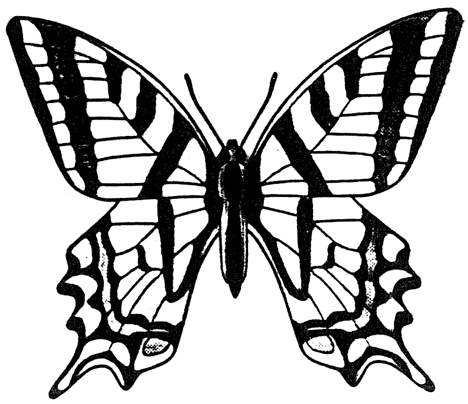
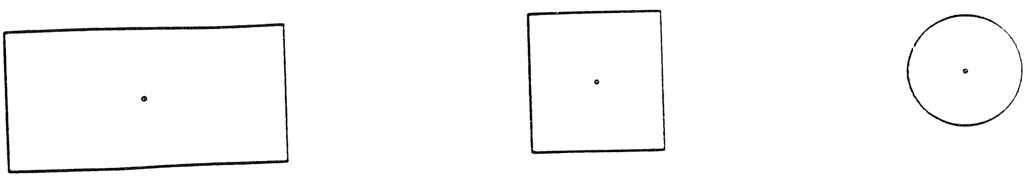
Symmetries in quantum mechanics

What is a symmetry?





A **symmetry** (or symmetry transformation) of an object is a transformation which maps the object into itself. If an object admits a certain symmetry it is said to be **invariant** under the transformation.

The set of all symmetry transformations of an object represents a group: the **symmetry group** of the object.

Symmetries of the laws of Nature

A **symmetry transformation** of a physical law is a change of the variables and/or the space-time coordinates (in terms of which it is formulated) such that the equations describing the law have the same form in terms of the new variables and coordinates as they had in terms of the old one: the equations are said to be **covariant** w.r.t. the symmetry transformation.

Ex. the wave equation

$$\Box \varphi(\vec{r}, t) = 0$$

$$\Box = -\nabla^2 + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

Lorentz boost

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

$$\gamma = \left(1 - (v/c)^2\right)^{-1/2}$$

$\Downarrow$ Lorentz

$$\Box' = \Box, \quad \varphi'(\vec{r}', t') = \varphi(\vec{r}, t)$$

$\Rightarrow$ THE WAVE EQUATION IS COVARIANT UNDER LORENTZ BOOSTS

Space-time vs. internal symmetries

2 $\rightarrow$

SYMMETRIES

SPACE-TIME SYMMETRIES

INVARIANCE UNDER

TRANSLATIONS
ROTATIONS
⋮
LORENTZ BOOSTS
⋮
SCALE CHANGES
⋮

INTERNAL SYMMETRIES

INVARIANCE UNDER

GLOBAL PHASE TRANSFORMATIONS

GAUGE TRANSFORMATIONS
⋮

Space-time symmetries

translation in time : $t \to t + a_0$

translations in space : $\vec{r} \to \vec{r} + \vec{a}$

rotations in space : $\vec{r} \to \mathbb{R}\vec{r}$

Induced transformations in Hilbert space

Galilei transformations : $\vec{r} \to \vec{r} - \vec{v}t$ (non-relativistic)

Lorentz boosts : previous page (relativistic)

Lorentz group $\mathcal{L}$ = Lorentz boosts
+ rotations in space
+ parity+ time reversal

$\mathcal{P} = \mathcal{L} \otimes \mathcal{T}_4$ POINCARE' GROUP

EXTENSIONS

CONFORMAL GROUP

$\mathcal{P} \otimes \{\text{SCALE CHANGES}\}$

$x \to \lambda x$

SUPER POINCARE'

$\mathcal{P} \otimes \{\text{SUSY}\}$

BOSONS $\leftrightarrow$ FERMIONS

. . . . .

IS THIS RIGHT ?

DO WE HAVE SCALE INVARIANCE IN A GRAVITATIONAL ENVIRONMENT ?

ARE SYMMETRIES ALWAYS MANIFEST ?

Internal symmetries

Global phase symmetries

$H\psi = i\hbar \frac{\partial \psi}{\partial t}$ INVARIANT UNDER $\psi \to e^{i\alpha}\psi$, $\alpha \in \mathbb{R}$

Gauge symmetries (local phase symmetries)

$H\psi = i\hbar \frac{\partial \psi}{\partial t}$ INVARIANT UNDER $\psi \to e^{i\alpha(\vec{r},t)}\psi$

local phase

PROVIDED ONE PROPERLY
INTRODUCES A GAUGE FIELD

$$A_\mu$$

IN H !

MAXWELL THEORY
("U(1) GAUGE SYMMETRY")

: $A_\mu = (\phi, -\vec{A})$

$$\vec{E} = -\nabla\phi - \partial_t \vec{A}$$

$$\vec{B} = \nabla \times \vec{A}$$

WHY ARE SYMMETRIES USEFUL IN PHYSICS ?

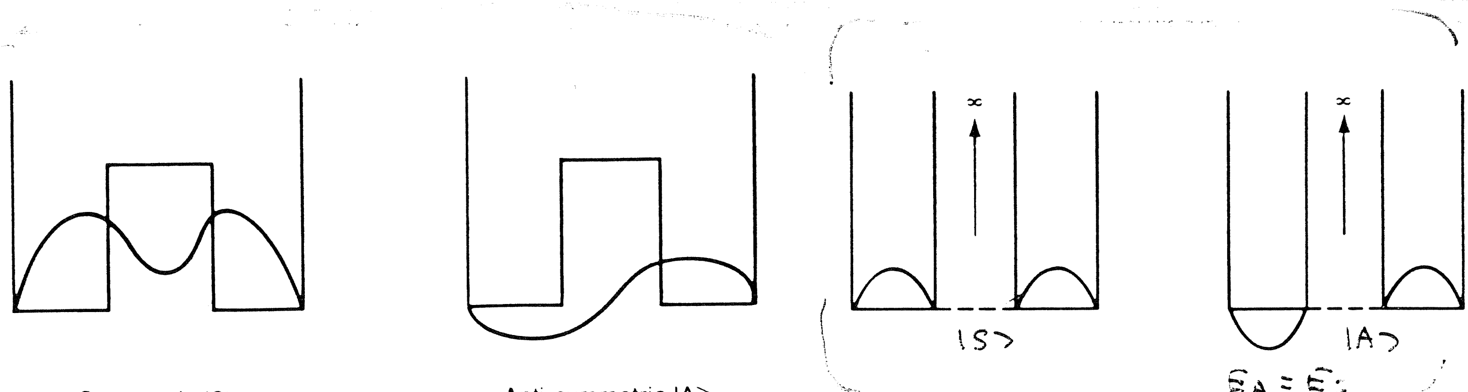
NOETHER'S THEOREM

Covariance of the equations of motion with respect to a continuous transformation with n parameters implies the existence of n conserved quantities ('conserved charges' or 'integrals of motion'), i.e. it implies conservation laws.

Property	Invariance of equations	Conserved quantity
homogeneity of time	time translation invariance	energy
homogeneity of space	translational invariance	momentum
isotropy of space	rotational invariance	angular momentum

DISCRETE SYMMETRIES ARE ALSO USEFUL ...

SYMMETRIC DOUBLE-WELL POTENTIAL



Symmetric $|S\rangle$ Anti-symmetric $|A\rangle$

$E_A > E_S$

$$|R\rangle = \frac{1}{\sqrt{2}}\left(|S\rangle + |A\rangle\right) \longrightarrow \frac{1}{\sqrt{2}}\left(e^{-iE_S t/\hbar}|S\rangle + e^{-iE_A t/\hbar}|A\rangle\right)$$

$$= \frac{1}{\sqrt{2}} e^{-iE_S t/\hbar}\left(|S\rangle + e^{-i(E_A - E_S)t/\hbar}|A\rangle\right)$$

$$|L\rangle = \frac{1}{\sqrt{2}}\left(|S\rangle - |A\rangle\right) \longrightarrow \frac{1}{\sqrt{2}} e^{-iE_S t/\hbar}\left(|S\rangle - e^{-i(E_A - E_S)t/\hbar}|A\rangle\right)$$

$V(x) \rightarrow \infty \Rightarrow$ NO TUNNELING

$|S\rangle$ AND $|A\rangle$ DEGENERATE $\Rightarrow$ $|R\rangle$ AND $|L\rangle$ ALSO ENERGY EIGENSTATES

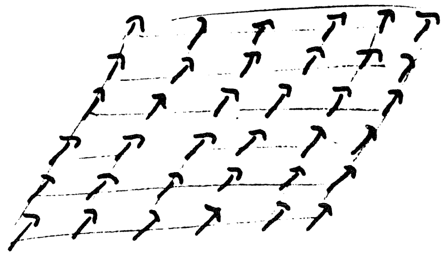
THE GROUND STATE MAY BREAK THE PARITY SYMMETRY ALTHOUGH THE HAMILTONIAN IS INVARIANT UNDER PARITY

EXAMPLE OF **SPONTANEOUSLY BROKEN SYMMETRY**

... THE GROUNDSTATE HAS LESS SYMMETRY THAN THE HAMILTONIAN

$\rightarrow$

Paradigm of a **broken symmetry:** **FERROMAGNET**

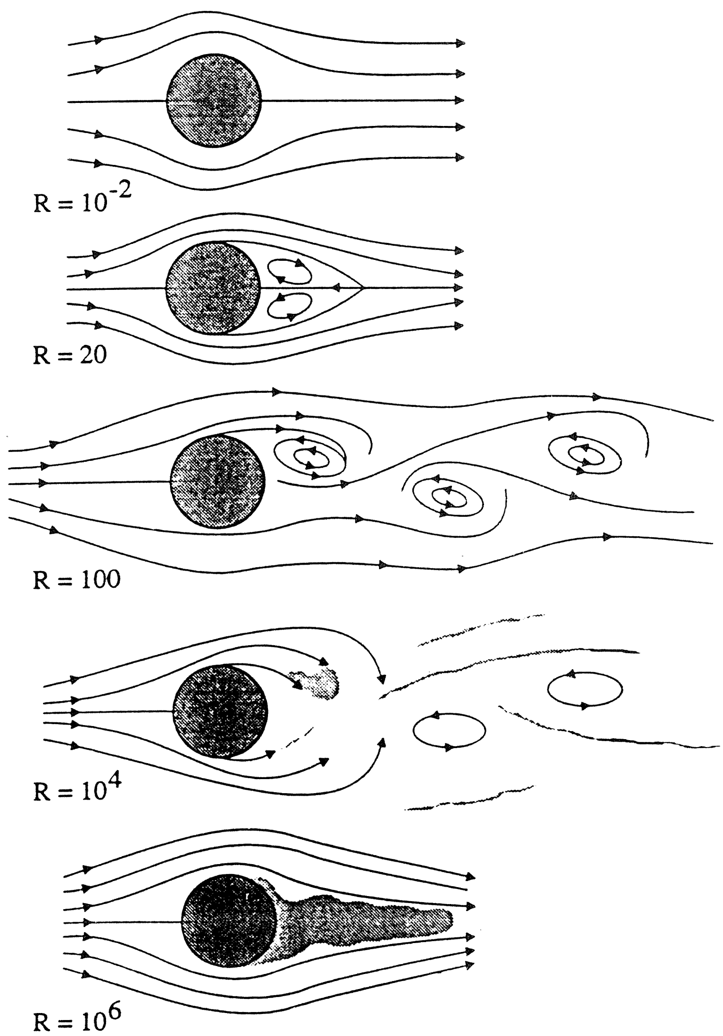


Hamiltonian $H = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$

Ground state $|\psi\rangle = |\uparrow\uparrow\uparrow \dots \uparrow\rangle$

breaks SU(2) ----> low-lying excitations are ***Goldstone*** *bosons*

Another example (classical physics!) is **TURBULENCE,** where different symmetries are broken spontaneously at different scales

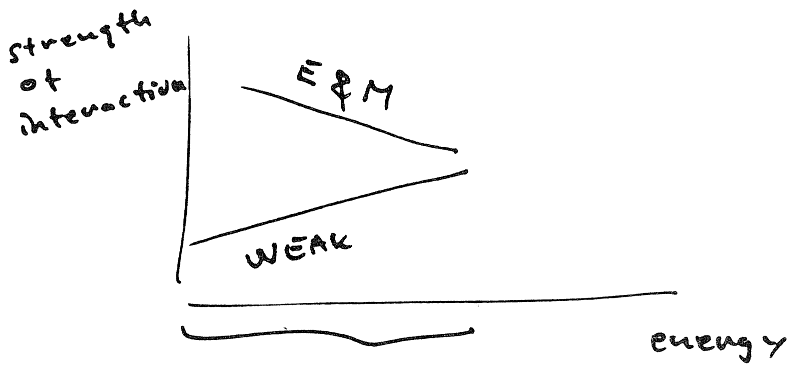


Yet another example of a broken symmetry...

Electroweak theory (Glashow, Weinberg, Salam, 1960s)

U(1) x SU(2) gauge symmetry spontaneously broken

$W^{\pm}$, Z^{0} acquire masses ("Anderson-Higgs mechanism", like a photon in a superconductor!)



$$\langle 0 | \varphi | 0 \rangle \neq 0$$

"Higgs field"

Cf $\langle \psi | \vec{S} | \psi \rangle \neq 0$

in a ferromagnet

THE DIFFERENT PHASES OF MATTER CAN BE CLASSIFIED ACCORDING TO WHICH SYMMETRIES THEY "SPONTANEOUSLY" BREAK

EXAMPLE :

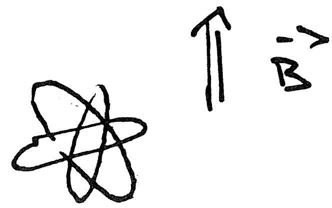
TABLE OF BROKEN-SYMMETRY PHENOMENA

Phenomenon	High Phase	Low Phase	Order Parameter (constant of motion)		Order-Parameter Dimensionality $T \to 0$	T_c	Common Transition Type (Can Always Be First Order	"Goldstone Bosons" (or "Higgs" Bosons)	Fluctuations	Collective Hydrodynamic Modes	Generalized Rigidity Phenomenon	Long-Range Forces Due to General. Rigidity	Singularities
Ferroelectricity (Pyroelectricity)	Non-Polar crystal	Polar crystal	$\vec{P}$	(no)	1	1 or 3	2nd or 1st nearly 2nd	no (optical phonons)	Soft Modes	No	Ferroelectric hysteresis	No	Domain walls (thin)
Ferromagnetism	Paramagnet	Ferromagnet	$\vec{M}$	(yes)	1, often ≈ 3	1 or 3	2nd (time reversal)	Spin waves, one branch $\omega \propto k^2$ (+ const)	Spin waves	No ?	Permanent magnetism: hysteresis	Suhl-Nakamura	Domain walls (very mobile, ≈ 3 dim)
Antiferromagnetism	Paramagnet	Antiferromagnet	$\vec{M}_{\text{sublattice}}$	(no)	1, often ≈ 3	1 or 3	2nd (or first)	Spin waves, 2 branches $\omega \propto k$ + const ("Fermions" in metal case?)	Spin waves (diffusion?)	No?	subtle effects in A.F. resonance	Suhl-Nakamura	Domain walls
Superconductivity	Normal Metal	Superconductor	$\langle \psi^*_\uparrow \psi^*_\downarrow \rangle = Fe^{i\varphi}$	(no)	2	2	2nd (Gauge; no 3rd order terms)	no (plasmons)	diffusive fluctuations of gap	Mostly not	super conductivity	No: Penetration depth	Flux lines (or normal domains in Type I)
He II	Normal liquid	Superfluid	$\langle \psi \rangle$	(no)	2	2	2nd (Gauge; no 3rd order terms)	phonons (1 branch)	diffusive fluctuations of $\langle \psi \rangle$	2nd sound, especially	superfluidity	Yes, vortex lines unscreened	vortex lines
Nematic liquid crystal	Normal liquid	oriented liquid	$\dot{d}$ (directrix)	(no)	3 {complex topology}	3	2nd	not stable at T = O (yes in principle)	Yes	usually overdamped	various peculiar properties: orientation elasticity	Yes	disclinations, points
Cholesteric, Smectic liquid crystal	Nematic	Density wave	$\rho(Q)$	(no)	> 3 {complex topology}	> 3	2nd or 1st	-	Yes		-	Yes	disclinations, points and dislocations
Crystal	liquid	solid	ρ_G, all G on recip. lattice	(no)	3(2 orient, 1 phase) at least	3	1st	yes: 3 kinds 2 transverse 1 longitudinal	phonons	2nd sound in solid, etc.	rigidity	Yes: elasticity effects	dislocations, grain boundaries, points (vacancies, interstitials)
He 3	normal liquid	anisotropic superfluid	$d_{ij} = \langle \psi \psi \rangle_{M_L, M_S}$	(no)	at least 3 {complex topology}	18	2nd	yes, several kinds	yes, complex	probably	superfluidity and orientation elasticities	Yes	Vortex lines, disgyrations, point defects
CDW's	normal electron gas (2-dim.)	Incommensurate Density wave + commensurate	ρ_G, G on triangular superlattice		2	2	1st	Yes, phasons		Yes	Yes, $NbSe_2$ sliding CDW's	?	discommensurations, dislocations

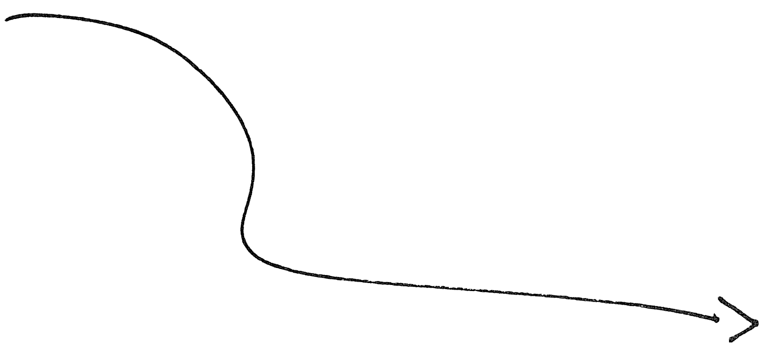
Symmetries can also be **explicitly broken**

examples:

atom in a magnetic field



parity!



CAN PARITY BE EXPLICITLY BROKEN ?

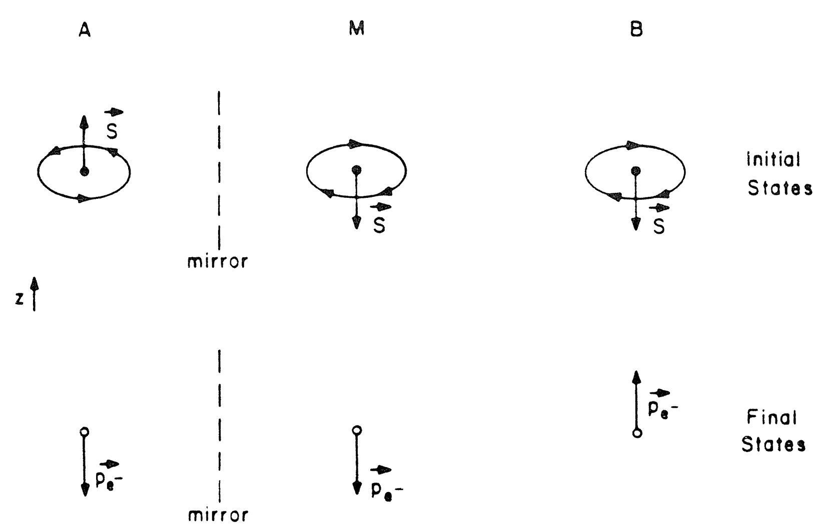
$$[\Pi, H] \neq 0 \quad ?$$

YES! IN WEAK INTERACTIONS!

$$\begin{array}{ccc} |\psi(0)\rangle & \xleftrightarrow{\text{PARITY}} & \Pi|\psi(0)\rangle \\ \Big\downarrow{\scriptstyle \text{WEAK INTERACTION}} & & \Big\downarrow \\ U(t)|\psi(0)\rangle & \xleftrightarrow{\text{NON PARITY}} & U(t)\,\Pi|\psi(0)\rangle \end{array}$$

<u>EX</u>. $\quad {}^{60}Co \longrightarrow {}^{60}Ni + e^- + \bar{\nu}$

flies out in a direction opposite to the spin of ${}^{60}Co$



Lee & Yang (1956) theory
Wu et al. (1957) experiment

STUDY CHAPTER 4
IN SAKURAI CAREFULLY !

4.1 - 4.3 EASY READ!

4.4 SEE THE COURSE HOME PAGE
FOR READING INSTRUCTIONS
AND SUPPLEMENTARY NOTES

TIME REVERSAL
SYMMETRY