Theory of angular momentum (cont'd)

What did we do yesterday?

relativistic invariance

$\Downarrow$

multicomponent wave functions possible

$$\vec{J} = \vec{L} + \vec{S} \quad \xleftarrow{} \text{SPIN}$$

$$J^2|jm\rangle = j(j+1)\hbar^2|jm\rangle, \quad j = 0, 1/2, 1, 3/2, \ldots$$

$$J_z|jm\rangle = m\hbar|jm\rangle, \quad m = -j, -j+1, \ldots, j$$

J_x, J_y, J_z are represented by block diagonal matrices in the $|jm\rangle$ basis. Each block, labelled by j, provides a *finite* (2j+1)-dimensional representation.

$$U[R(\vec{\theta})] = \exp(-i\vec{\theta}\cdot\vec{J}/\hbar) \longrightarrow \mathcal{D}^{(j)}[R(\vec{\theta})]$$

$$[J_i^{(j)}, J_j^{(j)}] = i\hbar\varepsilon_{ijk}J_k^{(j)}, \quad i, j, k = x, y \text{ or } z$$

satisfied within each j-block

IRREDUCIBLE REP OF $\vec{J}$

For a *rotationally invariant problem in 3D,* with $\vec{J} = \vec{L}$:

$$[L_z, H] = [L^2, H] = 0$$

common eigenfunctions to L_z, L^2, H :

$$\psi_{Elm}(r,\theta,\phi) = R(r)\, Y_l^m(\theta,\phi) \quad \xleftarrow{} \quad \textit{spherical harmonics}$$

Next... **Spin**

from multicomponent wave function

relativistic quantum theory

(electron spin: Dirac equation)

Let's generalize the example from the ***black board***

(infinitesimal rotation about the $\hat{z}$-axis)

$$\begin{pmatrix}\tilde{\psi}_1 \\ \vdots \\ \tilde{\psi}_n\end{pmatrix} = \left(\begin{pmatrix}1 & & 0 \\ & \ddots & \\ 0 & & 1\end{pmatrix} - \frac{i\varepsilon}{\hbar}\begin{pmatrix}-i\hbar\frac{\partial}{\partial\phi} & & 0 \\ & \ddots & \\ 0 & & -i\hbar\frac{\partial}{\partial\phi}\end{pmatrix} - \frac{i\varepsilon}{\hbar}\overset{n \times n \text{ matrix}}{S_z} \right) \times \begin{pmatrix}\psi_1 \\ \vdots \\ \psi_n\end{pmatrix}$$

$$|\tilde{\psi}\rangle = \left\{ \mathbb{1} - \frac{i\varepsilon}{\hbar}\underbrace{(L_z + S_z)}_{J_z} \right\} |\psi\rangle$$

$$\vec{J} = \vec{L} + \vec{S} \quad \sim \quad [S_i, S_j] = i\hbar\,\varepsilon_{ijk} S_k$$

$i, j, k = x, y$ or z

Electron spin

prepare the electron in a state with l = 0

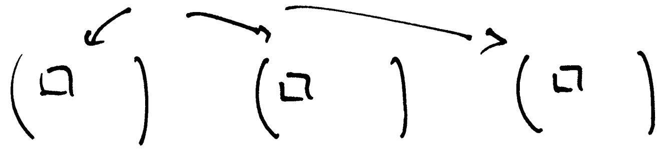
measure the angular momentum (or rather: the *magnetic moment)*

$$\Downarrow \qquad \text{STERN-GERLACH (Sakurai 1.1)}$$

$$m = s_z = \pm\hbar/2$$

D = 2 representations of S_z ?

Pick the 2x2 blocks ! $\mathcal{D}^{j=1/2}[R]$!



PAULI MATRICES

$$\Rightarrow \quad S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Electron wave function

$$\Psi_{\text{electron}}(x,y) = \begin{pmatrix} \Psi_+(x,y,z) \\ \Psi_-(x,y,z) \end{pmatrix} = \Psi_+(x,y,z)\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \Psi_-(x,y,z)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

EIGENBASIS IN $V_{s=1/2}$

$$S_z\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S_z\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S^2 = S_x^2 + S_y^2 + S_z^2 = \frac{3}{4}\hbar^2 \mathbb{1} \quad \Rightarrow \quad S^2\Psi(x,y,z) = \frac{3}{4}\hbar^2\Psi(x,y,z)$$

for <u>any</u> Ψ with spin-1/2

What about "ordinary" operators (which in a coordinate basis act on the *arguments* of the wave function), like **R, P, L, ...** ?

They are *diagonal* in the spin space $V_{s=1/2}$

Ex.

$$L_z|\psi_{\text{electron}}\rangle \xrightarrow[\text{COORDINATE BASIS}]{} \begin{pmatrix} -i\hbar\frac{\partial}{\partial\phi} & 0 \\ 0 & -i\hbar\frac{\partial}{\partial\phi}\end{pmatrix}\begin{pmatrix}\psi_+(r,\theta,\phi)\\ \psi_-(r,\theta,\phi)\end{pmatrix}$$

$$V_{\text{electron}} = \underbrace{V_{\text{orbital}}}_{\infty\text{-dimensional}} \otimes \underbrace{V_{\text{spin-}1/2}}_{2\text{-dimensional}}$$

$$|xyzs_z\rangle = |xyz\rangle \otimes |s_z, s=1/2\rangle$$

Orbital (="spatial")

Spin

If **O** and **S** separate in the Hamiltonian,

$$H = H_O + H_S$$

then

$$|\psi(0)\rangle_{\text{electron}} \xrightarrow{e^{-iHt/\hbar}} |\psi(t)\rangle_{\text{electron}} = |\psi_O(t)\rangle \otimes |\chi_S(t)\rangle$$

$\in V_{\text{orbital}}$ $\quad\in V_{\text{spin}}$

$$|\chi_s\rangle = \alpha \underbrace{|\tfrac{1}{2} +\tfrac{1}{2}\rangle}_{\begin{pmatrix}1\\0\end{pmatrix}} + \beta \underbrace{|\tfrac{1}{2} -\tfrac{1}{2}\rangle}_{\begin{pmatrix}0\\1\end{pmatrix}} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\langle \vec{S} \rangle = \langle \tfrac{1}{2} \pm \tfrac{1}{2} | \vec{S} | \tfrac{1}{2} \pm \tfrac{1}{2} \rangle = \pm \left(\tfrac{\hbar}{2}\right) \hat{z}$$ CHECK!

$$\Downarrow$$

$|\tfrac{1}{2} \pm \tfrac{1}{2}\rangle$ ARE STATES WITH THE SPIN "POINTING ALONG THE $\hat{z}$-AXIS"

MORE GENERAL

$$\hat{n} \cdot \vec{S} \, |\hat{n} \pm\rangle = \pm \left(\tfrac{\hbar}{2}\right) |\hat{n} \pm\rangle$$

WHERE $$|\hat{n} +\rangle = \begin{pmatrix} \cos(\theta/2)\, e^{-i\phi/2} \\ \sin(\theta/2)\, e^{i\phi/2} \end{pmatrix}$$

$$|\hat{n} -\rangle = \begin{pmatrix} -\sin(\theta/2)\, e^{-i\phi/2} \\ \cos(\theta/2)\, e^{i\phi/2} \end{pmatrix}$$

$$\Downarrow$$

$$\langle \vec{S} \rangle = \langle \hat{n} \pm | \vec{S} | \hat{n} \pm \rangle = \pm \left(\tfrac{\hbar}{2}\right) \hat{n}$$

↑ STATES WITH "SPIN POINTING ALONG $\hat{n}$"

Spin operators in $V_{s=1/2}$

$$\vec{S} = \frac{\hbar}{2}\vec{\sigma}\,, \qquad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Pauli matrices

Useful identities:

I. $[\sigma_i, \sigma_j]_+ \overset{i \neq j}{=} 0 \qquad (\sigma_i\sigma_j + \sigma_j\sigma_i = 0)$

II. $\sigma_x\sigma_y = \sigma_z$ and cyclic permutations

III. $\mathrm{Tr}\,\sigma_i = 0$

IV. $(\hat{n}\cdot\vec{\sigma})^2 = \mathbb{1} \quad \Longleftarrow \quad$ $(\sigma_z + \mathbb{1})(\sigma_z - \mathbb{1}) = 0$

$$\Downarrow$$

$$(\hat{n}\cdot\vec{\sigma} + \mathbb{1})(\hat{n}\cdot\vec{\sigma} - \mathbb{1}) = 0$$

V. $[\sigma_i, \sigma_j]_+ = 2\delta_{ij}\mathbb{1}$ (from (I) & (IV))

VI. For any $\overbrace{\text{vectors or vector operators}}^{\vec{\Omega},\, \vec{\Gamma}}$ that commute with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$:

$$(\vec{\Omega}\cdot\vec{\sigma})(\vec{\Gamma}\cdot\vec{\sigma}) = \vec{\Omega}\cdot\vec{\Gamma}\,\mathbb{1} + i(\vec{\Omega}\times\vec{\Gamma})\cdot\vec{\sigma} \quad \text{(from (II)+(V))}$$

VII. $\mathrm{Tr}(\sigma_i\sigma_j) = 2\delta_{ij}$

Finite spin rotations

$$U[R(\vec\theta)] = \exp(-i\vec\theta\cdot\vec S/\hbar) = \exp\left(-i\vec\theta\cdot\vec\sigma/2\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i\theta}{2}\right)^n \underbrace{(\hat\theta\cdot\vec\sigma)^n}_{\substack{n \text{ odd} \\ = \hat\theta\cdot\vec\sigma \qquad\qquad n \text{ even} \\ = \underline{1\!\!1}}}$$

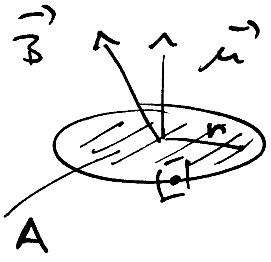
$(\underline{IV})$ with $\hat n = \hat\theta$

$$= \cos(\theta/2)\underline{1\!\!1} - i\sin(\theta/2)\,\hat\theta\cdot\vec\sigma$$

Time evolution ("spin dynamics")

important case: spin precession in a magnetic field

Classically:



$$\vec{T} = \vec{\mu} \times \vec{B} \Rightarrow H_{int} = -\vec{\mu} \cdot \vec{B}$$

$$\mu = \frac{IA}{c} = \frac{qv}{2\pi r}\frac{\pi r^2}{c} = \left(\frac{q}{2mc}\right) mvr$$

$$= \frac{q}{2mc}\vec{l} \Rightarrow \boxed{\vec{\mu} = \frac{q}{2mc}\vec{l}}$$

$$\vec{T} = \frac{d\vec{l}}{dt} = \vec{\mu} \times \vec{B} = \gamma(\vec{l} \times \vec{B})$$

$\gamma = \frac{q}{2mc}$ "gyromagnetic ratio"

$$\Rightarrow \Delta\vec{l} = \gamma(\vec{l} \times \vec{B})\Delta t$$

$$\Delta l = \gamma l B \sin\theta \Delta t, \quad \Delta\phi = \frac{\Delta l}{l\sin\theta} \Rightarrow \Delta\phi = (\gamma B)\Delta t$$

$$\Rightarrow \boxed{\vec{\omega} = -\gamma \vec{B}}$$

NEXT PAGE *

Q. M.

STERN - GERLACH (1924)

$\vec{S}$ comes with a magnetic moment $\vec{\mu}$

$$\vec{\mu} = \gamma\vec{S} = \underset{\text{quantum corrections}}{g}\underset{\text{classical } \gamma}{\left(\frac{-e}{2mc}\right)}\vec{S}$$

g-factor:
2.0023193048[8/4]
EXP
THEORY (QED)

$$H_{int} = -\vec{\mu} \cdot \vec{B} = \frac{ge}{2mc}\vec{S} \cdot \vec{B} = \frac{ge\hbar}{4mc}\vec{\sigma} \cdot \vec{B}$$

$$H_{int} = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{S}\cdot\vec{B} \quad , \quad \gamma = g \times (\text{classical } \gamma)$$

TIME EVOLUTION ?

$$|\psi(0)\rangle \xrightarrow{U(t)} |\psi(t)\rangle = U(t)\,|\psi(0)\rangle$$

$$U(t) = e^{-iHt/\hbar} = e^{i\gamma t(\vec{S}\cdot\vec{B})/\hbar}$$

COMPARE TO $e^{-i\vec{\theta}\cdot\vec{S}/\hbar}$

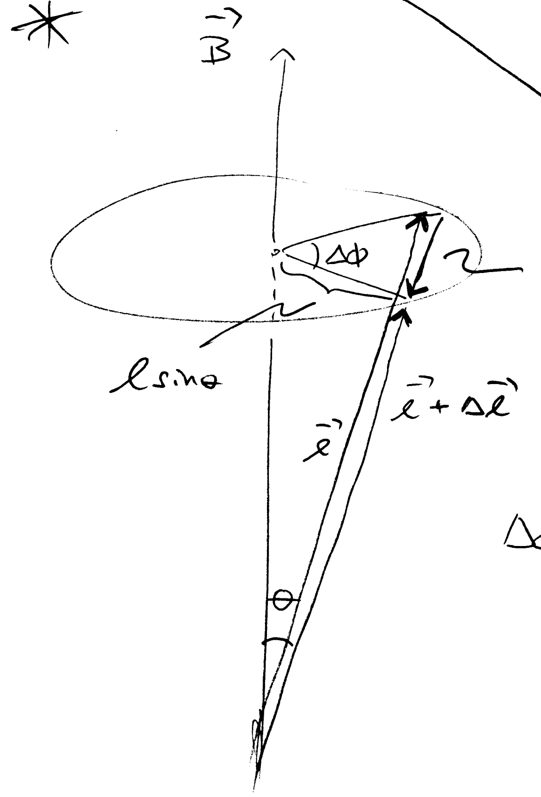
OPERATOR FOR ROTATIONS BY AN ANGLE θ ABOUT $\hat{\theta}$

⇓

U(t) ROTATES THE STATE BY AN ANGLE $\vec{\theta}(t) = -\gamma\vec{B}t$

⇓

$$\dot{\vec{\theta}} = \boxed{\vec{\omega} = -\gamma\vec{B}}$$



$$\Delta\phi = \frac{\Delta l}{l\sin\theta}$$

Ex. $\vec{B} = B\hat{z}$, $\omega_0 = \gamma B$

$$U(t) = \exp(i\gamma t S_z B/\hbar) = \exp(i\omega_0 t \sigma_z/2)$$

$$\xrightarrow[|S_z\rangle \text{ basis}]{} \begin{pmatrix} e^{i\omega_0 t/2} & 0 \\ 0 & e^{-i\omega_0 t/2} \end{pmatrix}$$

$$|\psi(0)\rangle = |\hat{n},+\rangle \xrightarrow[\text{c.f. p}^5]{} \begin{pmatrix} \cos(\theta/2)\, e^{-i\phi/2} \\ \sin(\theta/2)\, e^{i\phi/2} \end{pmatrix}$$

$$\Downarrow$$

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle = \begin{pmatrix} e^{i\omega_0 t/2} & 0 \\ 0 & e^{-i\omega_0 t/2} \end{pmatrix} \begin{pmatrix} \cos(\theta/2)\, e^{-i\phi/2} \\ \sin(\theta/2)\, e^{i\phi/2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta/2)\, e^{-i(\phi-\omega_0 t)/2} \\ \sin(\theta/2)\, e^{i(\phi-\omega_0 t)/2} \end{pmatrix}$$

$\Rightarrow$ ϕ decreases by the rate $\omega_0 t$

Addition of angular momenta

sec 3.6, Sakurai

simple example: two particles with spin-1/2

BASIS $\quad |s_1 m_1\rangle \otimes |s_2 m_2\rangle \equiv |s_1 m_1, s_2 m_2\rangle$

$$\vec{S}_1 = \vec{S}_1^{(1)} \otimes \mathbb{1}^{(2)} \qquad \vec{S}_2 = \mathbb{1}^{(1)} \otimes \vec{S}_2^{(2)}$$

$$S_i^2 |s_1 m_1, s_2 m_2\rangle = \hbar^2 s_i(s_i+1) |s_1 m_1, s_2 m_2\rangle$$

$$S_{iz} |s_1 m_1, s_2 m_2\rangle = \hbar m_i |s_1 m_1, s_2 m_2\rangle , \qquad i=1,2$$

$$s_1 = s_2 = 1/2 \ , \ m_1 = \pm 1/2 \ , \ m_2 = \pm 1/2$$

$\Downarrow$ 4 possible states

"FACTORIZED BASIS"

$$|++\rangle, \ |+-\rangle, \ |-+\rangle, \ |--\rangle$$

states with well-defined values for the magnitude and z-component of the *individual spins* of the particles

What is the spin of the two-particle system as a whole?

Problem of "addition of angular momentum"!

$\vec{S} = \vec{S}_1 + \vec{S}_2$ total spin operator

$[S_i, S_j] = i\hbar\, \varepsilon_{ijk} S_k$, $i, j, k = x, y$ or z

Eigenvalue problem: **eigenvalues and eigenstates of S^2 and S_z ?**

$$S_z = S_{1z} + S_{2z}$$

$$S_z |++\rangle = (S_{1z} + S_{2z})|++\rangle = \left(\frac{\hbar}{2} + \frac{\hbar}{2}\right)|++\rangle = \hbar\,|++\rangle$$

$$S_z |+-\rangle = 0\,|+-\rangle$$

$$S_z |-+\rangle = 0\,|-+\rangle$$

$$S_z |--\rangle = -\hbar\,|--\rangle$$

$$S_z \xrightarrow[\substack{\{|s_1 m_1, s_2 m_2\rangle\} \\ \text{basis}}]{} \hbar \begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \end{array} \begin{array}{c} \\ ++ \\ +- \\ -+ \\ -- \end{array}$$

SIMILARLY FOR $S^2 = (\vec{S}_1 + \vec{S}_2)\cdot(\vec{S}_1 + \vec{S}_2) = S_1^2 + S_2^2 + 2\,\vec{S}_1 \cdot \vec{S}_2$

$$S^2 \xrightarrow{\qquad} \hbar \begin{array}{c} \begin{array}{cccc} ++ & +- & -+ & -- \end{array} \\ \left(\begin{array}{cccc} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{array}\right) \end{array} \begin{array}{c} \\ ++ \\ +- \\ -+ \\ -- \end{array}$$

Note: $|+-\rangle$ and $|-+\rangle$ are **not** eigenstates of S^2

But,

$$|S=1\rangle = \frac{|+-\rangle + |-+\rangle}{\sqrt{2}} \qquad S=1,\ m=0$$

$$|S=0\rangle = \frac{|+-\rangle - |-+\rangle}{\sqrt{2}} \qquad S=0,\ m=0$$

are eigenstates!

↓

"Total-S basis" for two spin-1/2 particles:

$$\left.\begin{array}{l} |S=1, m=1\rangle = |++\rangle \\ |S=1, m=0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right) \\ |S=1, m=-1\rangle = |--\rangle \end{array}\right\} \textit{spin-1 (triplet)}$$

$$|S=0, m=0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) \quad \textit{spin-0 (singlet)}$$

$$\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$$

The direct product of two spin-1/2 Hilbert spaces is a direct sum of a spin-1 and a spin-0 space.

Total wave function

$$|\Psi\rangle = |\Psi_{\text{Orbital}}\rangle \otimes |\chi_{\text{Spin}}\rangle$$

$$\uparrow \qquad\qquad \uparrow \text{ASSUME} \Rightarrow \uparrow$$

anti-symmetric **symmetric** **anti-symmetric**

$\Uparrow$

required for *identical*
spin-1/2 particles

(SPIN-STATISTICS THEOREM)

$\Downarrow$ $s = 0, m = 0$

$$\boxed{\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)}$$

"entangled state"

↯

we cannot assign a quantum (spin) state (even an unknown one!) to each of the particles individually

$\Downarrow$

NON-LOCAL QUANTUM CORRELATIONS

"spooky action at a distance..."

Einstein-Podolsky-Rosen (1935)

Experimental test possible via "BELL'S INEQUALITY"
(more about this after the fall recess!)

The general problem

$$\vec{J} = \vec{J}_1 + \vec{J}_2$$

What are the eigenvalues and eigenkets?

One way we could try: copy the procedure above for two spin-1/2 particles, i.e. construct the $(2j_1+1)x(2j_2+1)$ matrices J_z and J^2, and diagonalize them.

boring and time consuming...

A faster track:

black board !

⟶

But why do we have to worry about the TOTAL ANGULAR MOMENTUM ?
Why not keep working with the individual angular momenta, i.e. use a factorized basis ?

SYMMETRY and INTERACTIONS

force us to go to the "total-J basis".

DISCUSSION

Possible values of j ?

CONJECTURE (guided by the spin-1/2 case)

$$j = j_1+j_2,\ j_1+j_2-1,\ \ldots,\ j_1-j_2 \quad , \quad j_1 \geq j_2$$

?

\# of product kets $\underbrace{|j_1 m_1\rangle \otimes |j_2 m_2\rangle}_{|j_1 m_1 , j_2 m_2\rangle} = (2j_1+1)(2j_2+1)$

$\updownarrow$ Ok !

\# of kets in the "total-J basis":

$$\sum_{j=j_1-j_2}^{j_1+j_2} (2j+1) = \sum_{j=0}^{j_1+j_2} (2j+1) - \sum_{j=0}^{j_1-j_2-1} (2j+1) = (2j_1+1)(2j_2+1)$$

$$\uparrow$$

$$\sum_{n=0}^{N} n = \frac{N(N+1)}{2}$$

We take this as proof of our conjecture !

$\Downarrow$

$$j_1 \otimes j_2 = (j_1+j_2) \oplus (j_1+j_2-1) \oplus \ldots \oplus (j_1-j_2)$$

$\Downarrow$

KETS IN THE "TOTAL-J BASIS"

$$|jm, j_1 j_2\rangle \qquad j_1-j_2 \leq j \leq j_1+j_2 \quad , \quad -j \leq m \leq j$$

$\xrightarrow{j}$

$m \downarrow$

$m \downarrow$ \ $j \rightarrow$	j_1+j_2	j_1+j_2-1	$\dots$	j_1-j_2
	$\vert j_1+j_2, j_1+j_2\rangle$			
	$\vert j_1+j_2, j_1+j_2-1\rangle$	$\vert j_1+j_2-1, j_1+j_2-1\rangle$		
	$\vert j_1+j_2, j_1+j_2-2\rangle$	$\vert j_1+j_2-1, j_1+j_2-2\rangle$		$\vert j_1-j_2, j_1-j_2\rangle$
	$\vdots$	$\vdots$	$\dots$	$\vdots$
	$\vert j_1+j_2, -(j_1+j_2-2)\rangle$	$\vert j_1+j_2-1, -(j_1+j_2-2)\rangle$		$\vert j_1-j_2, -(j_1-j_2)\rangle$
	$\vert j_1+j_2, -(j_1+j_2-1)\rangle$	$\vert j_1+j_2-1, -(j_1+j_2-1)\rangle$		
	$\vert j_1+j_2, -(j_1+j_2)\rangle$			

EXPRESS EACH OF THESE KETS AS A LINEAR COMBINATION OF PRODUCT KETS $|j_1 m_1\rangle \otimes |j_2 m_2\rangle$

This is the key problem!

Look at the table on the previous page !

Look at the first column !

Ⓐ The top state ("highest-weight state") with $m = j_1 + j_2$ (= max value of m) can be built out of only one product ket, each factor having its maximum value of m_i :

$m_1 = j_1, \; m_2 = j_2$

$$\Downarrow$$

phase factor = +1 for the top state in each column ("Condon–Shortley convention")

$$\Big|\underbrace{j_1+j_2}_{j}, \underbrace{j_1+j_2}_{m}, \overset{\text{drops from now on}}{\cancel{j_1 j_2}}\Big\rangle = \underbrace{|j_1 j_1, j_2 j_2\rangle}$$

RECALL: Short hand for $|j_1 m_1 = j_1\rangle \otimes |j_2 m_2 = j_2\rangle$

Ⓑ Next, let's step down to the next state in the first column by applying J_- to the top state

see yesterday's lecture

$$J_- |j_1+j_2, j_1+j_2\rangle = \hbar\sqrt{2(j_1+j_2)}\, |j_1+j_2, j_1+j_2-1\rangle$$

$$\Rightarrow |j_1+j_2, j_1+j_2-1\rangle = \frac{1}{\hbar\sqrt{2(j_1+j_2)}} J_- |j_1+j_2, j_1+j_2\rangle$$

$$(\text{from Ⓐ}) = \frac{1}{\hbar\sqrt{}} (J_{1-} + J_{2-}) |j_1 j_1, j_2 j_2\rangle$$

$$= \frac{1}{\cancel{\hbar}\sqrt{}} \Big\{ \cancel{\hbar}\sqrt{2j_1}\, |j_1 (j_1-1), j_2 j_2\rangle + \cancel{\hbar}\sqrt{2j_2}\, |j_1 j_1, j_2 (j_2-1)\rangle \Big\}$$

Ⓒ Continue this way to $m = -(j_1+j_2)$

THIS EXHAUSTS THE FIRST COLUMN

Now to the second column :

Ⓐ

$m = j_1 + j_2 - 1 = m_1 + m_2 \Rightarrow m_1 = j_1 , m_2 = j_2 - 1$

$m_1 = j_1 - 1 , m_2 = j_2$

$\Downarrow$

two building blocks

$|j_1 j_1 , j_2 (j_2-1)\rangle$

$|j_1 (j_1-1) , j_2 j_2\rangle$

Require that $|j_1+j_2-1 , j_1+j_2-1\rangle$ is ORTHOGONAL to $|j_1+j_2 , j_1+j_2-1\rangle$

(+ the Condon-Shortley condition)

This uniquely fixes the linear combination of $|j_1 j_1 , j_2 (j_2-1)\rangle$ and $|j_1 (j_1-1) , j_2 j_2\rangle$ that defines $|j_1+j_2-1 , j_1+j_2-1\rangle$!

Ⓑ Climb down the second column by making repeated use of J_- (as for the first column).

THIS EXHAUSTS THE
SECOND COLUMN

AND SO ON ...

We can summarize the outcome of our procedure using the **Clebsch-Gordan** coefficients.

It's easy: use the completeness of the $|j_1m_1, j_2m_2\rangle$ basis:

$$|jm\rangle = \sum_{m_1,m_2} |j_1m_1, j_2m_2\rangle \underbrace{\langle j_1m_1, j_2m_2|jm\rangle}_{\text{Clebsch-Gordan coefficients}}$$

can be looked up in tables

Properties of the **Clebsch-Gordan coefficients**

(1) $\langle j_1m_1, j_2m_2|jm\rangle = 0$ unless $m = m_1+m_2$
and $j_1-j_2 \leq j \leq j_1+j_2$

(2) real by convention

(3) $\langle j_1j_1, j_2(j-j_1)|jj\rangle \geq 0$ by convention

(4) $\langle j_1m_1, j_2m_2|jm\rangle = (-1)^{j_1+j_2-j}\langle j_1(-m_1), j_2(-m_2)|j(-m)\rangle$

(5) $$\overset{|jm\rangle}{\begin{pmatrix} & \\ & \\ & \end{pmatrix}} = \begin{pmatrix} \text{CLEBSCH-GORDAN} \\ \text{MATRIX} \end{pmatrix} \overset{|j_1m_1, j_2m_2\rangle}{\begin{pmatrix} & \\ & \\ & \end{pmatrix}}$$

REAL AND UNITARY (=ORTHOGONAL) MATRIX

EXPECTED, SINCE IT CONNECTS TWO $O(N)$ BASES

So far we've looked at the addition of spin or total angular momentum for two particles.

What about the addition of orbital and spin angular momentum for a *single particle* ?

(Sakurai tretas these two classes of problems in the reverse order.)

The same formalism applies!

$$\vec{J} = \vec{L} + \vec{S}$$

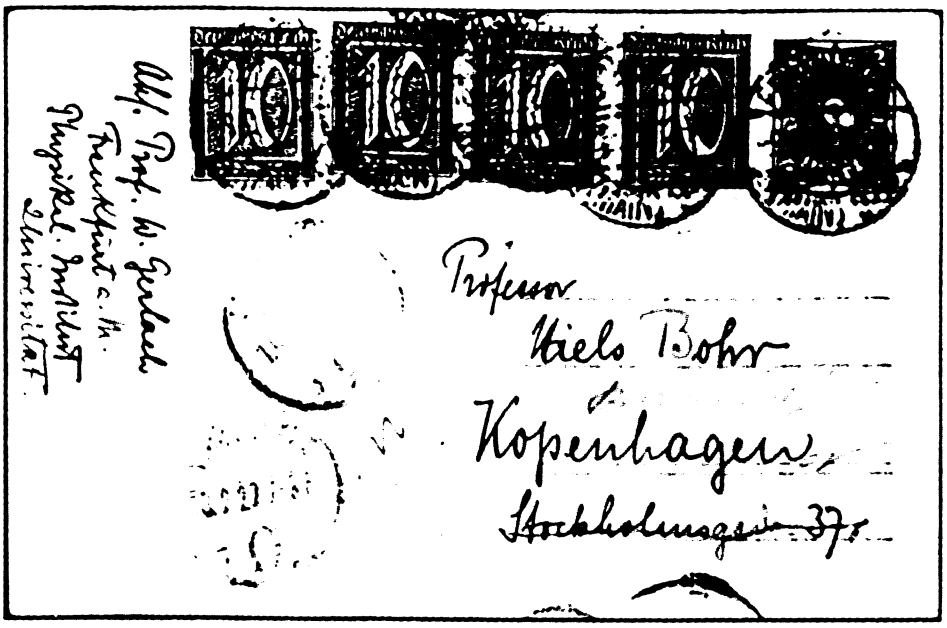
$$\underset{\substack{\nearrow \\ \text{ORBITAL}}}{|\ell m_0\rangle} \otimes \underset{\substack{\uparrow \\ \text{SPIN}}}{|s\, m_s\rangle} \equiv |\ell m_0, s\, m_s\rangle$$

"Total-J basis" particularly convenient when studying spin-orbit interactions
(cf. chapter 17 in Sakurai)

Final topic on angular momentum:

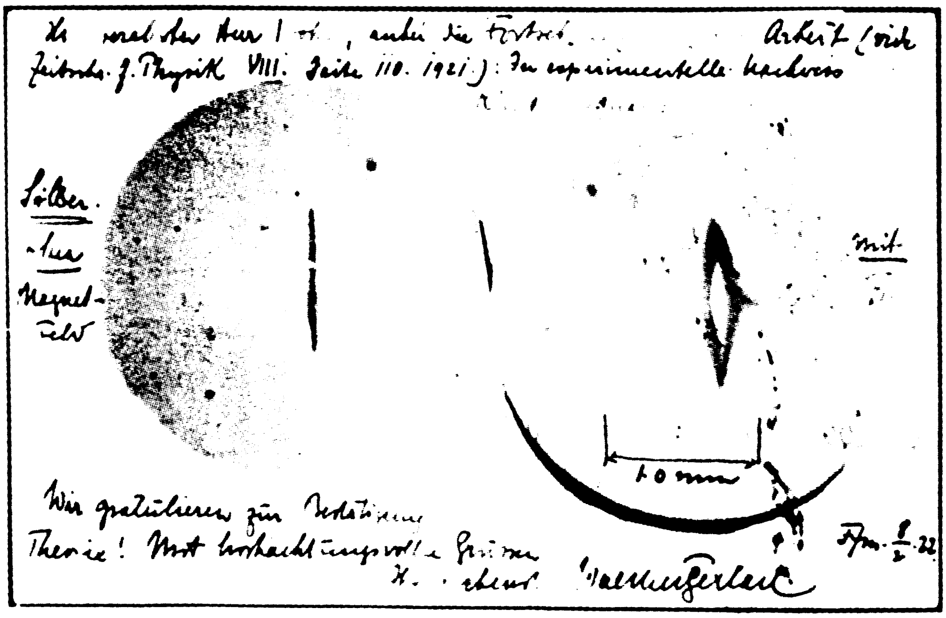
Irreducible tensor operators and the Wigner-Eckart theorem

black board!



Abs. Prof. W. Gerlach
Frankfurt a. M.
Physikal. Institut
Universität

Professor
Niels Bohr
Kopenhagen
~~Stockholmsgade 37~~



[illegible] verehrter Herr [illegible], anbei die Fortset[illegible] Arbeit (vide Zeitschr. f. Physik VIII. Seite 110. 1921): Der experimentelle Nachweis

Wir gratulieren zur Bestätigung
Theorie! Mit hochachtungsvolle Grüssen
Ihr ergebener Walther Gerlach

Ffm. 8/2.22

photograph of the postcard sent by Gerlach, Stern's collaborator in their famous experiment on 'space quantization' to Niels Bohr, announcing their discovery.