x')}\, dx' = x f(x)$$

$$= \langle x|\tilde{f}\rangle = \tilde{f}(x) \qquad \Rightarrow \tilde{f}(x) = x f(x)$$

What about the matrix elements of X in the $|k\rangle$-basis?

Let's find out!

$$\langle k | X | k' \rangle = \int_{-\infty}^{\infty} \underbrace{\langle k | X | x \rangle}_{\frac{x}{\sqrt{2\pi}} e^{-ikx}} \underbrace{\langle x | k' \rangle}_{\frac{1}{\sqrt{2\pi}} e^{ikx'}} dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x \, e^{i(k'-k)x} \, dx = i \frac{d}{dk} \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(k'-k)x} \, dx \right)$$

Fourier's integral

$$= \underbrace{-i \delta'(k-k')}_{-i\delta(k-k') \frac{d}{dk'}} , \text{ using that } \delta(k'-k) = \delta(k-k')$$

Thus, if $|g(k)\rangle$ is the ket corresponding to the function $g(k)$ in the $|k\rangle$-basis, it follows that

$$X \, |g(k)\rangle = \left| -i \frac{dg(k)}{dk} \right\rangle \qquad \star\star$$

using $K = -iD$

In summary, from $\star$ (p. 3) and $\star\star$:

IN THE EIGENBASIS OF X, X ACTS AS x AND K AS $-i\frac{d}{dx}$,
WHILE IN THE EIGENBASIS OF K, K ACTS k AND X AS $-i\frac{d}{dk}$

X AND K ARE CONJUGATE TO EACH OTHER

The conjugate operators X and K are INCOMPATIBLE, i.e. $[X, K] \neq 0$

$$X|f\rangle = X\int |x\rangle\langle x|f\rangle\, dx = \int x\,|x\rangle \underbrace{\langle x|f\rangle}_{f(x)}$$

$$\xrightarrow[\text{IN THE X-BASIS}]{\text{EFFECT ON THE FUNCTION}} \quad f(x) \to x\, f(x)$$

$$K|f\rangle \xrightarrow[\text{THE FUNCTION IN THE X-BASIS}]{\text{EFFECT ON}} \quad f(x) \longrightarrow -i\,\frac{df(x)}{dx}$$

$$XK|f\rangle \Longrightarrow \quad f(x) \longrightarrow -ix\,\frac{df(x)}{dx}$$

$$KX|f\rangle \Longrightarrow \quad f(x) \longrightarrow -i\,\frac{d}{dx}\left(x\, f(x)\right)$$

$$\Downarrow$$

$$[X, K]\,|f\rangle \Longrightarrow \quad f(x) \to -ix\,\frac{df}{dx} + ix\,\frac{df}{dx} + if = if$$

$$\Downarrow$$

$$[X, K] = i\mathbb{1}$$

As disussed in Sakurai, sec. 1.7, MOMENTUM in quantum mechanics is represented by the operator $P = \hbar K$

$$\Rightarrow \quad [X, P] = i\hbar\,\mathbb{1}$$

"CANONICAL COMMUTATION RELATION"